Original Research Article

Golden Section and Evolving Systems

Abstract

We present a body of results that demonstrate that the golden section, one of the most important intelligence signatures in the universe, can be implemented in the design of evolving systems. Albeit we only develop the mathematics, both mathematician and computer expert should find the concept of this manuscript clear. Perhaps of particular interest to the number theorist is the observation that the sequence $T_n = 1,4,7,10,13, \ldots$ defined by $t_{n+1} = t_n + 3, n \geq 1, t_1 = 1$, known as the Teleois number system, crops up in our results. Having shown in previous works how this sequence is closely related to the golden section, this manuscript gives further confirmation and the fact that Teleois numbers penetrate the golden section renders it a proportion of great splendour. Our results can find a wide range of applications from information technology through manufacturing to military purposes.

Keywords: Cassini identity; evolving systems; Fibonacci sequences; golden section; Teleois numbers; transformation vector; zero transformation.

1 Introduction

Let an integer x satisfy

$$\begin{cases} y = round(x\varphi) \\ y - x = z \\ x \neq round(z\varphi) \\ \varphi = \frac{1+\sqrt{5}}{2} \end{cases} \quad (1.1)$$

x is called a parent number and is a seed value of a Fibonacci sequence. Let a sequence H_n satisfy the relation

$$h_{n+1} = round(\varphi h_n), n \geq 1 \quad (1.2)$$

H_n is called a Fibonacci sequence [1]. As a natural consequence of relation (1.2), H_n also satisfies

$$h_{n+2} = h_{n+1} + h_n, n \geq 1 \quad (1.3)$$

The work at hand is devoted to mathematically demonstrating that the golden section can be implemented in (the design of) evolving systems. The concept makes much reliance upon the Cassini identity for H_n given generally as

$$\left.\begin{matrix} h_n h_{n+2} - {h_{n+1}}^2 = c(-1)^a, n \geq 1 \\ a = n \text{ or } n+1 \end{matrix}\right\} \quad (1.4)$$

We introduce transformations based on the Cassini rule and we obtain useful results. For analysis purposes, a Fibonacci sequence H_n shall be represented in the mechanical form

$$h_i = g_{n+i-1} \pm f_i, i \geq 1, n \geq 4 \quad (1.5)$$

where G_n is a Fibonacci sequence and

$$F_n = 1,2,3,5,8, \ldots \quad (1.6)$$

Hereinafter, the designation F_n is reserved for the sequence (1.6).

The concept of parent number as defined above has enabled us to regenerate not only the sequence (1.6) but also the "Lucas numbers" through the Fibonacci sequence

$$H_n = 7,11,18,29,47,\ldots \quad (1.7)$$

We would like to assemble a sequence L_n defined by

$$l_n = f_i + f_{i+2}, n \geq 1 \quad (1.8)$$

thus

$$L_n = 4,7,11,18,29,\ldots \quad (1.9)$$

Notice in passing that the sequence (1.9) is the sequence (1.7) extended backward by one step. Now let

$$2l_n = l_n{}', n \geq 1 \quad (1.10)$$

It follows

$$L_n{}' = 8,14,22,36,58,\ldots \quad (1.11)$$

The sequence (1.9) has profound significance to the results of this work. The sequence (1.11), call it the "double Lucas numbers", is important in the study of symmetry. One may find the result in theorem 1.1 interesting.

Theorem 1.1

Consider two Fibonacci sequences P_n and Q_n such that

$$\left.\begin{matrix} p_i = h_{n+i-1} + f_i \\ q_i = h_{n+i-1} - f_i \end{matrix}\right\} i \geq 1, n \geq 5 \quad (1.12)$$

$$(p_i + p_{i+2}) - (q_i + q_{i+2}) = l_i{}' \quad (1.13)$$

Proof

$$\begin{aligned} p_i + p_{i+2} &= (h_{n+i-1} + f_i) + (h_{n+i+1} + f_{i+2}) \\ &= h_{n+i-1} + f_i + h_{n+i+1} + f_{i+2} \end{aligned} \quad (1.14)$$

$$\begin{aligned} q_i + q_{i+2} &= (h_{n+i-1} - f_i) + (h_{n+i+1} - f_{i+2}) \\ &= h_{n+i-1} - f_i + h_{n+i+1} - f_{i+2} \end{aligned} \quad (1.15)$$

$$\begin{aligned} (1.14) - (1.15) &= h_{n+i-1} + f_i + h_{n+i+1} + f_{i+2} - h_{n+i-1} + f_i - h_{n+i+1} + f_{i+2} \\ &= 2(f_i + f_{i+2}) \\ &= 2l_i \\ &= l_i{}' \end{aligned} \quad (1.16)$$

2 Results

2.1 Zero transformation

Axiom 2.1

Let a Fibonacci sequence H_n be defined by $h_i = g_{n+i-1} - f_i, i \geq 1, n \geq 4$. The Cassini identity is given by

$$h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^{i+1}, i \geq 1 \tag{2.1}$$

where c is a constant.

Axiom 2.2

Let a Fibonacci sequence H_n be defined by $h_i = g_{n+i-1} + f_i, i \geq 1, n \geq 4$. The Cassini identity is given by

$$h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^{i}, i \geq 1 \tag{2.2}$$

where c is a constant.

Let the constant *c* be the Cassini value of H_n. Take $h_i, i \geq 5$. Let

$$\left.\begin{array}{l} h_i - 1 = p_1 \\ h_i + 1 = q_1 \end{array}\right\} \tag{2.3}$$

where P_n and Q_n are Fibonacci sequences. Now take p_j and $q_j, j \geq 4$. Consider the sequences

$$\left.\begin{array}{l} p_j - 1, p_{j+1} - 2, p_{j+2} - 3, p_{j+3} - 5, \dots : c = c_1 \\ p_j + 1, p_{j+1} + 2, p_{j+2} + 3, p_{j+3} + 5, \dots : c = c_2 \\ q_j - 1, q_{j+1} - 2, q_{j+2} - 3, q_{j+3} - 5, \dots : c = c_3 \\ q_j + 1, q_{j+1} + 2, q_{j+2} + 3, q_{j+3} + 5, \dots : c = c_4 \end{array}\right\} \tag{2.4}$$

Let

$$(h_i)_j = (c_1, c_2, c_3, c_4) \tag{2.5}$$

Let

$$z = c_1 + c_2 + c_3 + c_4 \tag{2.6}$$

Theorem 2.1

For an arbitrary Fibonacci sequence H_n, let equations (2.3) to (2.6) hold. $z = 4(h_{i+j-4} + h_{i+j-2}$.

Proof

$$c_1 + c_2 = 6f_j + 2f_{j+2} + 8h_{i+j} - 8f_{j+1} - 6h_{i+j-1} - 2h_{i+j+1} \tag{2.7}$$

$$c_3 + c_4 = 8h_{i+j} + 8f_{j+1} - 6f_j - 2f_{j+2} - 6h_{i+j-1} - 2h_{i+j+1} \tag{2.8}$$

$$\begin{aligned} z &= 16h_{i+j} - 12h_{i+j-1} - 4h_{i+j+1} \\ &= 4(4h_{i+j} - 3h_{i+j-1} - h_{i+j+1}) \\ &= 4(4h_{i+j} - 3h_{i+j-1} - h_{i+j} - h_{i+j-1}) \\ &= 4(3h_{i+j} - 4h_{i+j-1}) \\ &= 4(h_{i+j-4} + h_{i+j-2}) \end{aligned} \tag{2.9}$$

Proof is complete.

Now let $(h_j)_i = (c_1', c_2', c_3', c_4')$. The transformation vector for the mapping

$$(h_{2i+2})_{2i+3} \to (h_{2i+3})_{2i+2}, i \geq 1 \tag{2.10}$$

is given by

$$v = (c_1 - c_4', c_2 - c_1', c_3 - c_2', c_4 - c_3') \tag{2.11}$$

The transformation vector for the mapping

$$(h_{2i+3})_{2i+4} \to (h_{2i+4})_{2i+3}, i \geq 1 \tag{2.12}$$

is given by

$$v = (c_1 - c_2', c_2 - c_3', c_3 - c_4', c_4 - c_1') \tag{2.13}$$

Theorem 2.2: zero transformation

Consider an arbitrary Fibonacci sequence H_n *. For the mapping* $(h_n)_{n+1} \to (h_{n+1})_n, n \geq 4$, *let the transformation vector be given by* $v = (w_1, w_2, w_3, w_4)$. $\sum_{k=1}^{k=4} w_k = 0$.

Proof

$(h_n)_{n+1} = (c_1, c_2, c_3, c_4)$
From theorem 2.1,

$$z = 4\big(h_{n+(n+1)-4} + h_{n+(n+1)-2}\big) = 4(h_{2n-3} + h_{2n-1}) \tag{2.14}$$

$(h_{n+1})_n = (c_1', c_2', c_3', c_4')$
Again from theorem 2.1,

$$z' = 4\big(h_{(n+1)+n-4} + h_{(n+1)+n-2}\big) = 4(h_{2n-3} + h_{2n-1}) \tag{2.15}$$

Notice that (2.14) = (2.15). Therefore (2.14)-(2.15)=0, thus transformation vector v = 0.

2.2 Direct computation of transformation vector

We first need to state theorems 2.3 and 2.4.

Theorem 2.3

Let a Fibonacci sequence H_n *be defined by* $h_i = g_{n+i-1} - f_i, i \geq 1, n \geq 4$. *The Cassini identity is given by* $h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^{i+1}, i \geq 1$. *The constant c is given by*

$$\left.\begin{aligned} c &= g_{n-1} + g_{n-3} + c_g - 1 \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \tag{2.16}$$

Proof

$$\begin{aligned} c &= h_1 h_3 - h_2^2 \\ &= (g_n - 1)(g_{n+2} - 3) - (g_{n+1} - 2)^2 \\ &= g_n g_{n+2} - 3g_n - g_{n+2} + 3 - g_{n+1}^2 + 4g_{n+1} - 4 \\ &= 4(3g_{n-1} - g_{n-3}) - 3(2g_{n-1} - g_{n-3}) - (5g_{n-1} - 2g_{n-3}) + c_g - 1 \\ &= g_{n-1} + g_{n-3} + c_g - 1 \end{aligned}$$

Theorem 2.4

Let a Fibonacci sequence H_n *be defined by* $h_i = g_{n+i-1} + f_i, i \geq 1, n \geq 4$. *The Cassini identity is given by* $h_i h_{i+2} - {h_{i+1}}^2 = c(-1)^i, i \geq 1$. *The constant c is given by*

$$\left.\begin{aligned} c &= g_{n-1} + g_{n-3} - c_g + 1 \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \end{aligned}\right\} \tag{2.17}$$

Proof

$$\begin{aligned} &h_1 h_3 - h_2^2 \\ &= (g_n + 1)(g_{n+2} + 3) - (g_{n+1} + 2)^2 \end{aligned}$$

$= g_n g_{n+2} + 3g_n + g_{n+2} + 3 - g_{n+1}^2 - 4g_{n+1} - 4$
$= 5(g_{n-1}) - 2(g_{n-3}) + 3(2g_{n-1} - g_{n-3}) - 4(3g_{n-1} - g_{n-3}) + c_g - 1$
$= -g_{n-1} - g_{n-3} + c_g - 1$ (2.18)

From axiom 2.2, $h_1 h_3 - h_2^2$ is negative. This means
$c = -(2.18)$
$= g_{n-1} + g_{n-3} - c_g + 1$

Theorems 2.3 and 2.4 are crucial to theorems 2.5 to 2.12 that deal with the direct computation of transformation vector.

Theorem 2.5

Let a Fibonacci sequence H_n be such that $h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n$ is even. Consider the transformation $(h_i)_{i+1} \to (h_{i+1})_i$. Let the transformation vector be given by $v = (w_1, w_2, w_3, w_4)$. When i is even,

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \quad (2.19)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \quad (2.20)$$

Proof

<u>Scenario I:</u> i is even
Given $(h_i)_{i+1} \to (h_{i+1})_i$, let
$(h_i)_{i+1} = (c_1, c_2, c_3, c_4)$
$(h_{i+1})_i = (c_1', c_2', c_3', c_4')$
Since i is even, equation (2.11) gives the transformation vector as $v = (c_1 - c_4', c_2 - c_1', c_3 - c_2', c_4 - c_3')$. Consider $(h_i)_{i+1}$. Using equation (2.5), $j = i + 1$. It follows $p_1 = h_i - 1$. This means

$$\left.\begin{aligned} p_{i+1} &= h_{2i} - f_{i+1} \\ p_{i+2} &= h_{2i+1} - f_{i+2} \\ p_{i+3} &= h_{2i+2} - f_{i+3} \end{aligned}\right\} \quad (2.21)$$

$(p_{i+1} - 1)(p_{i+3} - 3) - (p_{i+2} - 2)^2$
$= (h_{2i} - f_{i+1} - 1)(h_{2i+2} - f_{i+3} - 3) - (h_{2i+1} - f_{i+2} - 2)^2$
$= h_{2i}h_{2i+2} - f_{i+3}h_{2i} - 3h_{2i} - f_{i+1}h_{2i+2} + f_{i+1}f_{i+3} + 3f_{i+1} - h_{2i+1}^2 + 2f_{i+2}h_{2i+1} + 4h_{2i+1} - 4f_{i+2}$
$-\ f_{i+2}^2 - 4 - h_{2i+2} + f_{i+3} + 3$ (2.22)

With even i, from axion 2.1 and theorem 2.3,
$h_{2i}h_{2i+2} - h_{2i+1}^2 = -(g_{n-1} + g_{n-3} + c_g - 1)$
Also notice that
$f_{i+1}f_{i+3} - f_{i+2}^2 = -1$, *therefore equation (2.22) reduces to*

$$2f_{i+2}h_{2i+1} + 4h_{2i+1} - f_{i+3}h_{2i} - 3h_{2i} - f_{i+1}h_{2i+2} - h_{2i+2} - (3f_i + f_{i+2} - f_{i+3}) - (g_{n-1} + g_{n-3} + c_g) - 1 \tag{2.23}$$

Since i is even, it follows $c_1 = (2.23)$. *Now consider* $(h_{i+1})_i$.

$q'_1 = h_{i+1} + 1$

$q'_i = h_{2i} + f_i, q'_{i+1} = h_{2i+1} + f_{i+1}, q'_{i+2} = h_{2i+2} + f_{i+2}$

$$(q'_i + 1)(q'_{i+2} + 3) - (q'_{i+1} + 2)^2 = (h_{2i} + f_i + 1)(h_{2i+2} + f_{i+2} + 3) - (h_{2i+1} + f_{i+1} + 2)^2 \tag{2.24}$$

$$= h_{2i}h_{2i+2} + f_{i+2}h_{2i} + 3h_{2i} + f_ih_{2i+2} + f_if_{i+2} + 3f_i + h_{2i+2} + f_{i+2} + 3 - h_{2i+1}^2 - 2f_{i+1}h_{2i+1} - 4h_{2i+1} - 4f_{i+1} - f_{i+1}^2 - 4$$

Since i is even, it follows $c'_4 = -(2.24)$, *therefore,*

$$c'_4 = 4h_{2i+1} + 4f_{i+1} + f_{i+1}^2 - f_if_{i+2} + 1 + h_{2i+1}^2 - h_{2i}h_{2i+2} + 2f_{i+1}h_{2i+1} - f_{i+2}h_{2i} - 3h_{2i} - f_ih_{2i+2} - 3f_i - f_{i+2} - h_{2i+2}$$

$$= 4h_{2i+1} + f_{i+1} + 3f_{i-1} - f_{i+2} - (h_{2i}h_{2i+2} - h_{2i+1}^2) + 2f_{i+1}h_{2i+1} - f_{i+2}h_{2i} - 3h_{2i} - f_ih_{2i+2} - h_{2i+2}$$
$$= 2f_{i+1}h_{2i+1} + 4h_{2i+1} - f_{i+2}h_{2i} - 3h_{2i} - f_ih_{2i+2} - h_{2i+2} + 3f_{i-1} - f_i + (g_{n-1} + g_{n-3} + c_g) - 1 \tag{2.25}$$

$$\begin{aligned} w_1 &= c_1 - c'_4 \\ &= 2f_ih_{2i+1} - f_{i+1}h_{2i} - f_{i-1}h_{2i+2} - (f_{i-1} + f_{i+1}) - 2(g_{n-1} + g_{n-3} + c_g) \\ &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \end{aligned} \tag{2.26}$$

Having given a detailed geometric proof for w_1 *for scenario I of the theorem, it is assumed that the reader may be able to follow the same procedure in proving all eight scenarios.*

It is in the interest of space that we state theorems 2.6 to 2.12 below without proof. The interested reader shall follow proof to theorem 2.5 in proving these theorems.

Theorem 2.6

Let a Fibonacci sequence H_n *be such that* $h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n\ is\ odd$. *Consider the transformation* $(h_i)_{i+1} \to (h_{i+1})_i$. *Let the transformation vector be given by* $v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_ng_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \tag{2.27}$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_ng_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \tag{2.28}$$

Theorem 2.7

Let a Fibonacci sequence H_n *be such that* $h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ even.$ *Consider the transformation* $(h_i)_{i+1} \rightarrow (h_{i+1})_i$. *Let the transformation vector be given by* $v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \quad (2.29)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \quad (2.30)$$

Theorem 2.8

Let a Fibonacci sequence H_n *be such that* $h_i = g_{n+i-1} - f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ odd.$ *Consider the transformation* $(h_i)_{i+1} \rightarrow (h_{i+1})_i$. *Let the transformation vector be given by* $v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \quad (2.31)$$

when i is odd,

$$\left.\begin{aligned} w_1 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \quad (2.32)$$

Theorem 2.9

Let a Fibonacci sequence H_n *be such that* $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n\ is\ even.$ *Consider the transformation* $(h_i)_{i+1} \rightarrow (h_{i+1})_i$. *Let the transformation vector be given by* $v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 &= h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g &= g_n g_{n+2} - g_{n+1}^2 \\ L_n &= (1.9) \end{aligned}\right\} \quad (2.33)$$

when i is odd,

$$\left.\begin{array}{l} w_1 = h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 = h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 = l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 = l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ \qquad c_g = g_n g_{n+2} - g_{n+1}^2 \\ \qquad L_n = (1.9) \end{array}\right\} \quad (2.34)$$

Theorem 2.10

Let a Fibonacci sequence H_n *be such that* $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} + f_n, m, n \geq 4, n\ is\ odd.$ *Consider the transformation* $(h_i)_{i+1} \rightarrow (h_{i+1})_i$. *Let the transformation vector be given by* $v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{array}{l} w_1 = h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 = h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 = l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 = l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ \qquad c_g = g_n g_{n+2} - g_{n+1}^2 \\ \qquad L_n = (1.9) \end{array}\right\} \quad (2.35)$$

when i is odd,

$$\left.\begin{array}{l} w_1 = h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 = h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 = l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 = l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ \qquad c_g = g_n g_{n+2} - g_{n+1}^2 \\ \qquad L_n = (1.9) \end{array}\right\} \quad (2.36)$$

Theorem 2.11

Let a Fibonacci sequence H_n *be such that* $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ even.$ *Consider the transformation* $(h_i)_{i+1} \rightarrow (h_{i+1})_i$. *Let the transformation vector be given by* $v = (w_1, w_2, w_3, w_4)$. *When i is even,*

$$\left.\begin{array}{l} w_1 = h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 = h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 = l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 = l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} + c_g) \\ \qquad c_g = g_n g_{n+2} - g_{n+1}^2 \\ \qquad L_n = (1.9) \end{array}\right\} \quad (2.37)$$

when i is odd,

$$\left.\begin{array}{l} w_1 = h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_2 = h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ w_3 = l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} + c_g) \\ w_4 = l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} + c_g) \\ c_g = g_n g_{n+2} - g_{n+1}^2 \\ L_n = (1.9) \end{array}\right\} \quad (2.38)$$

Theorem 2.12

Let a Fibonacci sequence H_n be such that $h_i = g_{n+i-1} + f_i, i \geq 1,\ g_n = q_{m+n-1} - f_n, m, n \geq 4, n\ is\ odd$. Consider the transformation $(h_i)_{i+1} \rightarrow (h_{i+1})_i$. Let the transformation vector be given by $v = (w_1, w_2, w_3, w_4)$. When i is even,

$$\left.\begin{array}{l} w_1 = h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 = h_{i-2} + h_{i-4} - l_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 = l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 = l_{i-4} - h_{i-2} - h_{i-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g = g_n g_{n+2} - g_{n+1}^2 \\ L_n = (1.9) \end{array}\right\} \quad (2.39)$$

when i is odd,

$$\left.\begin{array}{l} w_1 = h_{i-2} + h_{i-4} - l_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_2 = h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 = l_{i-4} - h_{i-2} - h_{i-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 = l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ c_g = g_n g_{n+2} - g_{n+1}^2 \\ L_n = (1.9) \end{array}\right\} \quad (2.40)$$

2.3 Systems Evolution

A series of the transformations $(h_i)_{i+1} \rightarrow (h_{i+1})_i, i \geq 4$, for a given Fibonacci sequence H_n yields very important results. For illustration we give the first ten such transformations for the sequence

$$H_n = 9,15,24,39,63, \ldots \quad (2.41)$$

We only require the transformation vector, the reason for developing theorems 2.5 to 2.12. These transformations are therefore:

$$\left.\begin{array}{c} (h_4)_5 \rightarrow (h_5)_4 \\ v = (92,2,-60,-34) \end{array}\right\} \quad (2.42)$$

$$\left.\begin{array}{c} (h_5)_6 \rightarrow (h_6)_5 \\ v = (45,107,-13,-139) \end{array}\right\} \quad (2.43)$$

$$\left.\begin{array}{c} (h_6)_7 \rightarrow (h_7)_6 \\ v = (215,31,-183,-63) \end{array}\right\} \quad (2.44)$$

$$\left.\begin{array}{c} (h_7)_8 \rightarrow (h_8)_7 \\ v = (92,306,-60,-338) \end{array}\right\} \quad (2.45)$$

$$\left.\begin{array}{c}(h_8)_9 \rightarrow (h_9)_8 \\ v = (537,107,-505,-139)\end{array}\right\} \tag{2.46}$$

$$\left.\begin{array}{c}(h_9)_{10} \rightarrow (h_{10})_9 \\ v = (215,827,-183,-859)\end{array}\right\} \tag{2.47}$$

$$\left.\begin{array}{c}(h_{10})_{11} \rightarrow (h_{11})_{10} \\ v = (1380,306,-1348,-338)\end{array}\right\} \tag{2.48}$$

$$\left.\begin{array}{c}(h_{11})_{12} \rightarrow (h_{12})_{11} \\ v = (537,2191,-505,-2223)\end{array}\right\} \tag{2.49}$$

$$\left.\begin{array}{c}(h_{12})_{13} \rightarrow (h_{13})_{12} \\ v = (3587,827,-3555,-859)\end{array}\right\} \tag{2.50}$$

$$\left.\begin{array}{c}(h_{13})_{14} \rightarrow (h_{14})_{13} \\ v = (1380,5762,-1348,-5794)\end{array}\right\} \tag{2.51}$$

Let $v = (w_1, w_2, w_3, w_4)$. w_1, w_3 in equation (2.45) equals w_1, w_3 in equation (2.42); w_2, w_4 in equation (2.46) equals w_2, w_4 in equation (2.43); w_1, w_3 in equation (2.47) equals w_1, w_3 in equation (2.44); w_2, w_4 in equation (2.48) equals w_2, w_4 in equation (2.45); etc. Theorems 2.13 and 2.14 structure this result.

Theorem 2.13

For an arbitrary Fibonacci sequence H_n, let the transformation vector for the transformation $(h_i)_{i+1} \to (h_{i+1})_i, i \geq 4$, be given by $v = (w_1, w_2, w_3, w_4)$. If I is even, then $(h_{i+3})_{i+4} \to (h_{i+4})_{i+3}$ has vector $v' = (w_1', w_2', w_3', w_4')$ such that $w_1' = w_1;\ w_3' = w_3$.

Proof

Assume theorem 2.9 applies. In the transformation $(h_i)_{i+1} \to (h_{i+1})_i$, if i is even, equation (2.33) gives

$$\left.\begin{aligned} w_1 &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3 &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \end{aligned}\right\} \quad (2.52)$$

Since i is even, it follows $(i + 3)$ is odd. Let $j = (i + 3)$. We need the transformation $(h_j)_{j+1} \to (h_{j+1})_j$. Since j is odd, from equation (2.34),

$$\left.\begin{aligned} w_1' &= h_{j-2} + h_{j-4} - l_{j-4} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3' &= l_{j-4} - h_{j-2} - h_{j-4} + 2(g_{n-1} + g_{n-3} - c_g) \end{aligned}\right\} \quad (2.53)$$

But $j = (i + 3)$, it follows

$$\left.\begin{aligned} w_1' &= h_{i+1} + h_{i-1} - l_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \\ w_3' &= l_{i-1} - h_{i+1} - h_{i-1} + 2(g_{n-1} + g_{n-3} - c_g) \end{aligned}\right\} \quad (2.54)$$

Notice that (2.54) = (2.52), therefore result is true.

Remark
In our proof we have assumed that theorem 2.9 applies. The same can be done with any of theorems 2.5 to 2.12.

Theorem 2.14

For an arbitrary Fibonacci sequence H_n, let the transformation vector for the transformation $(h_i)_{i+1} \to (h_{i+1})_i, i \geq 4$, be given by $v = (w_1, w_2, w_3, w_4)$. If I is even, then $(h_{i+3})_{i+4} \to (h_{i+4})_{i+3}$ has vector $v' = (w_1', w_2', w_3', w_4')$ such that $w_1' = w_1;\ w_3' = w_3$.

Proof

Assume theorem 2.12 applies. In the transformation $(h_i)_{i+1} \to (h_{i+1})_i$, when i is odd, equation (2.40) gives

$$\left.\begin{aligned} w_2 &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4 &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \end{aligned}\right\} \quad (2.55)$$

Since i is odd, it follows $(i + 3)$ is even. Let $j = (i + 3)$. For the transformation $(h_j)_{j+1} \to (h_{j+1})_j$, since j is even, from equation (2.39),

$$\left.\begin{aligned} w_2' &= h_{j-2} + h_{j-4} - l_{j-4} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4' &= l_{j-4} - h_{j-2} - h_{j-4} - 2(g_{n-1} + g_{n-3} - c_g) \end{aligned}\right\} \quad (2.56)$$

With $j = i + 3$, we have

$$\left.\begin{aligned} w_2' &= h_{i+1} + h_{i-1} - l_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \\ w_4' &= l_{i-1} - h_{i+1} - h_{i-1} - 2(g_{n-1} + g_{n-3} - c_g) \end{aligned}\right\} \quad (2.57)$$

Notice that (2.57)=(2.55), therefore result is true.

Here we have a system that retains and modifies certain attributes as it evolves. But equally striking is the fact that this concept provides once again a link between the golden section and the Teleois numerical system:

$$\left.\begin{matrix} T_n = 1,4,7,10,13,\ldots \\ t_{n+2} = t_{n+1} + 3, n \geq 1, t_1 = 1 \end{matrix}\right\} \quad (2.58)$$

As implied by theorems 2.13 and 2.14, replication and modification of attributes occurs at Teleois positions. Therefore by coding a system following the concept of this paper one does not only implement the golden section, but the Teleois also, about which Hardy et al. [2], cited by Sherbon [3] say, *"Understand the electromagnetic frequencies of the atom and you understand why the Teleois proportions were used."*

3 Conclusion

The concept of this manuscript should be appreciated by both geometer and computer expert. Applications range from information technology through manufacturing to military purposes. The paper also improves the reader's appreciation of the Cassini identity in Fibonacci sequences. Various authors have worked on the application of the Cassini identity and Fibonacci numbers in computing science, especially cryptography, see [4-13].

References

[1] Mamombe L. Proportiones Perfectus Law and the Physics of the Golden Section. *Asian Research Journal of Mathematics. 2017; 7(1):1-41.*

[2] Hardy D. et al. Pyramid Energy. *Clayton G.A: Cadake Industries*; 1987.

[3] Sherbon M. Wolfgang Pauli and the Fine-Structure Constant. *Journal of Science. 2012; 2(3).*

[4] Sudha K.R., Sekhar A.C., Reddy P. Cryptography Protection of Digital Signals Using Some Recurrence Relations. *International Journal of Computer Science and Network Security. 2007; 7(5).*

[5] Tahghighi M., Turaev S., Jaafar A., Mahmod R., Said M. Md. On the Security of Golden Cryptosystems. *Int. J. Cont. Math. Sciences. 2012; 7(7): 327 – 35.*

[6] A. M. del Rey and Sanchez G.R. On the Security of "Golden" Cryptography. *International Journal of Network Security. 2008; 7(3):448 – 50.*

[7] Mohamed M.H., Mahdy Y.B., and Shaban W. Confidential Algorithm for Golden Cryptography Using Haar Wavelet. *International Journal of Computer Science and Information Security. 2014; 12(8).*

[8] Stakhov A.P. The golden matrices and a new kind of cryptography. *Chaos, Solitons and Fractals. 2007; 32, 1138 – 46.*

[9] Johnson R.C. Fibonacci numbers and matrices.
Available: http://maths.dur.ac.uk/~dma0rcj/PED/fib.pdf
[Last accessed October 2017].

[10] Ray P.K., Dila G.K., Patel B.K. Application of Some Recurrence Relations to Cryptography using Finite State Machine. *International Journal of Computer Science and Electronics Engineering (IJCSEE). 2014; 2(4).*

[11] Schroeder M.R. Number Theory in Science and Communication, with Applications in Cryptography. *Springer-Verlag; 1986.*

[12] Koshy T. Fibonacci and Lucas Numbers with Applications. *Wiley-Interscience; 2001*.

[13] Vajda S. Fibonacci and Lucas Numbers, and the Golden Section, Theory and Applications. *Ellis Horwood, Chichester; 1989.*