Pullback Absorbing set for the stochastic lattice Selkov equations

Abstract: In this paper, a transformation of addition involved with O-U process is exploited to deal with the challenge in proving the pullback absorbing property for the stochastic reversible Selkov system in an infinite lattice with additive white noises.
Keywords: Random dynamical system; additive noise; Selkov system

1 Introduction

For the stochastic reversible Selkov system with the cubic nonlinearity and additive white noise on an infinite lattice:

$$\begin{cases} du_i = [d_1(u_{i+1} - 2u_i + u_{i-1}) - a_1 u_i + b_1 u_i^2 v_i - b_2 u_i^3 + f_{1i}]dt + \alpha_i dw_i, \quad i \in \mathbb{Z}, \ t > 0, \\ dv_i = [d_2(v_{i+1} - 2v_i + v_{i-1}) - a_2 v_i - b_1 u_i^2 v_i + b_2 u_i^3 + f_{2i}]dt + \alpha_i dw_i, \quad i \in \mathbb{Z}, \ t > 0, \end{cases} \tag{1.1}$$

with initial conditions

$$u_i(0) = u_{i,0}, \qquad v_i(0) = v_{i,0}, \qquad i \in \mathbb{Z}, \tag{1.2}$$

where $\mathbb{Z}$ denotes the integer set, $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, d_1, d_2, a_1, a_2, b_1, b_2 are positive constants, $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$, $\{w_i | i \in \mathbb{Z}\}$ is independent Brownian motions. We have obtained the random dynamical system, see [7]. Pullback absorbing property is very important to describe the long-time behavior of the equations for the mathematics and physics, especially, to prove the existence of random attractor. Therefore, in this paper, we prove the pullback absorbing property for the Selkov equations (1.1).

2 Preliminaries

In this section, we introduce the relevant definitions of absorbing property, which are taken from [2], [4], [6], [8].

Let (H, d) be a complete separable metric space, $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F}, \mathcal{F})$ measurable, $\theta_0 = I, \theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{R}$, and $\theta_t\mathcal{P} = \mathcal{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous random dynamical system (RDS) on H over a metric dynamical system $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times H \to H, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable and satisfies, for every $\omega \in \Omega$,

(i) $\varphi(0, \omega, \cdot)$ is the identity on H;

(ii) Cocycle property: $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot))$ for all $t, s \in \mathbb{R}^+$;

(iii) $\varphi(\cdot, \omega, \cdot) : \mathbb{R}^+ \times H \to H$ is strongly continuous.

Definition 2.3. A random bounded set $B(\omega) \subset X$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for every $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{\beta t} d(B(\theta_{-t}\omega)) = 0 \text{ for all } \beta > 0,$$

where $d(B) = \sup_{x\in B} \|x\|_X$.

Definition 2.4. A random set $K(\omega)$ is called a pullback absorbing set in $\mathcal{D}$, where $\mathcal{D}$ is a collection of random sets of H, if for all $B \in \mathcal{D}$ and every $\omega \in \Omega$, there exists a $t_B(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \text{ for all } t \geq t_B(\omega).$$

3 Ornstein-Uhlenbeck process

To convert the stochastic wave equation to a deterministic one with random parameters, we introduce an Ornstein-Uhlenbeck process (O-U process) in ℓ^2 on the metric dynamical systems $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ given by the Wiener process:

$$y(\theta_t\omega) = -(a_1 + a_2)\int_{-\infty}^{0} e^{(a_1+a_2)s}(\theta_t\omega)(s)ds, \qquad t \in \mathbb{R}, \qquad \omega \in \Omega.$$

The above integral exists for any path ω with a subexponential growth, and y solve the following Itô equations respectively:

$$dy + (a_1 + a_2)ydt = dw(t), \qquad t > 0.$$

Furthermore, there exists a θ_t-invariant set $\Omega' \subset \Omega$ of full $\mathcal{P}$ measure such that

(1) the mappings $s \to y(\theta_s\omega)$, is continuous for each $\omega \in \Omega$;

(2) the random variables $\|y(\theta_t\omega)\|$ is tempered.

Let

$$\tilde{u}(t) = u(t) - y(\theta_t\omega), \qquad \tilde{v}(t) = v(t) - y(\theta_t\omega).$$

Then we get

$$\begin{cases} \tilde{u}_t = -d_1A(\tilde{u} + y(\theta_t\omega)) - a_1\tilde{u} + a_2y(\theta_t\omega) + b_1(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)) \\ \qquad -b_2(\tilde{u} + y(\theta_t\omega))^3 + f_1 \\ \tilde{v}_t = -d_2A(\tilde{v} + y(\theta_t\omega)) - a_2\tilde{v} + a_1y(\theta_t\omega) - b_1(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)) \\ \qquad +b_2(\tilde{u} + y(\theta_t\omega))^3 + f_2 \end{cases} \tag{3.1}$$

with the initial value condition

$$\tilde{u}(0, \omega, \tilde{u}_0) = \tilde{u}_0(\omega) = u_0 - y(\omega), \qquad \tilde{v}(0, \omega, \tilde{v}_0) = \tilde{v}_0(\omega) = v_0 - y(\omega).$$

4 pullback absorbing property

Lemma 4.1. *There exists a θ_t-invariant set $\Omega' \subset \Omega$ of full $\mathcal{P}$ measure and an absorbing random set $K(\omega), \omega \in \Omega'$, for the random dynamical system $\varphi(t, \omega)$, i.e. for all $B \in \mathcal{D}$ and all $\omega \in \Omega'$, there exists $T_B(\omega) > 0$ such that*

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \qquad \text{for all } t \geq T_B(\omega).$$

Moreover, $K \in \mathcal{D}$.

Proof. Taking the inner product (3.1) with $(\tilde{u}, \tilde{v}, \tilde{z})^T$ in E, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2 &= -d_1\langle A\tilde{u}, \tilde{u}\rangle - d_1\langle Ay(\theta_t\omega), \tilde{u}\rangle - a_1\|\tilde{u}\|^2 + b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{u}\rangle \\ &\quad -b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{u}\rangle + \langle f_1, \tilde{u}\rangle + a_2\langle y(\theta_t\omega), \tilde{u}\rangle, \\ \frac{1}{2}\frac{d}{dt}\|\tilde{v}\|^2 &= -d_2\langle A\tilde{v}, \tilde{v}\rangle - d_2\langle Ay(\theta_t\omega), \tilde{v}\rangle - a_2\|\tilde{v}\|^2 - b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{v}\rangle \\ &\quad +b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{v}\rangle + \langle f_2, \tilde{v}\rangle + a_1\langle y(\theta_t\omega), \tilde{v}\rangle. \end{aligned} \tag{4.1}$$

Summing the three equations up, we get that

$$
\begin{aligned}
& \frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2]+2d_1\langle A\tilde{u},\tilde{u}\rangle+2d_2\langle A\tilde{v},\tilde{v}\rangle+2a_1\|\tilde{u}\|^2+2a_2\|\tilde{v}\|^2 \\
= \; & -2d_1\langle Ay(\theta_t\omega),\tilde{u}\rangle-2d_2\langle Ay(\theta_t\omega),\tilde{v}\rangle+2\langle f_1,\tilde{u}\rangle+2\langle f_2,\tilde{v}\rangle+2a_2\langle y(\theta_t\omega),\tilde{u}\rangle+2a_1\langle y(\theta_t\omega),\tilde{v}\rangle \\
& +2b_1\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{u}\rangle-2b_1\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{v}\rangle \\
& -2b_2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{u}\rangle+2b_2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{v}\rangle. \qquad (4.2)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& 2b_1\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{u}\rangle-2b_1\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{v}\rangle \\
& -2b_2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{u}\rangle+2b_2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{v}\rangle \\
= \; & 2b_1\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)),\tilde{u}-\tilde{v}\rangle-2b_2\langle(\tilde{u}+y(\theta_t\omega))^3,\tilde{u}-\tilde{v}\rangle \\
\le \; & 2\max\{b_1,b_2\}\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{v}+y(\theta_t\omega)-\tilde{u}-y(\theta_t\omega)),\tilde{u}-\tilde{v}\rangle \\
= \; & -2\max\{b_1,b_2\}\langle(\tilde{u}+y(\theta_t\omega))^2(\tilde{u}-\tilde{v}),\tilde{u}-\tilde{v}\rangle \\
= \; & -2\sum_{i\in\mathbb{Z}}(\tilde{u}_i+y_i(\theta_t\omega))^2(\tilde{u}_i-\tilde{v}_i)^2\le 0. \qquad (4.3)
\end{aligned}
$$

By Young's inequality, we have the following estimate

$$
\begin{aligned}
& -2d_1\langle Ay(\theta_t\omega),\tilde{u}\rangle\le\frac{a_1}{3}\|\tilde{u}\|^2+\frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2, \\
& -2d_2\langle Ay(\theta_t\omega),\tilde{v}\rangle\le\frac{a_2}{3}\|\tilde{v}\|^2+\frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2, \\
& 2a_2\langle y(\theta_t\omega),\tilde{u}\rangle\le\frac{a_1}{3}\|\tilde{u}\|^2+\frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2, \\
& 2a_1\langle y(\theta_t\omega),\tilde{v}\rangle\le\frac{a_2}{3}\|\tilde{v}\|^2+\frac{3a_1^2)^2}{a_2}\|y(\theta_t\omega)\|^2, \\
& 2\langle f_1,\tilde{u}\rangle\le\frac{a_1}{3}\|\tilde{u}\|^2+\frac{3}{a_1}\|f_1\|^2, \\
& 2\langle f_2,\tilde{v}\rangle\le\frac{a_2}{3}\|\tilde{v}\|^2+\frac{3}{a_2}\|f_2\|^2. \qquad (4.4)
\end{aligned}
$$

By (4.2)-(4.4), we obtain that

$$
\begin{aligned}
& \frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2]+2d_1\langle A\tilde{u},\tilde{u}\rangle+2d_2\langle A\tilde{v},\tilde{v}\rangle+2a_1\|\tilde{u}\|^2+2a_2\|\tilde{v}\|^2 \\
\le \; & \frac{a_1}{3}\|\tilde{u}\|^2+\frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2+\frac{a_2}{3}\|\tilde{v}\|^2+\frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2+\frac{3}{a_1}\|f_1\|^2+\frac{a_1}{3}\|\tilde{u}\|^2 \\
& +\frac{3}{a_2}\|f_2\|^2+\frac{a_2}{3}\|\tilde{v}\|^2+\frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2+\frac{a_1}{3}\|\tilde{u}\|^2+\frac{3a_1^2}{a_2}\|y(\theta_t\omega)\|^2+\frac{a_2}{3}\|\tilde{v}\|^2 \\
= \; & a_1\|\tilde{u}\|^2+\frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2+a_2\|\tilde{v}\|^2+\frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2+\frac{3}{a_1}\|f_1\|^2 \\
& +\frac{3}{a_2}\|f_2\|^2+\frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2+\frac{3a_1^2}{a_2}\|y(\theta_t\omega)\|^2,
\end{aligned}
$$

hence we have,

$$
\begin{aligned}
& \frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2]+a_1\|\tilde{u}\|^2+a_2\|\tilde{v}\|^2 \\
\le \; & \frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2+\frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2+\frac{3}{a_1}\|f_1\|^2+\frac{3}{a_2}\|f_2\|^2+\frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2+\frac{3a_1^2}{a_2}\|y(\theta_t\omega)\|^2 \\
\le \; & C_1\|Ay(\theta_t\omega)\|^2+C_2\|y(\theta_t\omega)\|^2+C_3(\|f_1\|^2+\|f_2\|^2) \\
\le \; & C_4(\|y(\theta_t\omega)\|^2+\|Ay(\theta_t\omega)\|^2+\|f_1\|^2+\|f_2\|^2+\|f_3\|^2), \qquad (4.5)
\end{aligned}
$$

where

$$C_1 = \max\{\frac{3d_1^2}{a_1}, \frac{3d_2^2}{a_2}\}, \qquad C_2 = \max\{\frac{3a_2^2}{a_1}, \frac{3a_1^2}{a_2}\},$$
$$C_3 = \max\{\frac{3}{a_1}, \frac{3}{a_2}\}, \qquad C_4 = \max\{C_1, C_2, C_3\}.$$

By Gronwall's inequality, it follows that

$$\begin{aligned} & \|\tilde{u}(t,\omega,\tilde{u}_0(\omega))\|^2 + \|\tilde{v}(t,\omega,\tilde{v}_0(\omega))\|^2 \\ \leq\ & e^{-\min\{a_1,a_2\}t}[\|\tilde{u}_0(\omega)\|^2 + \|\tilde{v}_0(\omega)\|^2] + \frac{C_4}{\min\{a_1,a_2\}}(\|f_1\|^2 + \|f_2\|^2) \\ & + C_4 \int_0^t e^{-\min\{a_1,a_2\}(t-s)}(\|y(\theta_s\omega)\|^2 + \|Ay(\theta_s\omega)\|^2)ds. \end{aligned} \tag{4.6}$$

Let $c_1 = \min\{a_1, a_2\}$. Note that the random variable $y(\theta_t\omega)$ is tempered and $y(\theta_t\omega)$ is continuous in t. Therefore, it follows from Proposition 4.3.3 in [1] that there exists a tempered function $l(\omega) > 0$ such that

$$\|y(\theta_t\omega)\|^2 + \|Ay(\theta_t\omega)\|^2 \leq l(\theta_t\omega) \leq l(\omega)e^{\frac{c_1}{2}|t|}. \tag{4.7}$$

Replacing ω by $\theta_{-t}\omega$ in (4.6) and using (4.7), we obtain

$$\begin{aligned} & \|\tilde{u}(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2 + \|\tilde{v}(t,\theta_{-t}\omega,\tilde{v}_0(\theta_{-t}\omega))\|^2 \\ \leq\ & e^{-c_1 t}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2 + \|\tilde{v}_0(\theta_{-t}\omega)\|^2] + \frac{C_4}{c_1}(\|f_1\|^2 + \|f_2\|^2) \\ & + C_4 \int_0^t e^{-c_1(t-s)}(\|y(\theta_{s-t}\omega)\|^2 + \|Ay(\theta_{s-t}\omega)\|^2)ds \\ \leq\ & e^{-c_1 t}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2 + \|\tilde{v}_0(\theta_{-t}\omega)\|^2] + \frac{C_4}{c_1}(\|f_1\|^2 + \|f_2\|^2) \\ & + C_4 \int_{-t}^0 e^{c_1\tau}(\|y(\theta_\tau\omega)\|^2 + \|Ay(\theta_\tau\omega)\|^2)d\tau \\ \leq\ & e^{-c_1 t}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2 + \|\tilde{v}_0(\theta_{-t}\omega)\|^2] + \frac{C_4}{c_1}(\|f_1\|^2 + \|f_2\|^2) + \frac{2C_4 l(\omega)}{c_1}. \end{aligned} \tag{4.8}$$

Define $R^2(\omega) = 2[C_4(\|f_1\|^2 + \|f_2\|^2) + 2C_4 l(\omega)]/c_1$; since $l(\omega)$ is a tempered function, then $R(\omega)$ is also tempered.

Define

$$\tilde{K}(\omega) = \{(\tilde{u},\tilde{v}) \in \ell^2 \times \ell^2, \|\tilde{u}\|^2 + \|\tilde{v}\|^2 \leq R^2(\omega)\}.$$

Then $\tilde{K}(\omega)$ is an absorbing set for the random dynamical system $(\tilde{u}(t,\omega,\tilde{u}_0), \tilde{v}(t,\omega,\tilde{v}_0))$, which follows from Theorem 4.2 in [7], that is, for every $B \in \mathcal{D}$ and every $\omega \in \Omega'$, there exists $T_B(\omega)$ such that

$$\Phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) \subset \tilde{K}(\omega) \quad \text{for} \quad t \geq T_B(\omega).$$

Let

$$K(\omega) = \{(u,v) \in \ell^2 \times \ell^2, \|u\|^2 + \|v\|^2 \leq R_1^2(\omega)\},$$

where

$$R_1^2(\omega) = 2R^2(\omega) + 4\|y(\theta_t\omega)\|^2.$$

Then, $K(\omega)$ is an absorbing random set for the random dynamical system $\varphi(t,\omega)$ since

$$\begin{aligned} & \varphi(t,\omega,(u_0,v_0,z_0)) \\ =\ & \Phi(t,\omega,(u_0 - y(\omega), v_0 - y(\omega))) + (y(\theta_t\omega), y(\theta_t\omega)) \\ =\ & (\tilde{u}(t,\omega,u_0 - y(\omega)) + y(\theta_t\omega), \tilde{v}(t,\omega,v_0 - y(\omega)) + y(\theta_t\omega)) \end{aligned}$$

and $K \in \mathcal{D}$. This completes the proof of Lemma 4.1. $\Box$

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