Method Article

A Novel Dynamic Gray Action GM(1,1,b) Model And Its Application

Abstract: The classical GM(1,1) model treats the gray action as an invariant constant, but changes have occurred within the system as time and space change. If the fixed grey action is still used for modeling, the model will have errors. Aiming at this shortcoming, this paper proposes a GM(1,1,b) model in which the gray action can be dynamically changed. Starting from the background value formula, the model solves the gray action at different time points by the development coefficient, and fits the sequence with the DGM(1,1) model, then brings the obtained time response sequence into the classical GM (1, 1) to replaces the gray action constant, so as to establish a GM(1,1,b) model with dynamic change of gray action. Finally, the model is applied to the example of China's rural residents' consumption index. The numerical example shows that the GM(1,1,b) model proposed in this paper effectively improves the prediction accuracy of the model and verifies the effectiveness and practicability of the improved model.

Key words: Rural residents' consumption level index; Grey action; GM(1,1,b); Prediction accuracy

1 Introduction

As one of the main driving forces for China's economic growth, consumption has affected the growth and speed of China's economy to a certain extent. Under the background of the continuous development of China's social economy, the consumption level of residents can effectively promote the growth of China's economy on the one hand, and the mutual influence with China's economic growth on the other. Both have a relationship of promotion and causality. At the same time, with the universal application of the Internet and the deepening of national policies in recent years, the quality of life of rural residents in China has been improved, and the level of consumption has also undergone significant changes. Therefore, scientific and reasonable prediction and analysis of China's rural residents' consumption level index can provide theoretical guidance for the macro-control of rural residents' consumption level.

In the prediction of the consumption level index, many scholars have conducted research. Among them, Meng [1] predicted the national consumption level index of China from 2010 to 2014 by establishing ARIMA model and residual autoregressive model. The results show that the ARIMA (2,2,0) model can achieve high prediction accuracy in short-term prediction. Li [2] used regression analysis to predict the consumption level of urban residents in Inner Mongolia, and applied computer to analyze the data. Wei [3] of Jilin university combined trend extrapolation method and exponential smoothing method, took the consumption level of urban residents in China from 1978 to 2007 as the analysis object, and comprehensively predicted the consumption level of urban residents by using deterministic time series analysis, and adopted the method of equal weight combination to improve the prediction accuracy. Ma et al. [4]

improved a traditional PCA method and established a new evaluation model and applied it to the study of Qinghai urban residents' consumption level index, which can better evaluate the consumption level of residents. Li [5] used SAS software to construct ARIMA model for time series analysis and short-term forecast of rural residents' consumption level in China. The forecast results show that the consumption level of rural residents in China will be further improved rapidly, and then propose a development strategy that takes into account overall planning. Li et al. [6] used the GM (1,1) model and the buffer operator axioms, combined with qualitative analysis, to predict the development prospects of China's consumption level from 2000 to 2005.Cheng [7] believes that the ordinary grey prediction model can't accurately predict the consumption level index of rural residents in China, so an improved GM (1,1) model based on initial value correction is proposed to predict the consumption level index of rural residents in China in the future, and the model is validated with 2012 data.

Since the introduction of the grey system theory by the famous Chinese scholar Deng [8] in 1982, after more than 30 years of development, the grey prediction model has been widely used in economic, agricultural, military, ecological and other fields [9-13].The grey prediction model has good performance in predicting "small sample" and "poor information" data sequences. The improvement of the GM(1,1) model has never stopped, mainly in the following five aspects: (1) preprocessing of raw data [14-16]; (2) improvement of background value [17-19]; (3) initial value improvement [20]; (4) parameter improvement of the model [21-22]; (5) combination improvement of the above method [23].

In summary, there are not many literatures on the consumption index of rural residents, and there are no systematic methods and models. Starting from the classical GM (1,1) model, this paper proposes a novel GM (1,1) model with dynamic features. The discrete DGM(1,1) model is used to model and predict the gray action sequence, which further improves the prediction accuracy of the GM(1,1) model and broadens the application range of the model. It provides a new modeling idea for the prediction of rural residents' consumption level index.

2 Basic model

2.1. GM (1,1) model

Let $X^{(0)}=\left(x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right)$ as a non-negative raw sequence. Grey differential equations can be established for sequences satisfying smooth conditions. After a one-order accumulated, $X^{(1)}=\left(x^{(1)}(1),x^{(1)}(2),\cdots,x^{(1)}(n)\right)$ is generated. $X^{(1)}$ is $1-AGO$ sequence of $X^{(0)}$, where

$$x^{(1)}(k)=\sum_{i=1}^{k}x^{(0)}(i)\left(k=1,2,\cdots,n\right). \tag{1}$$

And $Z^{(1)}=\left(z^{(1)}(1),z^{(1)}(2),\cdots,z^{(1)}(n)\right)$ is the sequence mean generated of consecutive neighbors of $X^{(1)}$, where

$$z^{(1)}(k)=0.5\left(x^{(1)}(k)+x^{(1)}(k-1)\right),\quad k=2,3,\cdots,n. \tag{2}$$

Thus, the gray differential equation is defined as

$$x^{(0)}(k)+az^{(1)}(k)=b, k=2,3,\cdots,n, \tag{3}$$

where, a is the development coefficient and b is the grey action. If $\hat{u}=\left[\hat{a},\hat{b}\right]^T$ is a parameter column and there is

$$Y_1=\begin{pmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{pmatrix}, B_1=\begin{pmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{pmatrix}. \tag{4}$$

Then the least squares parameter estimate of equation (3) satisfies

$$\hat{u}=\left(B_1^T B_1\right)^{-1} B_1^T Y_1. \tag{5}$$

If $B_1, Y_1, \hat{u}$ satisfies equations (4) and (5), the corresponding whitening differential equation is

$$\frac{dx^{(1)}(t)}{dt}+ax^{(1)}(t)=b. \tag{6}$$

Let $x^{(0)}(1)=\hat{x}^{(0)}(1)$, get the time response sequence of the GM(1,1) model as

$$\hat{x}^{(1)}(k+1)=\left(x^{(0)}(1)-\frac{b}{a}\right)e^{-ak}+\frac{b}{a},\ k=1,2,\cdots,n-1,\cdots. \tag{7}$$

After restoration, the predicted value $\hat{x}^{(0)}$ of $x^{(0)}$ is

$$\begin{aligned} \hat{x}^{(0)}(k+1) &= \hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k) \\ &= \left(1-e^{a}\right)\left(x^{(0)}(1)-\frac{b}{a}\right)e^{-ak},\ k=1,2,\cdots,n-1,\cdots. \end{aligned} \tag{8}$$

2.2 DGM (1,1) model

Let the non-negative sequence $X^{(0)}$ and the $1-AGO$ sequence $X^{(1)}$ as described in the above definition, called

$$\hat{x}^{(1)}(k+1)=\beta_1\hat{x}^{(1)}(k)+\beta_2, \tag{9}$$

as a DGM (1,1) model, or as a discrete form of the GM (1,1) model [24].

If $\hat{\beta}=\left[\beta_1,\beta_2\right]^T$ is a sequence of parameters; and

$$Y_2=\begin{pmatrix} x^{(1)}(2) \\ x^{(1)}(3) \\ \vdots \\ x^{(1)}(n) \end{pmatrix},\ B_2=\begin{pmatrix} x^{(1)}(1) & 1 \\ x^{(1)}(2) & 1 \\ \vdots & \vdots \\ x^{(1)}(n-1) & 1 \end{pmatrix}. \tag{10}$$

Then the least squares estimation sequence of parameters $\hat{\beta}=\left[\beta_1,\beta_2\right]^T$ of the discrete gray prediction model $\hat{x}^{(1)}(k+1)=\beta_1\hat{x}^{(1)}(k)+\beta_2$ satisfies

$$\hat{\beta}=\left(B_2^T B_2\right)^{-1} B_2^T Y_2. \tag{11}$$

Let $\hat{x}^{(1)}(1)=x^{(0)}(1)$, get the recursive function as

$$\hat{x}^{(1)}(k+1)=\beta_1^k\left(x^{(0)}(1)-\frac{\beta_2}{1-\beta_1}\right)+\frac{\beta_2}{1-\beta_1},k=1,2,\cdots n-1,\cdots. \tag{12}$$

Restore value is

$$\begin{aligned}\hat{x}^{(0)}(k+1)&=\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)\\&=(\beta_1-1)\left(x^{(0)}(1)-\frac{\beta_2}{1-\beta_1}\right)\beta_1^k,k=1,2,\cdots n-1,\cdots.\end{aligned} \tag{13}$$

3 Dynamics of gray action b and construction of GM(1,1,b) model

Regarding the optimization and improvement of the GM(1,1) model, the optimization of the gray action is rare. The gray action is mined from the background value, which is exactly gray, which reflects the relationship of the data sequence. The traditional GM (1,1) model regards the gray action b as an invariant constant, that is, the external disturbance is regarded as constant, so the purpose of the processing is to facilitate the solution of the subsequent model and simplify the calculation of the model. process. However, as can be seen from the literature [25-31], the amount of gray action will change with time and space. If it is regarded as an invariant constant, this will inevitably lead to errors in the GM(1,1) model. The prediction accuracy of the model has a certain impact.

In general, when the original sequence $X^{(0)}$ changes from the first data $x^{(0)}(1)$ through time $t=1,2,\cdots,n$ to the *nth* data $x^{(0)}(n)$, at this stage the environment of the entire system may have changed greatly with time. It can be seen from the gray differential equation $x^{(0)}(k)+az^{(1)}(k)=b,k=2,3,....,n$ of the classical GM (1,1) model that the classical GM (1,1) model lacks dynamic adjustment of the gray action, which causes the model to not dynamically adapt to changes in the system over time. This is also one of the key reasons for the low accuracy of GM(1,1) model fitting and predicting.

Starting from the gray differential equation of the GM(1,1) model, when $k=2,3,....,n$ is considered, the grey action of GM(1,1) model is respectively

$$k=2,\ \ b^{(0)}(1)=x^{(0)}(2)+az^{(1)}(2), \tag{14}$$

$$k=3,\ \ b^{(0)}(2)=x^{(0)}(3)+az^{(1)}(3), \tag{15}$$
$$\vdots$$

$$k=n,\ \ b^{(0)}(n-1)=x^{(0)}(n)+az^{(1)}(n). \tag{16}$$

The gray action sequence $B=\left(b^{(0)}(1),b^{(0)}(2),\cdots,b^{(0)}(n-1)\right)$ can be obtained by the above formulas (14), (15), and (16).Where, a is the development coefficient of the GM (1,1) model, and $z^{(1)}(k)$ is the sequence mean generated of consecutive neighbors of $X^{(1)}$.

Since the grey action sequence $B=\left(b^{(0)}(1),b^{(0)}(2),\cdots,b^{(0)}(n-1)\right)$ is also a time series, so the grey action sequence $B=\left(b^{(0)}(1),b^{(0)}(2),\cdots,b^{(0)}(n-1)\right)$ is used as the raw data sequence to establish a discrete DGM(1,1) model for fitting the grey action sequence and predicting the change trend of the grey action. Let $\hat{b}^{(1)}(1)=b^{(0)}(1)$,then the recursive function is

$$\hat{b}^{(1)}(k+1)=\beta_1^k\left(b^{(0)}(1)-\frac{\beta_2}{1-\beta_1}\right)+\frac{\beta_2}{1-\beta_1}, k=1,2,\cdots,n-2,\cdots. \tag{17}$$

Restore value is

$$\begin{aligned}\hat{b}^{(0)}(k+1)&=\hat{b}^{(1)}(k+1)-\hat{b}^{(1)}(k)\\&=(\beta_1-1)\left(b^{(0)}(1)-\frac{\beta_2}{1-\beta_1}\right)\beta_1^k, k=1,2,\cdots,n-2,\cdots.\end{aligned} \tag{18}$$

The fitted values obtained by the above formula (18) and the predictive values $\hat{b}^{(0)}(k)$ are brought into the above equations (7) and (8) to achieve dynamic change of the gray action.

$$\hat{x}^{(0)}(k+1)=\left(1-e^{a}\right)\left(x^{(0)}(1)-\frac{(\beta_1-1)\left(b^{(0)}(1)-\frac{\beta_2}{1-\beta_1}\right)\beta_1^k}{a}\right)e^{-ak},\ \ k=1,2,...,n,\cdots. \tag{19}$$

The above formula (19) is a GM (1, 1, b) model in which the gray action obtained by combining the GM (1, 1) model and the DGM (1, 1) model can be dynamically changed. The model implements a dynamic change of gray action that allows the model to adapt to changes in the external environment. In theory, the model should have better prediction accuracy than the static gray action.

4 Example application and analysis

In order to verify the validity and practicability of GM(1,1,b) model established in this paper. The consumption level index of rural residents' in China from 2002 to 2013 was obtained from the Statistical Yearbook of China, and used as an example to analyze the data for modeling and prediction analysis. Compare the fitted and predicted accuracy of the proposed GM(1,1,b) improved model and the classical GM(1,1) model. The raw data is shown in the following Table 1

Table 1: China's 2002-2013 Rural Residents Consumption Level Index (1978=100)

Year	2002	2003	2004	2005	2006	2007
	421.10	440.50	457.80	488.90	524.70	570.40
Year	2008	2009	2010	2011	2012	2013
	610.30	666.90	716.00	808.60	880.40	955.80

In this paper, the consumption level index of rural residents from 2002 to 2008 was selected as the fitting modeling data, and the consumption level index of rural residents from 2009 to 2013 was used as the predictive comparison data to establish the classical GM(1,1) model and GM(1,1,b) model respectively. After calculation, the estimated parameter of GM(1,1) model is

$$\hat{u}=\left(\hat{a},\hat{b}\right)^T=(-0.0685,387.7236)^T. \tag{20}$$

From the above estimated parameter column $\hat{u}$, the time response sequence of the GM(1,1) model can be obtained as

$$\hat{x}^{(0)}(k+1)=\left(1-e^{-0.0685}\right)\left(x^{(0)}(1)+\frac{387.7236}{0.0685}\right)e^{0.0685k}, k=1,2,\cdots,n,\cdots. \tag{21}$$

The gray action sequence

$$B=[396.59, 381.1, 383.51, 385.93, 388.38, 390.83, 393.31, 395.8, 398.3, 400.82, 403.36],$$

can be calculated by the development coefficient $\hat{a}$, and the DGM(1,1) model of the gray action amount is established by using the sequence to fit and predict the gray action amount. After calculation, the estimated parameters of the DGM(1,1) model is

$$\hat{\beta}=\left(\beta_1,\beta_2\right)^T=\left(1.0063,378.585\right)^T. \tag{22}$$

From the estimated parameter column $\hat{\beta}$ of the above DGM(1,1) model, the DGM(1,1) model recursive function can be calculated as

$$\hat{b}^{(1)}\left(k+1\right)=1.0063^k\left(b^{(0)}\left(1\right)-\frac{378.585}{1-1.0063}\right)+\frac{378.585}{1-1.0063},k=1,2,\cdots,n-2,\cdots. \tag{23}$$

Restore value is

$$\hat{b}^{(0)}\left(k+1\right)=\left(1.0063-1\right)\left(b^{(0)}\left(1\right)-\frac{378.585}{1-1.0063}\right)1.0063^k,k=1,2,\cdots,n-2,\cdots. \tag{24}$$

Bring the equation (24) into equation (21) to obtain the GM (1,1,b) model time response sequence as

$$\hat{x}^{(0)}\left(k+1\right)=\left(1-e^{-0.0685}\right)\left(x^{(0)}\left(1\right)+\frac{\left(1.0063-1\right)\left(b^{(0)}\left(2\right)-\frac{378.585}{1-1.0063}\right)1.0063^k}{0.0685}\right)e^{0.0685k},\quad k=1,2,\ldots,n,\cdots \tag{25}$$

In order to compare and analyze the GM (1,1) model and the GM (1,1,b) model, $MPAE_{pred}$ and $MAPE_{tol}$ are used as evaluation indicators of the model. Where $MPAE_{pred}$ is the mean absolute percentage error of the extrapolated predicted value and $MAPE_{tol}$ is the total mean absolute percentage error. The calculation formula are as follows

$$MAPE_{tol}=\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x^{(0)}\left(k\right)-\hat{x}^{(0)}\left(k\right)}{x^{(0)}\left(k\right)}\right|\times 100\%, \tag{26}$$

$$MPAE_{pred}=\frac{1}{n-N}\sum_{k=N+1}^{n}\left|\frac{x^{(0)}\left(k\right)-\hat{x}^{(0)}\left(k\right)}{x^{(0)}\left(k\right)}\right|\times 100\%. \tag{27}$$

Where n represents the total number of sample data and N represents the number of data used to fit the model.

The fitted and raw data and relative errors of the GM(1,1) and GM(1,1,b) models are shown in Table 2.The five-steps extrapolated prediction and prediction errors for the two models are shown in Table 3 below.

Table 2: GM(1,1) and GM(1,1,b) model fitted and relative error comparison (%)

Year	Raw data	GM(1,1)	Relative error	GM(1,1,b)	Relative error
2002	421.10	421.10	0.00	421.10	0.00
2003	440.50	431.14	2.12	440.32	0.04
2004	457.80	461.69	0.85	438.30	4.26
2005	488.90	494.40	1.13	494.57	1.16
2006	524.70	529.43	0.90	535.24	2.01
2007	570.40	566.95	0.61	579.07	1.52
2008	610.30	607.12	0.52	626.31	2.62

Table 3: Comparison of GM(1,1) and GM(1,1,b) model prediction and prediction errors (%)

Year	Raw data	GM(1,1)	Relative error	GM(1,1,b)	Relative error
2009	666.90	650.14	2.51	677.21	1.55
2010	716.00	696.20	2.77	732.05	2.24
2011	808.60	745.53	7.80	791.15	2.16
2012	880.40	798.36	9.32	854.83	2.90
2013	955.80	854.93	10.55	923.44	3.39

$MPAE_{pred}$	**6.5902**	**2.4471**
$MAPE_{tol}$	**3.2566**	**1.9873**

It can be seen from Table 3, $MPAE_{pred}$ and $MAPE_{tol}$ of the GM (1,1,b) model are only 2.4471% and 1.9873%, while $MPAE_{pred}$ and $MAPE_{tol}$ of the GM (1,1) model are as high as 6.5902% and 3.2566%, which is obviously higher than the GM (1,1,b) model proposed in this paper.

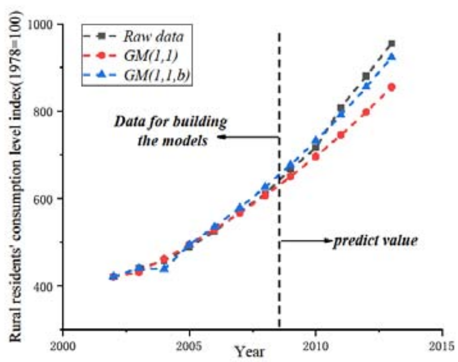

Fig.1 Comparison of fitted and predicted values of GM(1,1) and GM(1,1,b) models

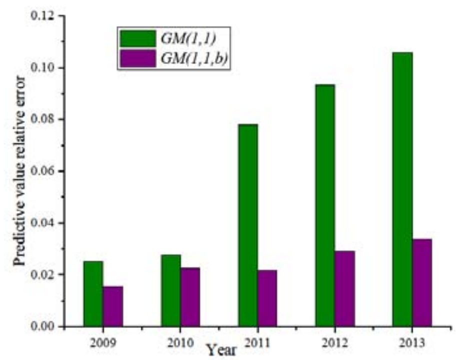

Fig.2 Prediction error of GM(1,1) and GM(1,1,b) models

Fig.1 visually compares the fitted and predictive values of the GM (1,1) and GM (1,1,b) models, and Fig.2 compares the prediction error of the GM (1,1) and GM (1,1,b) models. The above two figures clearly show that the GM(1,1,b) model proposed in this paper is superior to the classical GM(1,1) model.

5 Conclusion

This paper starts from the classical GM(1,1) model, aiming at the shortcoming that GM(1,1) model ignores the change of external environment and regards grey action as a constant. By studying the dynamic change property of grey action, the DGM(1,1) model was used to simulate and predict the grey action. Based on this, a novel GM(1,1,b) model with dynamic gray action can be proposed and applied to the prediction of rural residents' consumption level index. The results show that the proposed GM(1,1,b) model has higher prediction accuracy than the classical GM(1,1) model, which verifies the practicability and effectiveness of the GM(1,1,b) model. At the same time, it also provides relevant theoretical basis for the state to adjust the consumption index of rural residents in China.

6 References

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