Maximum Vanishing Moment of Compactly Supported B-spline Wavelets

Abstract:

Spline function is of very great interest in field of wavelets due to its compactness and smoothness property. As splines have specific formulae in both time and frequency domain, it greatly facilitates their manipulation. We have given a simple procedure to generate compactly supported orthogonal scaling function for higher order B-splines in our previous work. Here we determine the maximum vanishing moments of the formed spline wavelet as established by the new refinable function using sum rule order method.

Keywords: B-splines, Multiresolution analysis, vanishing moments, sum rule order.

Introduction:

One of the principle goal of wavelet theory has been the construction of useful orthonormal bases for $L^2(R)$. Orthonormal wavelet base have revealed to be powerful tool in applied mathematics and digital signal processing. Wavelet bases are usually constructed via multiresolution analysis(MRA) on R. MRA attraction is its utility as a powerful tool for efficiently representing functions at multiple levels of detail with many inherent advantages.

There is the great interest in the investigation of compactly supported wavelets. This interest is due to the computational capabilities of such wavelets and the wide range of their applications. The compactly supported orthonormal B-spline wavelets is been found to be powerful tool in many scientific and practical applications, the finite element method, image processing etc. Thanks to some of their exceptional properties defined in [6] and mathematical simplicity, they are also applied and give very good results in various areas of applied sciences in comparison to other known wavelets.

We already given in [11] a very simple procedure to generate orthonormal wavelet bases and using the sum rule approach as given in [10], calculate the maximum vanishing moment of ψ is m.

Prelimimaries:

1. B-Splines

The cardinal B-spline B_m defined in [1] of order $m \geq 1$ is

$$B_m = B_{m-1} * B_1 = \int_0^1 B_{m-1}(.-t)dt \qquad m \geq 1 \qquad (1)$$

with $B_1 = \chi_{[0,1)}$. It is known that $suppB_m = [0, m]$ and $B_m (m \geq 2)$ satisfies the recursion formula

$$(m-1)B_m(x) = xB_{m-1}(x) + (m-x)B_{m-1}(x-1) \qquad x \in R$$

By convolution property, the Fourier Transform of B_m is

$$\widehat{B_m}(\xi) = \int_{-\infty}^{\infty} B_m(x)\, e^{-i\xi x} dx = \left(\frac{1-e^{-i\xi}}{i\xi}\right)^m \qquad (2)$$

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The m[th] order cardinal B-spline B_m satisfies the following properties;

i) Supp $B_m = [0, m]$

ii) $B_m(x) > 0 \quad for\ 0 < x < m$

iii) $\sum_{k=-\infty}^{\infty} B_m(x-k) = 1 \quad \forall x$

iv) The Cardinal B-spline B_m and B_{m-1} are related by the identity $B_m(x) = \frac{x}{m-1} B_{m-1}(x) + \frac{m-x}{m-1} B_{m-1}(x-1)$

v) B_m is symmetric with respect to the centre of its support, namely $B_m\left(\frac{m}{2} + x\right) = B_m(\frac{m}{2} - x)$

2. Multiresolution Analysis

A MRA [7] in $L^2(R)$ (introduced by Mallat and Mayer) is given by a nested sequence of subspaces generated by dilates and translates of single function.

Definition:

Let a function $\emptyset \in L^2(R)$ generate spaces

$$V^0 = clos_{L^2}\langle \emptyset(.-k);\ k \in Z \rangle$$

$$V^j = clos_{L^2}\langle \emptyset_k^j;\ k \in Z \rangle$$

with

$$\emptyset_k^j(x) = 2^{j/2} \emptyset(2^j x - k) \qquad j, k \in Z$$

where <> denotes the linear span. The $\emptyset$ is said to generate a MRA if the subspaces V^j have the following properties

i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$

ii) $\overline{\bigcup_{j \in z} V_j} = L^2(R)$

iii) $\bigcap_{j \in Z} V_j = \{0\}$

iv) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$

v) There exist a function $\varphi \in V_0$ such that $\{\varphi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis.

The function φ defined in the last condition is called scaling function of MRA. For each subspace V_{j+1} there exist an orthonormal compliment W_j of V_{j+1} in V_j such that,

W_j is subspace of V_{j+1}

$W_j \perp V_j$

$V_{j+1} = V_j \oplus W_j$

and $W_k \perp W_l$ for all $k \neq l$

Hence under condition (i), (ii) (iii), it follows that $L_2 = \oplus_{k \in Z} W_k$. The spaces W_k , $k \in Z$ are called wavelet spaces of L_2 relative to the scaling function φ. A scaling function φ must be a

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function in $L_2(R)$ with $\int \varphi \neq 0$. Also since, $\varphi \in V_0$ is also in V_1 and $\{\varphi_{1,k} = 2^{j/2}(2x-k): k \in Z\}$ is a Riesz basis of V_1.

Construction of spline wavelets:

The Cardinal B-spline of order m generates a MRA of $L_2(R)$ in the sense that $V_k^m = clos_{L^2}\{B_m(2^k.-j): j \in Z\}$.

Since, $V_j \subset V_{j+1}$ and let $B_m(x) \in V_j$ then

$$B_m(x) = \sum_k p_k B_m(2x-k) \tag{3}$$

where p_k is some l^2 sequences. The Fourier transform of (3) is

$$\widehat{B_m}(\xi) = \frac{1}{2}\sum_k p_k e^{\frac{-ik\xi}{2}} \hat{B}_m\left(\frac{\xi}{2}\right) \tag{4}$$

$$\Rightarrow \frac{1}{2}\sum_k p_k e^{\frac{-ik\xi}{2}} = \frac{\widehat{B_m}(\xi)}{\hat{B}_m(\frac{\xi}{2})}$$

Using (2),

$$\frac{1}{2}\sum_k p_k e^{\frac{-ik\xi}{2}} = \left(\frac{1-e^{-i\xi}}{i\xi}\frac{\frac{i\xi}{2}}{1-e^{\frac{-i\xi}{2}}}\right)^m = \frac{1}{2^m}\left(1+e^{\frac{-i\xi}{2}}\right)^m$$

$$= \frac{1}{2^m}\sum_{k=0}^m \binom{m}{k} e^{\frac{-ik\xi}{2}}$$

$$\Rightarrow \frac{1}{2}\sum_k p_k = \frac{1}{2^m}\sum_{k=0}^m \binom{m}{k} \tag{5}$$

Therefore the two scale relation of cardinal B-spline is

$$B_m(x) = \sum_{k=0}^m 2^{-m+1}\binom{m}{k} B_m(2x-k) \tag{6}$$

and Fourier transform of it is,

$$\widehat{B_m}(\xi) = P\left(\xi/2\right)\widehat{B_m}(\xi/2) \tag{7}$$

where $P\left(\xi/2\right) = \frac{1}{2}\sum_{k=0}^m 2^{-m+1}\binom{m}{k} e^{\frac{-ik\xi}{2}} = \left(\frac{1+e^{-\frac{i\xi}{2}}}{2}\right)^m$ (8)

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$P(\xi/2)$is called the mask of the scaling function.

Also (7) can be written as,

$$\widehat{B_m}(2\xi) = P(\xi)\widehat{B_m}(\xi) \tag{9}$$

where $P(z) = P(\xi) = \frac{1}{2}\sum_{k=0}^{m} 2^{-m+1}\binom{m}{k} e^{-ik\xi} = \left(\frac{1+e^{-i\xi}}{2}\right)^m, \quad z = e^{-i\xi}$

Theorem2.1: A scaling function φ with refinement relation

$$\varphi(x) = \sum_{-\infty}^{\infty} p_k \varphi(2x-k),$$

forms an orthonormal basis if only if

$$|P(z)|^2 + |P(-z)|^2 = 1 \quad \text{for } z \in C \text{ with } |z| = 1 \tag{10}$$

and the $P(z)$ satisfies the following

1. $P(z) \in C^1$ and is 2π-periodic
2. $P(1) = 1$
3. $P(\xi) \neq 0 \qquad \forall \ \xi \in [-\pi, \pi]$

For B-splines of m[th] order,

$|P(z)|^2 + |P(-z)|^2 = cos^{2m}\frac{\xi}{2} + sin^{2m}\frac{\xi}{2} \leq cos^2\frac{\xi}{2} + sin^2\frac{\xi}{2} = 1.$

Hence (10) is satisfied only when m=1, thus B-spline form orthonormal basis for m=1 only. Therefore to induce orthogonality for $m \geq 2$, let us introduce a Laurentz's polynomial

$$Q(z) = \sum_{k=0}^{m} q_k z^k \qquad z = e^{-i\xi}$$

In such a manner that $|P(z)Q(z)|^2 + |P(-z)Q(-z)|^2 = 1$ (11)

$B = \max_{|z|=1}|Q(z)| < 2^{N-1}$ (12)

and $Q(z)$ must satisfied the three condition stated in theorem (2.1) for $P(z)$.

Putting z=1 in (9),

$$|P(1)Q(1)|^2 + |P(-1)Q(-1)|^2 = 1$$

$$|Q(1)|^2 = 1 \qquad \Rightarrow \ Q(1) = \pm 1$$

Therefore $\sum_{k=0}^{m} q_k = \pm 1$, but we consider only $Q(1) = 1$ in order to ensure the orthogonality of the scaling function.

Expression of $Q(z)$:

$$Q(z) = \sum_{k=0}^{m} q_k z^k$$

$$Q(z) = q_0 + q_1 cos\xi + q_2 cos2\xi + \cdots - i(q_1 sin\xi + q_2 sin2\xi + \cdots)$$

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$$|Q(z)|^2 = q_0^2 + q_1^2 + \cdots + \sum_{i=1}^{m} q_0 q_i cos i\xi + \sum_{i\neq j} q_i q_j \cos(i-j)2\xi$$

Thus Q is a polynomial function in $cos\xi$ with real coefficients [1], so we take $|Q(z)|^2 = \tilde{R}(cos\xi)$

Take $x = \frac{1-cos\xi}{2} = sin^2\frac{\xi}{2}$

and $R(x) = \tilde{R}(cos\xi) = \tilde{R}(1-2x) = \tilde{R}(sin^2\frac{\xi}{2})$

From (9),

$$|P(z)|^2|Q(z)|^2 + |P(-z)|^2|Q(-z)|^2 = 1$$

$$\left|\frac{1+z}{2}\right|^{2m} R(x) + \left|\frac{1-z}{2}\right|^{2m} R(1-x) = 1$$

$$\therefore\ (1-x)^m R(x) + x^m R(1-x) = 1 \qquad (13)$$

Since $(1-x)^m$ and x^m are co prime with $\gcd((1-x)^m, x^m) = 1$ therefore using the following lemma used in [8, 9], we find two polynomial $M(x) and\ N(x)$ of degree less than m such that

$$(1-x)^m M(x) + x^m N(x) = 1$$

Lemma2.2: Polynomial Extended Euclidean Algorithm,

If p and q are two non zero polynomial then the extended Euclidean algorithm produces the unique pair r and s such that $pr + qs = \gcd(p,q)$ where

$\deg(r) < deg(q) - \deg(gcd(p,q))$, $\deg(s) < \deg(p) - \deg$ (gcd (p,d)).

Now,

$$1 = (1-x+x)^{2m-1}$$

$$= (1-x)^{2m-1} + \binom{2m-1}{1}(1-x)^{2m-2}x + \cdots + (x)^{2m-1}$$

$$= \sum_{k=0}^{m-1}\binom{2m-1}{k}(1-x)^{2m-1-k}x^k + \sum_{k=0}^{m-1}\binom{2m-1}{k}x^{2m-1-k}(1-x)^k$$

$$= (1-x)^m \sum_{k=0}^{m-1}\binom{2m-1}{k}(1-x)^{m-1-k}x^k + x^m \sum_{k=0}^{m-1}\binom{2m-1}{k}(1-x)^k x^{m-1-k} \qquad (14)$$

Thus from (12) and (14),

$$R(x) = \sum_{k=0}^{m-1}\binom{2m-1}{k}(1-x)^{m-1-k}x^k$$

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$$\therefore\ |Q(z)|^2 = R(x) = \sum_{k=0}^{m-1} \binom{2m-1}{k} (1-x)^{m-1-k} x^k$$

$$= \sum_{k=0}^{m-1} \binom{2m-1}{k} cos^{m-1-k}\xi sin^k\xi \qquad (15)$$

Thus the new two scale symbol for the spline scaling function is given by $M(z) = P(z)Q(z)$

$$= \frac{1}{2}\sum_{l=0}^{2m} b_l\, z^l$$

generating orthonormal basis. And the required wavelet is

$$\hat{\psi}(2\xi) = \bar{M}(\xi)\hat{B}(\xi)$$

Vanishing Moments:

Before defining vanishing moments we define sum rule order L for a finite sequence $p = \{p_k\}$.

Definition: A finite sequence $p = \{p_k\}$ is said to have sum rule order L if L is the largest integer for which the two scale symbol $P(\xi)$ satisfies

$$P(0) = 1 \quad \& \quad \frac{d^l}{d\xi^l} P(\pi) = 0 \quad for\ l = 0,1,\dots,L-1$$

From (8), $P(0) = 1$

$$\frac{d^l}{d\xi^l} P(\pi) = \{\frac{d^l}{d\xi^l} (\frac{1+e^{-i\xi}}{2})^m\}_{\xi=\pi} = 0 \quad for\ l = 0,1,\dots,m-1$$

Also from (15), $Q(0) = 1$

Thus $\qquad M(0) = 1 \qquad (16)$

And $\quad \frac{d^l}{d\xi^l} M(\xi) = \sum_{j=0}^{l} \binom{l}{j} \frac{d^j}{d\xi^j} P(\xi) \frac{d^{l-j}}{d\xi^{l-j}} Q(\xi)$

$$= 0 \quad at\ \xi = \pi \qquad for\ \ l = 0,1,..,m-1 \qquad (17)$$

Thus the sequence $M(\xi)$ has sum rule order m. Next define the vanishing moments as,

A compactly supported function $\psi \in L^2(R)$ is said to have vanishing moments of order L if

$$\int_{-\infty}^{\infty} \psi(x)\, x^l dx = 0 \qquad for\ l = 0,1,\dots,L-1 \qquad (18)$$

Theorem: (Sum rule implies vanishing moments)

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Let $\emptyset$ be an orthogonal refinable function associated with polynomial $P(\xi)$. Let ψ be the function given by (15) with the sequence given by $M(\xi) = P(\xi)Q(\xi)$. If $P(\xi)$has sum rule order m, then ψ has maximum vanishing moments of order m.

Before proving this theorem we take a quick look on one of the property of Fourier Transform:

Let $f \in L_1(R)$ with fourier transform $\hat{f}(\omega)$. Then if $x^k f(x) \in L_1(R)$ for some

k>1, then $\hat{f} \in C^k(R)$with

$$\frac{d^k}{d\omega^k}\hat{f}(w) = (-i)^k \int_{-\infty}^{\infty} x^k f(x) e^{-i\omega x} dx, \quad \omega \in R$$

$$\int_{-\infty}^{\infty} x^k f(x) dx = (-i)^k \frac{d^k}{d\omega^k}\hat{f}(0), \quad for\ \omega = 0$$

Proof: From above,

$$\frac{d^l}{d\omega^l}\hat{\psi}(2\xi) = \frac{d^l}{d\omega^l}\mathcal{M}(\xi)\widehat{B_m}(\xi)$$

$$= \sum_{j=0}^{l} \binom{l}{j} \frac{d^j}{d\omega^j}\mathcal{M}(\xi) \frac{d^{l-j}}{d\omega^{l-j}}\widehat{B_m}(\xi)$$

Since $\mathcal{M}(\xi) = \overline{M(\xi)} = (-1)^k M(\pi - \xi)$ is a trgnometric polynomial and $M(\xi)$ has sum rule order m, therefore

$$\frac{d^l}{d\omega^l}\mathcal{M}(\xi) = \frac{d^l}{d\omega^l}(-1)^k M(\pi - \xi) = 0 \qquad at\ \xi = 0$$

Thus, $\frac{d^l}{d\omega^l}\mathcal{M}(0) = 0\,,$ $\qquad 0 \le l \le m$

Hence, $\frac{d^l}{d\omega^l}\psi(0) = 0$

or, $\int_{-\infty}^{\infty} x^l \psi(x) = \frac{d^l}{d\omega^l}\hat{\psi}\,(0) = 0 \qquad 0 \le l \le m$

Hence ψ has maximum vanishing moment m.

Conclusion:

The order of wavelet transform is typically given by the number of vanishing moments of the analysis wavelet. The number of vanishing moments of a wavelet is important when using

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wavelet for the analysis and synthesis of a function. The inner product of a function and a wavelet with many vanishing moments result in a smaller value, giving a better approximation for a fixed number of Fourier coefficients. Hence higher order wavelet transform usually result in better signal approximation.

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