A First Approach to Loop Quantum Gravity in the Momentum Representation

Abstract

We present a generalization of the first-order formalism used to describe the dynamics of a classical system. The generalization is then applied to the first-order action that describes General Relativity. As a result we obtain equations that can be interpreted as describing quantum gravity in the momentum representation.

Keywords: First-order formalism; Hamiltonian duality; General Relativity: Loop Quantum Gravity

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1 Introduction

As is well-known, Quantum Mechanics can be formulated in the configuration (or position) representation or in the momentum representation. This situation emerges from the two possible representations of the fundamental commutators in a quantum theory. To illustrate this, consider the simple example of the quantization of a one-dimensional system with a configuration variable q and a canonically conjugate momentum variable p. The corresponding quantum operators $\hat{q}$ and $\hat{p}$ must provide a representation of the fundamental commutator

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar \tag{1}$$

where $\hbar$ is Planck´s constant h divided by 2π. The usual way to represent the commutator (1) is to choose the operators

$$\hat{q} = q \qquad\qquad \hat{p} = -i\hbar\frac{d}{dq} \tag{2}$$

In this representation the quantum wave function will be a function of q, that is $\psi = \psi(q)$, and we will be in the configuration representation.

The other possibility of representing the commutator (1) is to choose the operators

$$\hat{q} = i\hbar \frac{d}{dp} \qquad\qquad \hat{p} = p \tag{3}$$

In this representation the quantum wave function will be a function of p, that is $\psi = \psi(p)$, and we will be in the momentum representation.

From a naive perspective, the operators (3) can be obtained from the operators (2) simply by substituting the letter q by p and the letter p by $-q$ in equations (2). However, in a deeper conceptual level, these two possibilities are related to the quantum mechanical wave-particle duality. The configuration representation is related to the particle aspect. Because of the De Broglie´s relation $\lambda = h/p$, the momentum representation is related to the wave aspect. The quantum wave-particle duality has a trace in classical mechanics in the form of a Hamiltonian duality. This duality interchanges position and momentum and leaves invariant the definition of the Poisson bracket. In this paper we will use this classical Hamiltonian duality to perform a first step towards the construction of a formulation of Loop Quantum Gravity (LQG) in the momentum representation.

At present time, a quantum theory for the gravitational interaction, based on the canonical quantization of General Relativity (GR) is under development. It is called Loop Quantum Gravity. This theory has already produced new interesting results, such as the quantization of the area and volume of a space-time region in terms of the Planck length $L_P = \sqrt{\frac{\hbar G}{c^3}} = 1,62 \times 10^{-35} m$. But with no present available way to test the theory against experimental results, the validity of LQG still remains an open question [1,2,3].

The motivation of this paper is to present an initial development that can be used to support the validity of LQG. This initial development is an indication that, as Quantum Mechanics, Loop Quantum Gravity can be equivalently formulated in the configuration or in the momentum representation. A moment of reflection leads us to the conclusion that, in spite of being only an initial development what is presented in this paper, it has a considerable importance because it can be used as a starting point for an entirely new line of research in Loop Quantum Gravity.

This paper is organized as follows. In section two we derive the two simple classical equations that allow transitions to LQG in the configuration and in the momentum representations. We also present a brief discussion about the consistency of our approach. In section three we review the basic equations of LQG in the configuration representation. In section four we present the basic equations of a quantum theory that can be interpreted as LQG in the momentum representation. Concluding remarks appear in section five.

2 The first-order formalism and the transition to quantum mechanics

The first-order formalism is in the interface between Lagrangian mechanics and Hamiltonian mechanics. According to Dirac [6], a Hamiltonian formalism is a first approximation to a corresponding quantum theory. Since, as we mentioned in the introduction, quantum mechanics can be formulated in the configuration or in the momentum representations, following Dirac´s idea, we need two first-order formalisms, one for each representation of quantum mechanics.

2.1 The first-order formalism for the configuration space formulation of quantum mechanics

The first-order formalism which can be considered as the classical limit of a configuration space formulation of quantum mechanics is the usual first-order formalism. It is based on the action functional

$$S = \int_{t_1}^{t_2} dt[p\dot{q} - H(q,p)] \tag{4}$$

where t is the time variable and $H(q,p)$ is the Hamiltonian. A dot denotes derivatives with respect to t. Varying action (4) we find

$$\delta S = \int_{t_1}^{t_2} dt[-\frac{\partial H}{\partial q}\delta q + p\delta\dot{q} + (\dot{q} - \frac{\partial H}{\partial p})\delta p] \tag{5}$$

Integrating by parts the second term we have

$$\int_{t_1}^{t_2} dtp\delta\dot{q} = p\delta q \mid_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\dot{p}\delta q$$

Inserting this result into the variation (5) we are left with a variation and a surface term

$$\delta S = \int_{t_1}^{t_2} dt[-(\dot{p} + \frac{\partial H}{\partial q})\delta q + (\dot{q} - \frac{\partial H}{\partial p})\delta p] + p\delta q \mid_{t_1}^{t_2}$$

The above variation of the action vanishes if Hamilton´s equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{6}$$

are satisfied. In this case the variation δS reduces to the surface term

$$\delta S = p\delta q \mid_{t_1}^{t_2}$$

Now we require that $\delta q(t_1) = 0$ and leave δq arbitrary at $t = t_2$. We therefore see that, as a function of the final point of the trajectory, action (4) satisfies

$$p = \frac{\delta S}{\delta q} \tag{7}$$

As we shall see below, an equation analogous to equation (7) plays a central role in the formalism leading to LQG in the configuration representation.

2.2 The first-order formalism for the momentum space formulation of quantum mechanics

We now introduce a first-order formalism which can be considered as the classical limit of a momentum space formulation of quantum mechanics. This formalism can be constructed using the Hamiltonian duality transformation

$$q \to p \qquad\qquad p \to -q \tag{8}$$

which leaves invariant the formal structure of the Hamilton´s equations (6) and the definition of the Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

which defines the algebraic structure in the phase space (q, p).

Applying the duality transformation (8) to action (4) we obtain the new action

$$S = \int_{t_1}^{t_2} dt[-q\dot{p} - \tilde{H}(q, p)] \tag{9}$$

Varying action (9) we have

$$\delta S = \int_{t_1}^{t_2} dt[-(\dot{p} + \frac{\partial \tilde{H}}{\partial q})\delta q - \frac{\partial \tilde{H}}{\partial p}\delta p - q\delta\dot{p}] \tag{10}$$

Integrating by parts the last term gives

$$-\int_{t_1}^{t_2} dtq\delta\dot{p} = -q\delta p \mid_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\dot{q}\delta p$$

Inserting this result into the variation (10) we find that

$$\delta S = \int_{t_1}^{t_2} dt[-(\dot{p} + \frac{\partial \tilde{H}}{\partial q})\delta q + (\dot{q} - \frac{\partial \tilde{H}}{\partial p})\delta p] - q\delta p \mid_{t_1}^{t_2} \tag{11}$$

Now, we see that, when Hamilton´s equations

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} \qquad\qquad \dot{p} = -\frac{\partial \tilde{H}}{\partial q}$$

are valid, the variation (11) reduces to the surface term

$$\delta S = -q\delta p \mid_{t_1}^{t_2}$$

We now impose that $\delta p(t_1) = 0$ and leave δp arbitrary at $t = t_2$. We now find that, as a function of the end point, action (9) satisfies

$$-q = \frac{\delta S}{\delta p} \tag{12}$$

Equation (12) is the central equation in this paper. We will describe below how equation (12) allows the construction of a formalism that can be interpreted as describing Loop Quantum Gravity in the momentum representation.

2.3 The transition to quantum mechanics

It is important to stress that the first-order formalism of section 2.2 was introduced only to be used as the classical limit of a momentum space formulation of quantum mechanics. Since the wave-particle duality disappears at the classical level, the classical Hamilton equations for the variables q and p derived from the Hamiltonian $\tilde{H}(q,p)$ will in general appear to be inconsistent. However, when we turn to quantum mechanics, the Schrödinger equation obtained from the quantum operator corresponding to $\tilde{H}(q,p)$ will be consistent.

The simplest example of the above situation is a free non-relativistic particle, described by the Hamiltonian

$$H = \frac{p^2}{2m}$$

The Hamilton equations for this system are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad\qquad \dot{p} = -\frac{\partial H}{\partial q} = 0$$

These equations tell us that the free particle moves in the position space and remains at a fixed point in the momentum space. Quantization of this system using the operators (2) leads to the Schrödinger equation

$$-\hbar^2 \frac{\partial^2 \psi(q,t)}{\partial q^2} = i\hbar \frac{\partial \psi(q,t)}{\partial t}$$

Now, from the duality transformation (8), we obtain the Hamiltonian

$$\tilde{H} = \frac{q^2}{2m}$$

As we saw in section (2.2), the Hamilton equations remain valid for this Hamiltonian. These equations now give

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} = 0 \qquad\qquad \dot{p} = -\frac{\partial \tilde{H}}{\partial q} = -\frac{q}{m}$$

which look counter-intuitive for us because the free particle now remains at a fixed point in the position space and moves in the momentum space. However, the existence of this dual situation is a consequence of the classical Hamiltonian duality, which we interpret as a residue of the quantum wave-particle duality. Quantization of this system using the operators (3) leads to the Schrödinger equation

$$-\hbar^2 \frac{\partial^2 \psi(p,t)}{\partial p^2} = i\hbar \frac{\partial \psi(p,t)}{\partial t}$$

which is perfectly consistent.

Obviously this illustrative example of a free non-relativistic particle should be looked at with some care. This is because a free particle is an idealized physical system. All particles are subjected to some kind of interaction due to the fact that they are part of the universe. In the following sections we study this same situation in the more realistic case of General Relativity.

3 Loop Quantum Gravity

In this section we review the basic equations that define LQG in the configuration representation.

In 1986 Ashtekar [4,5] introduced a new set of variables to describe General Relativity. In this new set of variables GR can be described by the first-order action (for details on this construction see ref. [1])

$$S = \frac{1}{8\pi i G} \int d^4x (E^a_i \dot{A}^i_a - \lambda^i D_a E^a_i - \lambda^a F^i_{ab} E^b_i - \lambda F^{ij}_{ab} E^a_i E^b_j) \qquad (13)$$

where

$$D_a V^i = \partial_a V^i + \epsilon^i_{jk} A^j_a V^k$$

is the covariant derivative on the tangent space $T(\Sigma)$ of a compact three-dimensional manifold Σ without boundaries.

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b$$

is the curvature of Σ and $F^{ij}_{ab} = \epsilon^{ij}_k F^k_{ab}$. The variables λ^i, λ^a and λ are Lagrange multipliers without dynamics.

Indices $i, j, ... = 1, 2, 3$ are internal $SU(2)$ indices and $a, b = 1, 2, 3$ are space indices. Comparing action (13) with action (4) we see that

a) the configuration variable is $A^i_a(\vec{x})$

b) the canonical momentum is $E^a_i(\vec{x})$

c) the total [6] Hamiltonian density is given by $H_T = \lambda^i D_a E^a_i + \lambda^a F^i_{ab} E^b_i + \lambda F^{ij}_{ab} E^a_i E^b_j$

Varying action (13) in relation to the variables λ^i, λ^a and λ we obtain the first-class [6] constraints

$$D_a E^a_i = 0 \qquad (14)$$

$$F^i_{ab} E^b_i = 0 \qquad (15)$$

$$F^{ij}_{ab} E^a_i E^b_j = 0 \qquad (16)$$

Equation (14) is the requirement of invariance of the theory under internal $SU(2)$ transformations. Equation (15) is the requirement of invariance of the theory under space diffeomorphisms. Equation (16) is the canonical Hamiltonian. Equations (14), (15) and (16) together are equivalent to the Einstein equations in vacuum [1].

Now, using the analog of equation (7), that is

$$E_i^a = \frac{\delta S}{\delta A_a^i}$$

we can make a transition to the Hamilton-Jacobi formalism as an intermediate step to the quantum theory. Equations (14), (15) and (16) then become [1]

$$D_a \frac{\delta S}{\delta A_a^i} = 0 \tag{17}$$

$$F_{ab}^i \frac{\delta S}{\delta A_b^i} = 0 \tag{18}$$

$$F_{ab}^{ij} \frac{\delta S}{\delta A_a^i} \frac{\delta S}{\delta A_b^j} = 0 \tag{19}$$

The transition to LQG in the configuration space is then obtained by substituting the classical action S by the wave functional $\Psi(A)$ in equations (17), (18) and (19). The final result is

$$D_a \frac{\delta}{\delta A_a^i} \Psi(A) = 0 \tag{20}$$

$$F_{ab}^i \frac{\delta}{\delta A_b^i} \Psi(A) = 0 \tag{21}$$

$$F_{ab}^{ij} \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} \Psi(A) = 0 \tag{22}$$

Equations (20), (21) and (22) are the basic quantum equations of LQG [1].

4 Loop Quantum Gravity in the momentum representation

In this section we derive the basic equations that we interpret to define LQG in the momentum representation. First we apply the duality transformation

$$A_a^i \to E_i^a \qquad\qquad E_i^a \to -A_a^i \tag{23}$$

to the first-order action (13) for GR. We obtain the action

$$S = \frac{1}{8\pi i G} \int d^4x(-A_a^i \dot{E}_i^a - \lambda_i \nabla^a A_a^i - \lambda_b R_i^{ab} A_a^i - R_{ij}^{ab} A_a^i A_b^j) \tag{24}$$

The covariant derivative ∇^a is now defined on the cotangent space $T^*(\Sigma)$ and is given by

$$\nabla^a V^i = \partial^a V^i + \epsilon^{ijk} E^a_j V^k$$

and the curvature R^{ab}_i in momentum space is given by

$$R^{ab}_i = \partial^a E^b_i - \partial^b E^a_i + \epsilon^{jk}_i E^a_j E^b_k$$

with $R^{ab}_{ij} = \epsilon_{ijk} R^{ab}_k$. It is important to mention here that we should not interpret action (24) as describing a real classical physical system. Rather, it should be interpreted as a formal action describing the classical limit of a real **quantum** physical system.

The equations of motion for the variables λ_i, λ_a and λ give the first-class [6] constraints

$$\nabla^a A^i_a = 0 \tag{25}$$

$$R^{ab}_i A^i_b = 0 \tag{26}$$

$$R^{ab}_{ij} A^i_a A^j_b = 0 \tag{27}$$

The next step towards the quantum theory is to use the general equation (12) we derived in section two. In the present case equation (12) becomes

$$A^i_a = -\frac{\delta S}{\delta E^a_i} \tag{28}$$

Substituting equation (28) into equations (25), (26) and (27) we obtain the Hamilton-Jacobi equations

$$\nabla^a \frac{\delta S}{\delta E^a_i} = 0 \tag{29}$$

$$R^{ab}_i \frac{\delta S}{\delta E^b_i} = 0 \tag{30}$$

$$R^{ab}_{ij} \frac{\delta S}{\delta E^a_i} \frac{\delta S}{\delta E^b_j} = 0 \tag{31}$$

Finally, the transition to the quantum theory is completed by substituting the classical action S by the wave functional $\Psi(E)$ in momentum space. This gives

$$\nabla^a \frac{\delta}{\delta E^a_i} \Psi(E) = 0 \tag{32}$$

$$R^{ab}_i \frac{\delta}{\delta E^b_i} \Psi(E) = 0 \tag{33}$$

$$R^{ab}_{ij} \frac{\delta}{\delta E^a_i} \frac{\delta}{\delta E^b_j} \Psi(E) = 0 \tag{34}$$

We interpret equations (32), (33) and (34) as the basic quantum equations of Loop Quantum Gravity in the momentum representation.

5 Results and discussion

In this paper, motivated by Dirac´s idea that a Hamiltonian formalism is a first approximation to a corresponding quantum theory, we presented a generalization of the first-order formalism used to describe the dynamics of a classical system. This generalization is based on the Hamiltonian duality, which interchanges the configuration and the momentum variables of phase space. The Hamiltonian duality leaves invariant the formal structure of the Poisson bracket, which defines the algebraic structure of phase space. It also leaves invariant the formal structure of Hamilton´s equations. As we saw in this paper, the generalization of the first-order formalism we presented also leaves Hamilton´s equations invariant. Our generalization was then used to derive a new equation (equation (12)) that allows a possible extension of the Hamilton-Jacobi formalism.

After a review of the basic equations that define Loop Quantum Gravity, the results described above were applied to the first-order action that describes General Relativity. For simplicity we considered only the case of a compact manifold. This eliminates the discussion of boundary terms. A new dual action, which should be interpreted as a formal action describing the classical limit of quantum General Relativity in the momentum representation, was then obtained.

As an intermediate step towards the quantum theory we used our new equation (12) to construct the Hamilton-Jacobi formalism for the dual action (24). Finally, using standard quantization techniques, we obtained quantum equations that can be interpreted as defining Loop Quantum Gravity in the momentum representation. These final quantum equations justify the interpretation of action (24) as describing the classical limit of quantum General Relativity in the momentum representation, in agreement with Dirac´s original idea mentioned above.

6 Conclusion

The conclusion of this paper is that the quantum theory for the gravitational interaction, based on the canonical quantization of General Relativity, can be formulated in the configuration or in the momentum representation. We think that this conclusion gives further support for the validity of Loop Quantum Gravity. In addition, this conclusion opens a new line of research in LQG. This new line of research is already well defined. It should start with a search for a possible geometrical meaning for the dual action (24) using the tools of differential geometry. A second step should be the use of the Poisson bracket in the (A_a^i, E_i^a) phase space to investigate the algebra defined by constraints (32), (33) and (34). After this one should try to elucidate the particular quantum dynamic features of the momentum representation and how these quantum dynamic features are related to the presently known ones in the configuration representation. A reasonable conjecture is that all these quantum dynamic features will be related via some type of gravitational wave-particle duality.

References

[1] C Rovelli *Quantum Gravity* (Cambridge Monographs on Mathematical Physics) (2004)

[2] T Thiemann *Modern Canonical Quantum General Relativity* (Cambridge Monographs on Mathematical Physics) (2008)

[3] C Rovelli and F Vidotto *Covariant Loop Quantum Gravity* (Cambridge University Press) (2015)

[4] A Ashtekar *Phys. Rev. Lett.* **57** 2244 (1986)

[5] A Ashtekar *Phys. Rev.* **D36** 1587 (1987)

[6] P A M Dirac *Lectures on Quantum Mechanics* (Yeshiva University) (1964)