Notes on "Constructions of triangular norms on lattices by means of irreducible elements"

Abstract

In this note, we show by an counterexample that a paper by Yılmaz and Kazancı (Ş. Yılmaz, O. Kazancı, Constructions of triangular norms on lattices by means of irreducible elements, Inform. Sci. 397–398 (2017) 110–117) suffers from certain mistakes.

Keywords: $\wedge$-semilattice, $\vee$-semilattice, Irreducible element, Triangular norm

1. Introduction

A poset $(L, \leq)$ is a $\vee$-semilattice (dually, $\wedge$-semilattice) iff $\sup\{x, y\}$ (dually, $\inf\{x, y\}$) exists for any two elements x and y. Denote $x \vee y = \sup\{x, y\}$ and call it the join of x and y. Dullary, Denote $x \wedge y = \inf\{x, y\}$ and call it the meet of x and y. A lattice is an ordered set $(E, \leq)$ which is both an $\vee$-semilattice and an $\wedge$-semilattice with respect to its order [1].

A sublattice is a non-empty subset S of a lattice L , such that S is closed under meet and join. A lattice is complete if for every subset there exist the meet and the join. A lattice which possesses the smallest (the bottom) and the greatest (the top) elements, 0 and 1, respectively is bounded. A lattice L is a chain if either $x \leq 1y$ or $y \leq x$ for all $x, y \in L$.

If L is a lattice then $a \in L$ (with $a \neq 0$ if L has a bottom element 0) is said to be $\vee$-irreducible if $x \vee y = a$ implies $x = a$ or $y = a$. Thus, $a \in L$ is $\vee$-irreducible if it cannot be expressed as the join of two elements that are strictly less than a. We denote the set of $\vee$-irreducible elements of a lattice

L by $J(L)$. Dually, we can define the set $M(L)$ of $\wedge$-irreducible elements. For further information see [1, 2].

Definition 1.1. [3] Let $(L, \leq, 0, 1)$ be a bounded lattice. The set $J(L)^* = J(L) \cup \{0, 1\}$ and $M(L)^* = M(L) \cup \{0, 1\}$ are defined as the extended set of $\vee$-irreducible elements and $\wedge$-irreducible elements of L, respectively.

We point out an assertion in [3] is incorrect by counterexamples.

2. Mistakes and counterexamples

On page 112 in [3], the last line, the authors claimed that $J(L_1 \times L_2)^* = J(L_1 \times L_2) \cup \{(0,0), (1,1)\} = J(L_1)^* \times J(L_2)^*$. It follows from Definition 1.1 that $J(L_1 \times L_2)^* = J(L_1 \times L_2) \cup \{(0,0), (1,1)\}$. However, the following example shows that $J(L_1 \times L_2)^* \neq J(L_1)^* \times J(L_2)^*$ in general.

Example 2.1. Let $L_1 = \{0, a, b, 1\}$ be a boolean lattice with two atoms, and let $L_2 = \{0, 1\}$ be a two-element chain. Then $L_1 \times L_2$ is a boolean lattice with three atoms. It is clear that $J(L_1) = \{a, b\}$, $J(L_1)^* = L_1$, $J(L_2) = \{1\}$ and $J(L_2)^* = L_2$. It follows that

$$J(L_1 \times L_2)^* = \{(0,0), (a,0), (b,0), (0,1), (1,1)\},$$

and $J(L_1)^* \times J(L_2)^* = L_1 \times L_2$. Thus $J(L_1 \times L_2)^* \neq J(L_1)^* \times J(L_2)^*$.

It is clear that $J(L_1 \times L_2)^*$ is a subposet of $L_1 \times L_2$. In Example 2.1, $(a, 0) \vee_{L_1 \times L_2} (b, 0) = (1, 0)$ and $(a, 0) \vee_{J(L_1 \times L_2)^*} (b, 0) = (1, 1)$. It follows that $J(L_1 \times L_2)^*$ is not a sublattice of $L_1 \times L_2$.

[1] G. Birkhoff, Lattice Theory, American Mathematical Society Colloquium Publications, Rhode Island, 1967.

[2] G. Gratzer , General Lattice Theory, second ed., Birkhauser Verlag, Basel, Switzerland, 1998 .

[3] Ş. Yılmaz, O. Kazancı, Constructions of triangular norms on lattices by means of irreducible elements, Inform. Sci. 397–398 (2017) 110–117.