Short Research Article

A Discourse on Completely Regular Space

Abstract: *This study is an investigation of some of the relationships which exist between various generalizations of completely regular spaces. The primary aim of the study is to look at the separation axioms and delve more into one of the claims about completely regular space; "Every completely regular space is a regular space as well".*

Keywords: Topological space, neighbourhood, open sets, closed sets, continuous map, Kolmogorov space, Fréchet space, Hausdorff space, regular space, normal space, completely regular space.

1. INTRODUCTION

In topology and related fields of mathematics, there are many restrictions that one often makes on the kinds of topological spaces that one wishes to work with. Some of these restrictions are given by the separation axioms [1]. The separation axioms are axioms only in the sense that, when defining the concept of topological space, one could add these conditions as extra axioms to get a greater restricted concept of what a topological space is. The separation axioms are denoted with the letter "T" after the German, Trennung which means "separation" [2]. The separation axioms are about the use of topological means to distinguish disjoint sets and distinct points. It is insufficient for elements of a topological space to be distinct; we may want them to be topologically distinguishable. Similarly, it is inadequate for subsets of a topological space to be distinct; we may want them to be separated in various ways.

In this paper, I will briefly discuss the separation axioms with more emphasis on completely regular space and some claims about it. The following preliminaries serve this purpose.

2. PRELIMINARIES

Definition 2.1:

Let X be a non – empty set. A set τ of subsets of X is said to be a topology on X if;

(i) X and the empty set, $\varnothing$ belong to τ .

(ii) the union of any (finite or infinite) number of sets in τ belongs to τ , and

(iii) the intersection of any two sets in τ belongs to τ .

The pair (X, τ) is called a topological space [3].

Definition 2.2:

A topological space (X, τ) is said to be T_o or Kolmogorov if given any two points $a, b \in X$, whenever $a \neq b$ there is ***either*** an open set containing *a* but not *b*, ***or*** an open set containing *b* but not *a* [4].

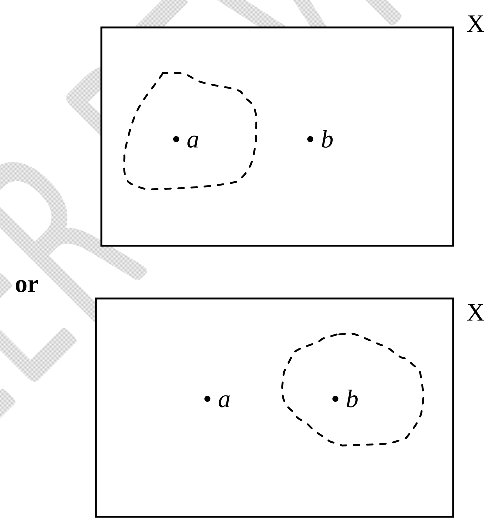

Fig. 1: T_o or Kolmogorov space

Definition 2.3:

A topological space (X, τ) is said to be T_1 or Fréchet space if given $a, b \in X$ and $a \neq b$, there exists open sets U_a, U_b $\in \tau$ containing *a, b* respectively, such that $b \notin U_a$ and $a \notin U_b$ [5].

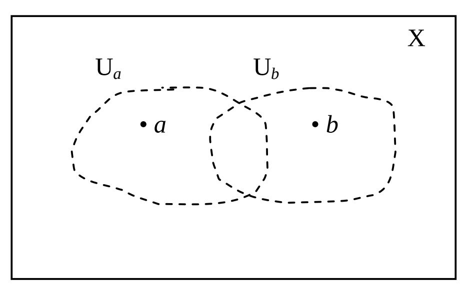

Fig. 2: T_1 or Fréchet space

Definition 2.4:
A topological space (X, τ) is said to be T_2 or Hausdorff space if given $a, b \in X$ and $a \neq b$, there exists ***disjoint*** open sets $U, V \in X$ such that $a \in U$ and $b \in V$ [5].

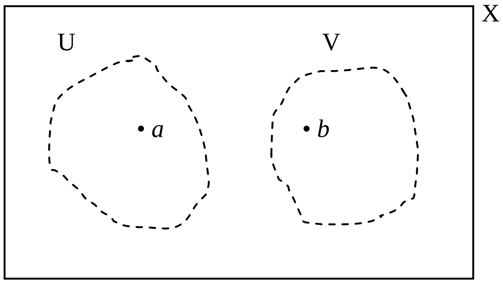

Fig. 3: T_2 or Hausdorff space

Definition 2.5:
A topological space (X, τ) is said to be T_3 or regular space if, given any closed set *F* and any point *x* such that $x \notin F$, there exists a neighbourhood U of *x* and a neighbourhood V of *F* that are ***disjoint*** (ie. $U \cap V = \varnothing$) [6].

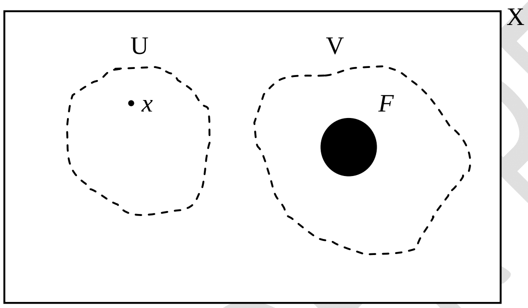

Fig. 4: T_3 or Regular space

Definition 2.6:
A topological space (X, τ) is said to be T_4 or normal space if *A* and *B* are ***disjoint*** closed sets in X, there exists ***disjoint*** open sets $U_A, U_B \in X$ containing *A* and *B* respectively

(ie. $U_A \cap U_B = \varnothing$).

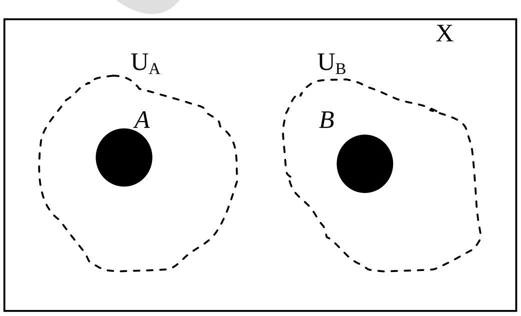

Fig. 5: T_4 or Normal space

3. MAIN RESULTS

Definition 3.1:
A topological space (X, τ) is said to be completely regular if given closed set C of X and a point $x \in X$ such that $x \notin C$, then there exists a continuous map $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(C) = \{1\}$.

Theorem: Every completely regular space is a regular space as well.

Proof (Suggested):
Let (X, τ) be a completely regular space, and let $x \in X$ and C be a closed subset of X such that $x \notin C$, then $f(x) = 0$ and $f(C) = \{1\}$. From this definition, it implies $f(x) \cap f(C) = \varnothing$.
Choose $U_{f(x)}$ and $V_{f(C)}$ as open sets for *f(x)* and *f(C)* respectively such that $U_{f(x)} \cap V_{f(C)} = \varnothing$. Then $f^{-1}(U_{f(x)})$ and $f^{-1}(V_{f(C)})$ are ***disjoint*** open sets of *x* and C respectively
(ie. $f^{-1}(U_{f(x)}) \cap f^{-1}(V_{f(C)}) = \varnothing$). Hence (X, τ) satisfies **Definition 2.5,** then (X, τ) is regular. □

4. CONCLUSION

From the proof above it can be concluded that any topological space which is completely regular is a regular space as well.

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