***Short Research Article***

# The Universal Coefficient Theorem for Homology and Cohomology an Enigma of Computations


## ABSTRACT

This article looks at the algebraic background leading to definition and explanation of some product topology as well as the kiinneth formula for computing the (co)homology group of product spaces. A detailed look at the four theorems considered as the Universal Coefficient Theorem is not ignored. Though this article does not proof the theorems, yet it states some properties of each of these products, which are enough for the calculation of (co)homology groups.



Keywords: Abelian Group**,** Homology, Cohomology, Exact Sequences, Tensor Product, Homomorphism, Isomorphism, Torsion Product, Extension, Kiinneth Formula and Cross Product.


## 1. INTRODUCTION

From the theory of topological spaces emerged algebraic topology. Objects are classified according to the nature of their connectedness [Obeng-Denteh, 2019]. At the elementary level, algebraic topology separates naturally into the two broad channels of homology and homotopy. With a simple dualization in the definition of homology, cohomology an algebraic variant of homology is formed [Hatcher, 2002]. It is therefore not surprise that cohomology groups $H^i(x)$ satisfy axioms much like the axioms for homology, except that induced homomorphisms go in the opposite direction as a result of the dualization. The basic difference between homology and cohomology is thus that cohomology groups are contravariant functors while homology groups are covariant. In terms of internal study, however, there is not much difference between homology groups and cohomology groups. The homology groups of a space determine its cohomology groups, and the converse holds at least when the homology groups are finitely generated. What is a little surprising being that contravariance leads to extra structure in cohomology.

## 2. PRELIMINARIES

### 2 .1 Exactness of a sequence

**Definition 1**. For a given pair of homomorphism $M \xrightarrow{\alpha} N \xrightarrow{\beta} Q$ is exact at N if

$im\,(x) = \ker(y)$. Hence a sequence $\ldots \to M_{i-1} \to M_i \to M_{i+1} \to M_{i+2} \to \cdots$ is exact if it actually exact at every $M_i$ that is between two homomorphisms.

**Proposition**. A sequence $0 \to M \xrightarrow{x} N$ is exact if provided is injective (1 to 1). furthermore, a sequence $N \xrightarrow{y} Q \to 0$ is exact if and only if g is surjective (onto).

**Proof**. A sequence being exact has its implication, that is kernel *x* is equal to the image of the homomorphism 0 → M, which is zero. There is an equivalence relation to the injectivity of homomorphism *x*. Similarly, the kernel of zero homomorphism Q → 0 is Q, and $y(N) = Q$ if and only if *y* is surjective

## 2 .2 Product Structures of Abelian Groups

### 2 .2. 1 Tensor product.

**Definition 2.** Let M and N be two abelian groups then the tensor product denoted by M ⊗ N is defined to be the abelian group with generators m ⊗ n for m ∈ M, n ∈ N, and relations

$(m + m') \otimes n = m \otimes n + m' \otimes n$ and

$m \otimes (n + n') = m \otimes n + m \otimes n'$.

So the zero element of M ⊗ N is $0 \otimes 0 = 0 \otimes n = m \otimes 0$, and

$-(m \otimes n) = -m \otimes n = m \otimes (-n)$.

Hence given the direct sums, $M = m_1 \oplus m_2 \oplus m_3 \oplus \ldots..$ and $N = n_1 \oplus n_2 \oplus n_3 \oplus \ldots..$

$M = \sum_i M^i$ and $N = \sum_j N^j$ then there exists an isomorphism $M \otimes N \cong \sum_{i,j} M^i \otimes N^J$ .

Tensor product satisfies the following elementary properties

1. *M⊗N ≈ M⊗N.*

2. *(M⊗N) ⊗ Q ≈ M ⊗ (⊗Q).*

3. $(\oplus_i M_i) \otimes N \approx \oplus(\mathrm{M}_i \otimes N)$

4. *$\mathbb{Z}$ ⊗M ≈. M⊗$\mathbb{Z}$ ≈ M.*

5. *$\mathbb{Z}_n$ ⊗M ≈ M/nM.*

6. *A pair of homomorphisms f: M→M' and g: N→N' induces a homomorphism*

   *f ⊗g: M⊗N→M'⊗N' via (f ⊗g) (m⊗n) = f(m)⊗g(n).*

7. *A bilinear map φ: M×N→Q induces a homomorphism M⊗N→Q sending m ⊗ n to φ(m, n).*

For the calculation of tensor products of finitely generated abelian groups, properties1 to 5 may be employed. properties 1,2,3,6 and 7 remain valid for tensor products of R-modules. [Hajime, 2000]

### 2 .3 Homomorphism

**Definition 3**: let M, N be two abelian groups. A mapping $\varphi: M \to N$ is called homomorphism if for all $x, y \epsilon M, \varphi(xy) = \varphi(x)\varphi(y)$.

For abelian groups M and N, we obtain the abelian group Hom(M, N) of the homomorphism of M and N. Particularly, given that $M = \sum_i M^i$ and $N = \sum_j N^j$ are direct sums as indicated, then

$$Hom(M,N) \cong \sum_{i,j} Hom(M^i, M^j)$$

Therefore, it is important to note that for any two finitely generated abelian groups M and N the following relations hold (over $\mathbb{Z}$): $\forall\, v, k \epsilon \mathbb{Z}$

1. $Hom(\mathbb{Z}, M) \cong M$
2. $Hom(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$
3. $Hom(\mathbb{Z}, \mathbb{Z}_v) \cong \mathbb{Z}_v$
4. $Hom(\mathbb{Z}_v, \mathbb{Z}) \cong 0$
5. $Hom(\mathbb{Z}_v, \mathbb{Z}_k) \cong \mathbb{Z}_{(v,k)}$

### 2 .4 Torsion Product

**Definition 4**: Given that M and N are abelian groups, the an abelian group called their torsion product over $\mathbb{Z}$, is given by $Tor(M,N)$ will be determined by the torsion part of M and N. That is their respective subgroups consisting of the elements whose integral multiples become 0 for some integers. [Hatcher, 2002]

Hence if M and N are $M = \sum_i M^i$ and $N = \sum_j N^j$ , then the torsion product

$$Tor(M,N) \cong \sum Tor(M^i, N^j).$$

It should be noted that for any abelian groups M and N, $Tor(M,N) \cong Tor(N,M)$.

For a given abelian group M

$$Tor(\mathbb{Z}, M) \cong Tor(M, \mathbb{Z}) = 0$$

Torsion product of two finitely generated abelian groups may be determined using the following relations.

$$Tor(\mathbb{Z}, \mathbb{Z}) = 0$$

$$Tor(\mathbb{Z},\mathbb{Z}_v) \cong Tor(\mathbb{Z}_v,\mathbb{Z}) = 0$$

$$Tor(\mathbb{Z}_v,\mathbb{Z}_k) \cong Tor\mathbb{Z}_{(v,k)}$$

### 2 .5 Extensions

**Definition 5**: given two abelian groups M and N, an extension of M by N is a group together with an exact sequence of the form:

$$0 \to N \to Q \to M \to 0$$

and is denoted by $Ext(M,N)$ for equivalent classes of extension of N by M which determine an abelian group. [Hatcher, 2002]

moreover, if are direct sums, $M = \sum_i M^i$ and $N = \sum_j N^j$ the it can be said that there exists an isomorphism

$$Ext(\mathrm{M},\mathrm{N}) \cong \sum_{i,j} Ext(M^i,\mathrm{N}^j)$$

**Lemma**: for any abelian group M,

$$Ext(\mathbb{Z},M) = 0$$

It also follows that the following relations are equivalent

$$Ext(\mathbb{Z},\mathbb{Z}) \cong Ext(\mathbb{Z},\mathbb{Z}_v) = 0$$

$$Ext(\mathbb{Z}_v,\mathbb{Z}) \cong \mathbb{Z}_v$$

$$Ext(\mathbb{Z}_v,\mathbb{Z}_k) \cong \mathbb{Z}_{(v,k)}$$

## 3. Main Thrust

### 3.1 The Kiinneth formula for (co)homology

Let $X \times Y$ be product spaces of topological spaces $X$ and $Y$ given their respective (co)homology groups.

**Theorem:** *For each p, there exists a natural isomorphism*

$$h_p(X) \cong h_p(C(X)),$$

*Where the left-hand side is the axiomatic homology of all the cell complex X which gives rise to the chain complex $C(X)$ computed algebraically*

The tensor product of the respective chains of $X$ and $Y$ can be regarded naturally as a chain on X x Y, which induces a homomorphism

$$\times: H_p(X;\mathbb{Z}) \otimes H_q(Y;\mathbb{Z}) \to H_{p+q}(X \times Y;\mathbb{Z})$$

Similarly, we now get the induced homomorphism

$$\times: H^p(X;\mathbb{Z}) \otimes H^q(Y;\mathbb{Z}) \to H^{p+q}(X \times Y;\mathbb{Z})$$

It can therefore be said that these maps are induced by the cross product. The map induce by the cross product is injective. The following theorems affirms that.

**Theorem:** $H_n(X \times Y; \mathbb{Z}) \cong \sum_{p+q=n} H_p(X; \mathbb{Z})$ for the homology kiinneth formula

$$\otimes H_q(Y; \mathbb{Z}) \otimes \sum_{p+q=n-1} Tor(H_p(X; \mathbb{Z}), H_q(Y; \mathbb{Z}))$$

**Theorem**: $H^n(X \times Y; \mathbb{Z}) \cong \sum_{p+q=n} H^p(X; \mathbb{Z})$ for the cohomology kiinneth formula

$$\otimes H^q(Y; \mathbb{Z}) \otimes \sum_{p+q=n+1} Tor(H^p(X; \mathbb{Z}), H^q(Y; \mathbb{Z}))$$

### 3.2 Cup Product

For a topological space X, the diagonal map

$$\Delta : X \to X \times X,$$

transforming $x \in X$ to $(x, x) \in X \times X$, is continuous. Hence the composition of the cross product and the induced map $\Delta^*$,

$$H^p(X; M) \times H^q(X; M) \xrightarrow{\times} H^{p+q}(X \times X; M) \xrightarrow{\Delta^*} H^{p+q}(X; M),$$

This defines a homomorphism

$$\cup : H^p(X; M) \times H^q(X; M) \to H^{p+q}(X, M)$$

Hence for $a \in H^p(X; M)$ and $b \in H^q(X; M)$, we define their cup product $a \cup b$ by

$$a \cup b \cup = \Delta^*(a \times b) \in H^{p+q}(X; M).$$

There is an implication in the definition that is the structure induced on a cohomology theory by the cup product is homotopy invariant. The cup products satisfy the following properties

For $a \in H^p(X; M), b \in H^q(X; M), c \in H^r(X; M)$

$$(a \cup b) \cup c = a \cup (b \cup c), \quad a \cup b = (-1)^{pq}(b \cup a);$$

For a map $f : X \to Y, \quad f^*(a \cup b) = f^*(a) \cup f^*(b).$

We see a product- preserving homomorphism in $f^*$. The cohomology group

$H^*(X; M) = \sum_p H^p(X, G)$ equipped with a product structure has become a ring.

### 3.3 The Universal Coefficient Theorem

In homology the universal coefficient theorem is a special case of the kiinneth theorem. Now let's look at these four formulae considered as the universal coefficient theorem. By reminding ourselves about the product structures of abelian groups, the easier it is for to comprehend these theorems.

**Theorem**: From the corresponding integral homology and the ***torsion product***, we can calculate homology over a general coefficient group M:

$$H_n(X; M) \cong H_n(X; \mathbb{Z}) \otimes M \oplus Tor(H_{n-1}(X; \mathbb{Z}); M).$$

**Theorem**: Using the corresponding integral homology and the ***extension product***, we may also calculate cohomology over a general coefficient group M:

$$H^n(X; M) \cong Hom(H_n(X; \mathbb{Z}) \otimes M \oplus Ext(H_{n-1}(X; \mathbb{Z}); M).$$

**Theorem**: We can compute cohomology over a general coefficient group M from the integral cohomology and the ***torsion product***:

$$H^n(X; M) \cong H^n(X; \mathbb{Z}) \otimes M \oplus Tor(H^{n+1}(X; \mathbb{Z}); M).$$

**Theorem**: from the integral cohomology and the ***extension product***, homology over a general coefficient group M can also be computed.

## 4. Conclusion

The general observation made so far is that, in our quest to look more into abelian groups such as M and N for the sake of this article as defined from the beginning the tensor product, Homomorphism, torsion product and extension has to be defined. It must also be noted that cohomology groups become rings using the structure of a cup product.

The identification of tensor products of respective homology and cohomology groups belonging to two topological spaces with the cohomology groups of the product spaces may be used to define the. Cohomology groups of product spaces fall out from kiinneth formula and can be inferred from the product structures that cross product homomorphism is injective

## References

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