

## Original Research Article

## OPEN STRING UNDER THE MODIFIED BORN-INFELD FIELD

**ABSTRACT.** In this article we consider the two end-points of the string to be attached to D-brane with the different Born-Infeld field strength  $\mathcal{F}$  and calculate the total momenta for the special case.

## 1. INTRODUCTION

We consider a string ending on a Dp-brane, the bosonic part of the action is

$$\begin{aligned} S_B = & \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma [g^{\alpha\beta} G_{\mu\nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} + \epsilon^{\alpha\beta} B_{\mu\nu} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu}] \\ & + \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau A_i(X) \partial_{\tau} X^i, \end{aligned}$$

where  $A_i$  ( $i = 0, 1, \dots, p$ ), is the  $U(1)$  gauge field living on the Dp-brane [1, 2, 3]. The string background is

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad \Phi = \text{constant}, \quad H = dB = 0.$$

Here we use the boundary condition of the action  $S_B$  so that we can get more specific equations of motion for a free field and the canonical momentum.

## 2. EQUATIONS OF MOTION AND THE CANONICAL MOMENTUM

Variation of the action yields the equations of motion for a free field

$$(1) \quad (\partial_{\tau}^2 - \partial_{\sigma}^2) X^{\mu} = 0$$

and the following boundary conditions at  $\sigma = 0$  :

$$(2) \quad \begin{aligned} \partial_{\sigma} X^i + \partial_{\tau} X^j \mathcal{F}_j^i &= 0, & i, j &= 0, 1, \dots, p, \\ X^a &= x_0^a, & a &= p+1, \dots, 9, \end{aligned}$$

and at  $\sigma = \pi$  :

$$(3) \quad \partial_{\sigma} X^i + \partial_{\tau} X^j \mathcal{F}'_j^i = 0, \quad i, j = 0, 1, \dots, p.$$

Here

$$\mathcal{F} = B - F \quad \text{and} \quad \mathcal{F}' = B' - F'$$

are the modified Born-Infeld field strength and  $x_0^a$ ,  $x_0^b$  are the location of the D-branes. Indices are raised and lowered by  $\eta_{ij} = (-, +, \dots, +)$ .

The general solution of  $X^k$  to the equations of motion in (1) is [1]

$$(4) \quad \begin{aligned} X^k &= x_0^k + (a_0^k \tau + b_0^k \sigma) + c_0^k \sigma \tau + d_0^k (\tau^2 + \sigma^2) \\ &+ \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (ia_n^k \cos n\sigma + b_n^k \sin n\sigma) \end{aligned}$$

and

$$X^a = x_0^a + b^a \sigma + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} a_n^a \sin n\sigma, \quad a = p+1, \dots, 9,$$

where  $x_0^a + \pi b^a$  is the location of the D-brane to which the other end-point of the open string is attached.

**Lemma 2.1.** *The coefficients  $c_0^k$  and  $d_0^k$  in Eq. (4) are*

(a)

$$c_0^k = \sum_{n \in \mathbb{Z}} (-1)^n \left( in(b_n^l + a_n^j \mathcal{F}'_j^l) + \frac{1}{\pi} (b_n^j + a_n^k \mathcal{F}'_k^j) \mathcal{F}'_j^l \right) (M'^{-1})_l^k,$$

(b)

$$d_0^k = \frac{1}{2} \sum_{n \in \mathbb{Z}} (-1)^{n-1} \left( \frac{1}{\pi} (b_n^l + a_n^j \mathcal{F}'_j^l) + in(b_n^j + a_n^k \mathcal{F}'_k^j) \mathcal{F}'_j^l \right) (M'^{-1})_l^k,$$

$$\text{where } M'_{ij} = \eta_{ij} - \mathcal{F}'_i^k \mathcal{F}'_{kj}.$$

*Proof.* By (3) and (4) we have

$$\begin{aligned} 0 &= \partial_\sigma X^k + \partial_\tau X^j \mathcal{F}'_j^k \\ &= \partial_\sigma \left( x_0^k + (a_0^k \tau + b_0^k \sigma) + c_0^k \sigma \tau + d_0^k (\tau^2 + \sigma^2) \right. \\ &\quad \left. + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (ia_n^k \cos n\sigma + b_n^k \sin n\sigma) \right) \\ &\quad + \partial_\tau \left( x_0^j + (a_0^j \tau + b_0^j \sigma) + c_0^j \sigma \tau + d_0^j (\tau^2 + \sigma^2) \right. \\ &\quad \left. + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (ia_n^j \cos n\sigma + b_n^j \sin n\sigma) \right) \mathcal{F}'_j^k \\ &= b_0^k + c_0^k \tau + 2d_0^k \sigma + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (-ina_n^k \sin n\sigma + nb_n^k \cos n\sigma) \\ &\quad + \left( a_0^j + c_0^j \sigma + 2d_0^j \tau + \sum_{n \neq 0} \frac{-ine^{-in\tau}}{n} (ia_n^j \cos n\sigma + b_n^j \sin n\sigma) \right) \mathcal{F}'_j^k \\ &= b_0^k + a_0^j \mathcal{F}'_j^k + (c_0^k + 2d_0^j \mathcal{F}'_j^k) \tau + (2d_0^k + c_0^j \mathcal{F}'_j^k) \sigma \\ &\quad - \sum_{n \neq 0} e^{-in\tau} \left( i \sin n\sigma (a_n^k + b_n^j \mathcal{F}'_j^k) - \cos n\sigma (b_n^k + a_n^j \mathcal{F}'_j^k) \right) \\ &= (c_0^k + 2d_0^j \mathcal{F}'_j^k) \tau + (2d_0^k + c_0^j \mathcal{F}'_j^k) \sigma \\ &\quad - \sum_{n \in \mathbb{Z}} e^{-in\tau} \left( i \sin n\sigma (a_n^k + b_n^j \mathcal{F}'_j^k) - \cos n\sigma (b_n^k + a_n^j \mathcal{F}'_j^k) \right) \end{aligned}$$

then, now since  $\sigma = \pi$  and using the Taylor series, this identity can be written as

$$\begin{aligned}
& (c_0^k + 2d_0^j \mathcal{F}'_j^k) \tau + (2d_0^k + c_0^j \mathcal{F}'_j^k) \pi + \sum_{n \in \mathbb{Z}} e^{-in\tau} (-1)^n (b_n^k + a_n^j \mathcal{F}'_j^k) \\
&= (c_0^k + 2d_0^j \mathcal{F}'_j^k) \tau + (2d_0^k + c_0^j \mathcal{F}'_j^k) \pi \\
&\quad + \sum_{n \in \mathbb{Z}} \left( \sum_{m=0}^{\infty} \frac{(-in\tau)^m}{m!} \right) (-1)^n (b_n^k + a_n^j \mathcal{F}'_j^k) \\
&= \left( (2d_0^k + c_0^j \mathcal{F}'_j^k) \pi + \sum_{n \in \mathbb{Z}} (-1)^n (b_n^k + a_n^j \mathcal{F}'_j^k) \right) \\
&\quad + \left( c_0^k + 2d_0^j \mathcal{F}'_j^k - i \sum_{n \in \mathbb{Z}} (-1)^n n (b_n^k + a_n^j \mathcal{F}'_j^k) \right) \tau \\
&\quad + \sum_{n \in \mathbb{Z}} \left( \sum_{m=2}^{\infty} \frac{(-in)^m}{m!} \right) (-1)^n (b_n^k + a_n^j \mathcal{F}'_j^k) \tau^m \\
&= 0.
\end{aligned}$$

Thus the above identical equation about  $\tau$  shows that

$$(5) \quad (2d_0^k + c_0^j \mathcal{F}'_j^k) \pi + \sum_{n \in \mathbb{Z}} (-1)^n (b_n^k + a_n^j \mathcal{F}'_j^k) = 0,$$

$$(6) \quad c_0^k + 2d_0^j \mathcal{F}'_j^k - i \sum_{n \in \mathbb{Z}} (-1)^n n (b_n^k + a_n^j \mathcal{F}'_j^k) = 0,$$

and

$$\sum_{n \in \mathbb{Z}} n^m (-1)^n (b_n^k + a_n^j \mathcal{F}'_j^k) = 0 \quad \text{for } m \geq 2.$$

(a) From (5) we can easily obtain

$$(7) \quad 2d_0^j \mathcal{F}'_j^k + c_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k + \frac{1}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n (b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k = 0.$$

Subtracting (7) from (6) we get

$$\begin{aligned}
& c_0^k - i \sum_{n \in \mathbb{Z}} (-1)^n n (b_n^k + a_n^j \mathcal{F}'_j^k) - c_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k \\
&\quad - \frac{1}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n (b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k = 0
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}} (-1)^n \left( i n (b_n^k + a_n^j \mathcal{F}'_j^k) + \frac{1}{\pi} (b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k \right) \\
&= c_0^k - c_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k \\
&= c_0^l \eta_l^k - c_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k \\
&= c_0^l (\eta_l^k - \mathcal{F}'_l^j \mathcal{F}'_j^k) \\
&= c_0^l M_l^{lk}
\end{aligned}$$

so

$$c_0^l = \sum_{n \in \mathbb{Z}} (-1)^n \left( in(b_n^k + a_n^j \mathcal{F}'_j^k) + \frac{1}{\pi} (b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k \right) (M'^{-1})_k^l.$$

(b) In a similar manner, by (6) we have

$$(8) \quad c_0^j \mathcal{F}'_j^k + 2d_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k - i \sum_{n \in \mathbb{Z}} (-1)^n n (b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k = 0.$$

After dividing (5) by  $\pi$ , we subtract (8) from (5) and obtain

$$\begin{aligned} & 2d_0^k + \frac{1}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n (b_n^k + a_n^j \mathcal{F}'_j^k) - 2d_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k \\ & + i \sum_{n \in \mathbb{Z}} (-1)^n n (b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \sum_{n \in \mathbb{Z}} (-1)^{n-1} \left( \frac{1}{\pi} (b_n^k + a_n^j \mathcal{F}'_j^k) + in(b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k \right) \\ & = d_0^k - d_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k \\ & = d_0^l \eta_l^k - d_0^l \mathcal{F}'_l^j \mathcal{F}'_j^k \\ & = d_0^l (\eta_l^k - \mathcal{F}'_l^j \mathcal{F}'_j^k) \\ & = d_0^l M_l'^k \end{aligned}$$

so

$$d_0^l = \frac{1}{2} \sum_{n \in \mathbb{Z}} (-1)^{n-1} \left( \frac{1}{\pi} (b_n^k + a_n^j \mathcal{F}'_j^k) + in(b_n^j + a_n^l \mathcal{F}'_l^j) \mathcal{F}'_j^k \right) (M'^{-1})_k^l.$$

□

**Remark 2.2.** Let us consider the two end-points of the string to be attached to D-brane with the same  $\mathcal{F}$  field. Then we can see that

$$b_n^k + a_n^j \mathcal{F}_j^k = 0, \quad \text{for all } n$$

in [1]. Applying this fact to Lemma 2.1, we simply have  $c_0^k = d_0^k = 0$ , which equates the result obtained in [1].

Now the canonical momentum is given by

$$2\pi\alpha' P^k(\tau, \sigma) = \partial_\tau X^k + \partial_\sigma X^j \left( \frac{\mathcal{F}_j^k + \mathcal{F}'_j^k}{2} \right)$$

So by (4), we note that

$$\begin{aligned}
(9) \quad & 2\pi\alpha' P^k(\tau, \sigma) \\
&= \partial_\tau \left( x_0^k + a_0^k \tau + b_0^k \sigma + c_0^k \sigma \tau + d_0^k (\tau^2 + \sigma^2) + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (ia_n^k \cos n\sigma + b_n^k \sin n\sigma) \right) \\
&\quad + \partial_\sigma \left( x_0^j + a_0^j \tau + b_0^j \sigma + c_0^j \sigma \tau + d_0^j (\tau^2 + \sigma^2) + \sum_{n \neq 0} \frac{e^{-in\tau}}{n} (ia_n^j \cos n\sigma + b_n^j \sin n\sigma) \right) \\
&\quad \times \left( \frac{\mathcal{F}_j^k + \mathcal{F}'_j^k}{2} \right) \\
&= a_0^k + c_0^k \sigma + 2d_0^k \tau - i \sum_{n \neq 0} e^{-in\tau} (ia_n^k \cos n\sigma + b_n^k \sin n\sigma) \\
&\quad + \left( b_0^j + c_0^j \tau + 2d_0^j \sigma - \sum_{n \neq 0} e^{-in\tau} (ia_n^j \sin n\sigma - b_n^j \cos n\sigma) \right) \left( \frac{\mathcal{F}_j^k + \mathcal{F}'_j^k}{2} \right) \\
&= \left( a_0^k + \frac{b_0^j (\mathcal{F}_j^k + \mathcal{F}'_j^k)}{2} \right) + \left( c_0^k + d_0^j (\mathcal{F}_j^k + \mathcal{F}'_j^k) \right) \sigma + \left( 2d_0^k + \frac{c_0^j (\mathcal{F}_j^k + \mathcal{F}'_j^k)}{2} \right) \tau \\
&\quad - \sum_{n \neq 0} e^{-in\tau} \left\{ i \left( b_n^k + \frac{a_n^j (\mathcal{F}_j^k + \mathcal{F}'_j^k)}{2} \right) \sin n\sigma - \left( a_n^k + \frac{b_n^j (\mathcal{F}_j^k + \mathcal{F}'_j^k)}{2} \right) \cos n\sigma \right\} \\
&= \left( c_0^k + d_0^j (\mathcal{F}_j^k + \mathcal{F}'_j^k) \right) \sigma + \left( 2d_0^k + \frac{c_0^j (\mathcal{F}_j^k + \mathcal{F}'_j^k)}{2} \right) \tau \\
&\quad - \sum_{n \in \mathbb{Z}} e^{-in\tau} \left\{ i \left( b_n^k + \frac{a_n^j (\mathcal{F}_j^k + \mathcal{F}'_j^k)}{2} \right) \sin n\sigma - \left( a_n^k + \frac{b_n^j (\mathcal{F}_j^k + \mathcal{F}'_j^k)}{2} \right) \cos n\sigma \right\}.
\end{aligned}$$

**Theorem 2.3.** If  $\mathcal{F}' = -\mathcal{F}$ , the total momenta

$$P_{tot}^k(\tau) = \frac{\pi}{4\alpha'} c_0^k + \frac{1}{\alpha'} d_0^k \tau + \frac{1}{2\alpha'} a_0^k + \frac{1}{2\pi\alpha'} \sum_{n \neq 0} \frac{ie^{-in\tau}}{n} ((-1)^n - 1) b_n^k,$$

where

$$\begin{aligned}
c_0^k &= \frac{i}{2} \sum_{n \in \mathbb{Z}} n \left( (1 + (-1)^n) b_n^k + (1 - (-1)^n) a_n^j \mathcal{F}_j^k \right), \\
d_0^k &= \frac{i}{4} \sum_{n \in \mathbb{Z}} n \left( (1 + (-1)^n) b_n^j \mathcal{F}_j^k + (1 - (-1)^n) a_n^i \mathcal{F}_i^j \mathcal{F}_j^k \right) \\
&\quad - \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} (-1)^n (b_n^k - a_n^j \mathcal{F}_j^k).
\end{aligned}$$

*Proof.* By the condition  $\mathcal{F}' = -\mathcal{F}$  and (9), we have

$$2\pi\alpha' P^k(\tau, \sigma) = c_0^k \sigma + 2d_0^k \tau - \sum_{n \in \mathbb{Z}} e^{-in\tau} (ib_n^k \sin n\sigma - a_n^k \cos n\sigma)$$

and so

$$\begin{aligned}
P_{tot}^k(\tau) &= \int_0^\pi d\sigma P^k(\tau, \sigma) \\
&= \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \left( c_0^k \sigma + 2d_0^k \tau - \sum_{n \in \mathbb{Z}} e^{-in\tau} (ib_n^k \sin n\sigma - a_n^k \cos n\sigma) \right) \\
&= \frac{1}{2\pi\alpha'} \left( \frac{\pi^2}{2} c_0^k + 2\pi d_0^k \tau + a_0^k \pi + \sum_{n \neq 0} \frac{ie^{-in\tau}}{n} ((-1)^n - 1) b_n^k \right) \\
&= \frac{\pi}{4\alpha'} c_0^k + \frac{1}{\alpha'} d_0^k \tau + \frac{1}{2\alpha'} a_0^k + \frac{1}{2\pi\alpha'} \sum_{n \neq 0} \frac{ie^{-in\tau}}{n} ((-1)^n - 1) b_n^k.
\end{aligned}$$

And using the boundary condition (2) and Taylor series for  $\tau$  we obtain

$$(10) \quad \sum_{n \in \mathbb{Z}} (b_n^k + a_n^j \mathcal{F}_j^k) = 0,$$

$$(11) \quad c_0^k + 2d_0^j \mathcal{F}_j^k - i \sum_{n \in \mathbb{Z}} n(b_n^k + a_n^j \mathcal{F}_j^k) = 0,$$

and

$$\sum_{n \in \mathbb{Z}} n^m (b_n^k + a_n^j \mathcal{F}_j^k) = 0 \quad \text{for } m \geq 2.$$

Also applying the assumption  $\mathcal{F}' = -\mathcal{F}$  to Eqs. (5) and (6), we have

$$(12) \quad (2d_0^k - c_0^j \mathcal{F}_j^k) \pi + \sum_{n \in \mathbb{Z}} (-1)^n (b_n^k - a_n^j \mathcal{F}_j^k) = 0,$$

$$(13) \quad c_0^k - 2d_0^j \mathcal{F}_j^k - i \sum_{n \in \mathbb{Z}} (-1)^n n(b_n^k - a_n^j \mathcal{F}_j^k) = 0.$$

Then by (11) and (13) we deduce that

$$2c_0^k - i \sum_{n \in \mathbb{Z}} n(b_n^k + a_n^j \mathcal{F}_j^k) - i \sum_{n \in \mathbb{Z}} (-1)^n n(b_n^k - a_n^j \mathcal{F}_j^k) = 0$$

so

$$c_0^k = \frac{i}{2} \sum_{n \in \mathbb{Z}} n ((1 + (-1)^n) b_n^k + (1 - (-1)^n) a_n^j \mathcal{F}_j^k).$$

Finally, substituting the above  $c_0^k$  into (12) we complete the proof.  $\square$

## REFERENCES

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