Original Research Article

Hypergeometric Functions on Cumulative Distribution Function

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ABSTRACT

Exponential functions have been extended to Hypergeometric functions. There are many functions which can be expressed in hypergeometric function by using its analytic properties. In this paper, we will apply a unified approach to the probability density function and corresponding cumulative distribution function of the noncentral chi square variate to extract and derive hypergeometric functions.

Key words: Generalized hypergeometric functions; Cumulative distribution theory; chi-square Distribution on Non-centrality Parameter.

I) INTRODUCTION

Higher-order transcendental functions are generalized from hypergeometric functions. Hypergeometric functions are special function which represents a series whose coefficients satisfy many recursion properties. These functions are applied in different subjects and ubiquitous in mathematical physics and also in computers as Maple and Mathematica. They can also give explicit solutions to problems in economics having dynamic aspects.

The purpose of this paper is to understand importance of hypergeometric function in different fields and initiating economists to the large class of hypergeometric functions which are generalized from transcendental function. The paper is organized in following way. In Section II, the generalized hypergeometric series is defined with some of its properties analytical properties and special cases. In Sections 3 and 4, hypergeometric function and Kummer's confluent hypergeometric function are discussed in detail which will be directly used in our results. In Section 5, main result is proved where we derive the exact cumulative distribution function of the noncentral chi sqaure variate.

An appendix is attached which summarizes notational abbreviations and function names.

The paper is introductory in nature presenting results reduced from general formulae. Much of the content is new unpublished formulae which are integrated with the mathematical literature. The hypergeometric functions are classified from Carlson (1976), Jahnke and Emde (1945) For integrals involving such functions, see Gradshteyn and Ryzhik (1994), Choi, Hasanov and Turaev(2012), Exton(1973), Joshi and Pandey(2013), Saran(1955), Seth and Sindhu(2005), Usha and Shoukat(2012). For the theory, we referred to Whittaker and Watson (1927), Erdélyi (1953, 1955) for a more comprehensive proof refer to Anderson(1984), Luke (1969), Olver (1974), Mathai (1993). Statistics/econometric theories was explained extensively in Cox and Hinkley(1974), Craig(1936), Feller(1971), Hardle and Linton(1994), Muellbauer(1983). The subject was developed by the three volumes edited by Erdélyi (1953, 1955). Further applications in statistics/econometrics and economic theory are suggested throughout. Here, Hypergeometric function is applied effectively to Distribution Theory.

II) THE GENERALIZED HYPERGEOMETRIC SERIES

Before introducing the hypergeometric function, we define the Pochhammer's symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{k=0}^{n-1}(a+k)$$

$$= (a)\,(a+1)..(a+n-1)$$

$$= \binom{a}{n}\; n!\; (-1)^n. \qquad \ldots..(1)$$

If we substitute $n \to k-n$ in eqn 1 we are left with

$$(a)_{k-n} = \frac{\Gamma(a+k-n)}{\Gamma(a)} \qquad \ldots..(2)$$

If we multiply and divide eqn (2) by $\Gamma(a+k)$, then we get

$$(a)_{k-n} = \frac{(a)_k}{(a+k-1)(a+k-2)\ldots\ldots\ldots\ldots(a+k-n)} = \frac{(-1)^n\,(a)_k}{(1-a-k)_n}. \qquad \ldots\ldots(3)$$

So if we set

$$k=0,\;\; (a)_{-n} = \frac{(-1)^{-n}}{(1-a)_n},$$

The general Hypergeometric function is given by

$${}_pF_q(a_1 \ldots\ldots a_p;\; b_1 \ldots\ldots b_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{p}(a_k)_n}{\prod_{k=1}^{q}(b_k)_n}\,\frac{z^n}{n!}, \quad = \sum_{n=0}^{\infty}\frac{(a_1)_j \ldots\ldots\ldots..(a_p)_j}{(b_1)_j \ldots\ldots\ldots..(b_q)_j}\,\frac{z^n}{n!}. \qquad \ldots..(4)$$

The $a_1, a_2 \ldots\ldots\ldots.., a_p$ and $b_1, b_2 \ldots\ldots\ldots.., b_q$ are called the numerator and denominator parameters respectively and z is called the argument. We deal within this paper will concern eqn (4) but for the most part we will deal with specific cases of the general hypergeometric function. For instance,

If we let $p = 1$ and $q = 0$ and by using eqn (1)

$${}_1F_0(a;z) = \sum_{j=0}^{\infty}(a)_j\,\frac{z^j}{j!} = 1 + az + \frac{a(a+1)}{2!}\,z^2 + \frac{a(a+1)(a+2)}{3!}\,z^3 + \cdots.$$

$$= \sum_{j=0}^{\infty}\binom{-a}{j}\,(-z)^j = (1-z)^{-\alpha}, \qquad \ldots\ldots(5)$$

which is the binomial expansion.

If we take $p = q = 0$ in eqn (4) we get

$${}_0F_0(-;-;x) = \sum_{j=0}^{\infty}\frac{z^j}{j!} = e^z, \qquad \ldots\ldots(6)$$

Some instant result follows from eqn (4), when one of the parameter of a_r is non negative integer, then

$${}_pF_q(0, a_2, a_3 \dots\dots\dots\dots. a_p;\ c_1 \dots\dots\dots\dots, c_q; z) \qquad \dots\dots(7)$$

Also,

$${}_pF_q(a_1, a_2 \dots\dots\dots\dots. a_p;\ c_1 \dots\dots\dots\dots, c_q; 0) \equiv 1, \qquad \dots\dots(8)$$

And interchanging elements separated by commas is feasible as multiplication is commutative.

$${}_pF_q(a_1, \dots\dots, a_r, \dots\dots. a_s \dots;\ b_1 \dots\dots, b_k, \dots\dots, b_r \dots..; z) \qquad \dots.(9)$$

$$\equiv\ {}_pF_q(a_1, \dots\dots, a_r, \dots\dots. a_s \dots;\ b_1 \dots\dots, b_r, \dots\dots, b_{k\dots\dots\dots}; z)$$

$$\equiv\ {}_pF_q(a_1, \dots\dots, a_s, \dots\dots. a_r \dots;\ b_1 \dots\dots, b_r, \dots\dots, b_{k\dots\dots\dots}; z)$$

$$\equiv\ {}_pF_q(a_1, \dots\dots, a_s, \dots\dots. a_r \dots;\ b_1 \dots\dots, b_k, \dots\dots, b_{r\dots\dots\dots}; z).$$

But, interchanging of semicolons (i.e, between a_i and b_j) is not allowed as division is not commutative.

Also, if $\exists\ a_r = b_m$, then

$${}_{p+1}F_{q+1}(a_1, \dots\dots\dots\dots. a_p, a_{p+1};\ c_1 \dots\dots\dots\dots, c_q, a_{p+1}; z)$$

$$\equiv\ {}_pF_q(a_1, \dots\dots\dots\dots. a_p;\ c_1 \dots\dots\dots\dots, c_q; z). \qquad \dots\dots(10)$$

An important property that we will need to use is the convergence criteria of the hypergeometric functions depending on the values of p and q. The radius of convergence of a series of variable $z$is defined as a value r_c such that the series converges if $|z - d_c| < r_c$ and diverges if $|z - d_c| > r_c$, where d_c, in the case 0, is the centre of the disc convergence. For hypergeometric function, provided a_j and b_j are not non negative integers for any j, the relevant convergence criteria stated below can be derived using the ratio test, which determines the absolute convergence of the series using the limit of the ratio of two consective terms.

a) If $p \le q$, then the ratio of coefficients of z^k in the taylor series of the hypergeometric function ${}_pF_q$ tends to 0 as $k \to \infty$; so the radius of convergence is ∞, so that the series converges for all values of $|z|$. Hence ${}_pF_q$ is entire. In particular, the radius of convergence for ${}_0F_1$ and ${}_1F_1$ is ∞.
b) If $p = q + 1$, the ratio of coefficients of z^k tends to 1 as $k \to \infty$, so the radius of convergence is 1, so that series converges only if $|z| < 1$. In particular, the radius of convergence for ${}_2F_1$ is 1.
c) If $p > q + 1$, the ratio of coefficients of z^k tends to ∞ as $k \to \infty$, so the radius of convergence is 0, so that the series does not converge for any value of $|z|$.

We will seek approximation to the relevant hypergeometric function for $|z|$ within radii of convergence. For $p = q + 1$, as given by Luke (1975), there is a restriction for convergence on the unit disc, the series only converges absolutely at $|z| = 1$ if

$$Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0 \qquad \dots\dots\dots(11)$$

So that the selection of values for a_j and b_j must reflect that.

III) THE HYPERGEOMETRIC FUNCTION

The Guass hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as

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$$ {}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}, \quad \ldots\ldots (12) $$

$$ = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2} + \ \ldots \ , $$

where z is in the radius of convergence of the series $|z| < 1$.

The first result is a representation of ${}_2F_1$ in terms of beta integral over [0,1]

$$ B(a,b) = \int_0^1 t^{\alpha-1} (1-t)^{b-1} \, dt \quad ..(13) $$

In terms of more familiar quantities, the hypergeometric function ${}_2F_1$ is

$$ {}_2F_1(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt \quad ..(14) $$

This expression can be obtained by expanding $(1-tz)^{-a}$ by binomial theorem and integrating termwise.

If we substitute the variable $y = tp$, this will give,

$$ \int_0^1 y^{b-1} (p-y)^{c-b-1} (p-xy)^{-a} \, dy \ = p^{c-a-1} B(b, c-b) \ {}_2F_1(a,b;c;x), \quad \ldots..(15) $$

The special case $c = b + 1$ produces,

$$ \int_0^p y^{b-1} (p-xy)^{-a} \, dy = \frac{1}{b} p^{b-a} \ {}_2F_1(a,b;b+1;x) \quad \ldots(16) $$

where we applied $B(b,1) = \frac{1}{b}$

To eliminate p^{-a}, we put $x = -pr$ to obtain,

$$ \int_0^p y^{b-1} \ (1+ry)^{-a} \, dy = \frac{1}{b} p^b \ {}_2F_1(a,b;b+1;-rp). \quad \ldots(17) $$

Also,

$$ \int_0^z x^a \ (1 + \alpha x)^b \, dx = \sum_{i=0}^{\infty} \binom{b}{i} \int_0^z x^a (\alpha x)^i \, dx \ , \quad \ldots.(18) $$

$$ = \frac{z^{a+1}}{a+1} \ {}_2F_1(-b, a+1; a+2; -\alpha z), $$

where $Re(a+1) \, \epsilon \, R_+$

This equation can also be obtained by using binomial theorem and integrating term by term. For $b \in N \, U \, \{0\}$, the series is finite with $b+1$ terms in it, and it can also be derived by successive integration (by parts),

$$ \log(1+z) = z \ {}_2F_1(1,1;2;-z) \quad = (-1) \sum_{j=0}^{\infty} \frac{1}{j+1} (-z)^{j+1}. \quad \ldots..(19) $$

If $z = 1$, on rearranging the above equation in such a way that negative term follows every two consecutive positive term then we get $\frac{3 \log 2}{2}$ for $\log 2$. See Wittaker and Watson(1927) for proof. This is the expansion of log function in infinite series. Series is absolutely convergent if $|z| < 1$ and conditionally convergent if $z = 1$,

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$(1+z)^a \equiv {}_1F_0(-a;-z) \equiv {}_2F_1(-a,b;b;-z),$ ... (20)

where b is arbitrary.

$(1+z)^a$ is infinite when $z=-1$ and $\alpha \in R^-$ or $|z| \to \infty$ and $\alpha \in R^+$.

So with these two exceptions, series expansion will give a finite value. For convergence of the hypergeometric series, we must have$1<|z|<\propto$. General formula for analytic continuation of Gauss Series is given in the volumes of Erdelyi (1955)

IV) KUMMER'S CONFLUENT HYPERGEOMETRIC FUNCTION

The confluent hypergeometric function denoted by ${}_1F_1(a;c;z)$ is defined by,

$\Phi(a;c;z) = {}_1F_1(a;c;z)$

$= \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}, \; c \neq \{0,-1,-2,\dots\}$ .... (21)

$= \lim_{b\to\infty} {}_2F_1(a,b;c;\frac{z}{b}).$

Following two formulas of Kummer are useful for our results.

${}_1F_1(a;c;z) = e^z \; {}_1F_1(c-a;c;-z)$ ..... (22)

${}_1F_1(a;2a;2z) = e^z \; {}_0F_1\left(\;;a+\frac{1}{2};\frac{z^2}{4}\right).$ .... (23)

$e^z \equiv {}_0F_0(z) \equiv {}_1F_1(a;a;z),$

where a is an arbitrary. ....(24)

The exponential function is the elementary example of the hypergeometric series. All the functions studied here can be considered as generalization of elementary transcendental function; e^z. Further, special cases arise when compared with Poisson process as discussed in Hardle and Linton (1994). We also have integral representation,

$\Phi(a,c.z)$

$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1} dt,$ ... (25)

$where \; Re(c) > Re(a) > 0$

$$I_v(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \; \Gamma(j+v-1)} \left(\frac{z}{2}\right)^{2j+v} \quad = \frac{\left(z/2\right)^v}{\Gamma(v+1)} \; {}_0F_1\left(\;;v+1;-\frac{z^2}{4}\right),$$

$= \frac{\left(z/2\right)^v}{\Gamma(v+1)} e^{-z} \; {}_1F_1\left(v+\frac{1}{2};2v+1;2z\right),$ $\quad -2v \notin \boldsymbol{\mathcal{N}}$ ..... (26)

where $I_v(z)$ is the modified Bessel function of the first kind with order v.

The incomplete gamma functions arise from Euler's integral for the gamma function,

$$\Gamma(a) = \int_0^{\infty} e^{-t} \, t^{a-1} \, dt.$$

By decomposing it into an integral from 0 to ∞,

$$\gamma(a,z) = \int_0^z e^{-t}\, t^{a-1}\, dt\,, \qquad Re(a) > 0$$

$$= z^a \sum_{j=0}^{\infty} \frac{(-z)^j}{j!(j+a)} = \frac{z^a}{a}\ {}_1F_1(a; 1+a; -z), \qquad \ldots\ldots(27)$$

$$Re\,(a) > 0$$

This derivation shows that integrals of elementary function leads to a geometric function.

Other special case is standard normal cumulative distribution function.

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-t^2/2}\, dt = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^z \sum_{j=0}^{\infty} \frac{\left(-t^2/2\right)^j}{j!}\, dt, = \frac{1}{2} + \frac{z}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{\left(-z^2/2\right)^j}{j!\,(2j+1)}$$

$$= \frac{1}{2} + \frac{z}{\sqrt{2\pi}}\ {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{-z^2}{2}\right), \qquad \ldots..(28)$$

$$= \frac{1}{2} + \frac{z}{2\sqrt{\pi}}\, \gamma\left(\frac{1}{2}, \frac{z^2}{2}\right),$$

$$= \frac{1}{2} + \frac{sgn\,(z)}{2\sqrt{\pi}}\, \gamma\left(\frac{1}{2}, \frac{z^2}{2}\right), \qquad \ldots\ldots(29)$$

where $sgn\ z$ is signum function, $sgn\ z = \begin{cases} -1, & if\ z < 0 \\ 0, & if\ z = 0 \\ 1, & if\ z > 0 \end{cases}$

This is a case incomplete gamma function is used to represent cumulative distribution function (cdf) of the standard normal distribution. Gamma distribution also have exponential pdf with negative value which is used in consumer theory by Delgado and Dumas(1992).

As we know Kummer's function satisfies a basic relation,

$${}_1F_1(a; c; z) = e^z\ \ {}_1F_1(c-a; c; -z).$$

This equation was derived by Sentana(1995) with help of Leibniz formula of fractional integral. In view this eqn (28) can be rewritten as

$$\Phi(z) = \frac{1}{2} + \frac{z}{\sqrt{2\pi}}\, e^{-z^2/2}\ \ {}_1F_1\left(1; \frac{3}{2}; \frac{z^2}{2}\right) \qquad = \frac{1}{2} + z\, f(z)\ {}_1F_1\left(1; \frac{3}{2}; \frac{z^2}{2}\right), \qquad \ldots..(30)$$

where $f(z) = \frac{1}{2}\exp\left(-\frac{1}{2}\, z^2\right),\ -\infty < z < \infty$ is standard normal density function.

V) APPLICATION IN DISTRIBUTION THEORY

If λ^2 follows a non central Chisquared distribution with d degree of freedom and non central parameter $\lambda^2, 0 \le \lambda \le \infty$, then

$$\gamma^2 \sim \chi_p^2(\lambda^2) \qquad \text{or} \qquad \gamma \sim \chi_p(\lambda) \qquad \ldots\ldots(31)$$

In most of the following notation p is fixed and will not be explicitly stated in the notation. Also, it is useful to set $\upsilon = \frac{p-2}{2}$.

The formula for the pdf involves Bessel Function $I_\upsilon(x)$ which can be limiting behavior,

$I_v(x) \sim \left(\frac{x}{2}\right)^v / \Gamma(v+1)$ as $x \to 0$

Consider,

$$h_{2v;2\delta}(u) = 2^{-v} e^{-(u+2\delta)/_2} \sum_{k\geq 0} \frac{u^{2v/2+k-1}\left(\frac{\delta}{2}\right)^k}{k!\ \Gamma(v+j)} \quad \text{....}(32)$$

$$= \frac{1}{2}\left(\frac{2\delta}{u}\right)^{\frac{1-v}{2}} e^{-\delta-\frac{u}{2}} I_v(\sqrt{2u\delta})$$

$$= \left(\frac{u}{2}\right)^v \frac{1}{u\,\Gamma(v)} e^{-\delta-\frac{u}{2}}\ {}_0F_1\left(v; \frac{u\delta}{2}\right)$$

$$= \left(\frac{u}{2}\right)^v \frac{1}{u\,\Gamma(v)} e^{-\delta-\frac{u}{2}-\sqrt{2u\delta}}\ {}_1F_1\left(v-\frac{1}{2}; 2v-1; \sqrt{8u\delta}\right), 2v \neq 1$$

Cox and Minkley(1974) had given the definition of equ (32) and further equations follows from eqn(26). When $2\delta = 0$, the above distribution reduces to

$$h_{\ 2v;0}(u) = \frac{u^{2v/2-1} e^{-u/2}}{2^v\ \Gamma(v)} \quad \text{......}(33)$$

Equation (32) can be rewritten as,

$$h_{\ 2v;2\delta}(u) = e^{-\delta} \sum_{k\geq 0} \frac{\delta^k}{k!} \frac{\left(\frac{u}{2}\right)^{k+v} e^{-u/2}}{u\,\Gamma(k+v)} \quad = e^{-\delta} \sum_{k\geq 0} \frac{\delta^k}{k!} h_{\ 2v+2k;0}(u), \quad \text{......}(34)$$

with the weights $e^{-\delta}\ \frac{\delta^k}{k!}$ is from Poisson density.

We will get corresponding cdf by term wise integration of equation (32) as,

$$H_{2v;2\delta}(u) = \int_0^u h_{\ 2v;2\delta}(x)dx = e^{-\delta} \sum_{k\geq 0} \frac{\delta^k}{\Gamma(k+v).\,k!} \int_0^{u/2} x^{k+v-1} e^{-x}\ dx,$$

$$= e^{-\delta} \sum_{k\geq 0} \frac{\delta^k}{k!\,\Gamma(k+v)}\ \gamma\left(k+v, {}^{u}/_{2}\right). \quad \text{....}(35)$$

We must recall that $\gamma(k+v,\infty) \equiv \Gamma(k+v)$, so if $u \to \infty$, then $H_{2v;2\delta}(u) \equiv 1$.

In addition, if $V \sim \chi^2_{2n}$ is independent from U, then $W \equiv \frac{mU}{nV} \sim F_{2m,2n}(2\lambda)$, with noncentral F distribution with $2m$ degree of freedom and $2n$ in denominator with noncentral parameter 2λ.

$$f(w; 2m, 2n, 2\lambda) = e^{-\lambda} \sum_{r\geq 0} \frac{\lambda^r}{r!} \left(\frac{m}{n}\right)^{m+r} \frac{w^{m+r-1}}{\left(1+\frac{mw}{n}\right)^{m+n+r}} \frac{\Gamma(m+n+r)}{\Gamma(m+r)\Gamma(n)} \quad \text{.....}(36)$$

$$= \frac{e^{-\lambda}}{w\,\beta(m,n)} \left(\frac{mw}{n}\right)^m \left(1+\frac{mw}{n}\right)^{-m-n}\ {}_1F_1\left(m+n; m; \frac{mw\lambda}{n+mw}\right)$$

$$= e^{\lambda\left(\frac{-2n+mw}{2n+2mw}\right)} \left(\frac{1}{2\lambda}\right)^{m+n-\frac{1}{2}} \left(\frac{\pi}{mw}\right)^{1/2} \left(\frac{n}{mw}\right)^n \frac{\Gamma(2m+2n)}{\Gamma(m)\Gamma(n)} I_{m+n-\frac{1}{2}}\left(\frac{mw\lambda}{2n+2mw}\right)$$

This gives the first definition and next one follows from using beta function where $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. The very last line follows from eqn (18)

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The cumulative distribution function may be found by

$$F_{2m,2n;2\lambda}(u) = \int_0^u x^k f_{2m,2n;2\lambda}(x)dx$$

$$= e^{-\lambda} \sum_{r\geq 0} \frac{\lambda^r}{r!} \left(\frac{m}{n}\right)^{m+r} \frac{\Gamma(m+n+r)}{\Gamma(m+r)\Gamma(n)} \int_0^u \frac{x^{m+r+k-1}}{\left(1+\frac{mx}{n}\right)^{m+n+r}}\, dx\,, \qquad \ldots\ldots(37)$$

$$= e^{-\lambda} \sum_{r\geq 0} \frac{\lambda^r}{r!} \frac{\beta_s(m+r,n)}{\beta(m+r,n)}\,,$$

$$= e^{-\lambda} \sum_{r\geq 0} \frac{\lambda^r}{r!} I_s(m+r,n),$$

$$= e^{-\lambda} \sum_{r\geq 0} \frac{\lambda^r}{r!} \frac{s^{m+r}}{(m+r)\beta(m+r,n)}\ {}_2F_1(m+r, 1-n; m+r+1; s) \qquad \ldots\ldots(38)$$

with $s = \frac{\frac{mu}{n}}{1+\frac{mu}{n}}$,

Here we have applied the definition of Incomplete beta function and in the last step we have applied the eqn (18)

CONCLUSION & EXTENSIONS

Mathematical extension for the content of this paper is possible in at least three ways. First of all, Meijer's G and Fox's functions are special functions of generalized hypergeometric function. Fox's H function is especially convenient as its analytic manipulation of asymptotic and power series is easy.In second place, we had discussed hypergeometric function of one argument but it can be extended to multiple series with more than one variable. We can rewrite hypergeometric function for two variables, instead of single variable functions. Finally, we make an assumption that z is a scalar but we can consider it to be non scalar and pursue accordingly. We can define hypergeometric functions even if we have the argument as a square matrix. If we define a matrix function whose output is a scalar, we get the type of hypergeometric functions used in multivariate distribution theory.

Hypergeometric functions are able to occur in fractional calculus [e.g. Cox and Hinkley (1974)]. The nature of implementation of these functions is in data (e.g. fractionally integrated) of time series and further area of economics. Given the determination of unemployment and price rises, this relation seems to have significance for economist. We had derived the exact cumulative distribution function by using Bessel function, incomplete gamma, Gauss hypergeometric function or other relevant functions.

A final statement on hypergeometric functions. They have now become so important in many areas of applied mathematics that they can be found in many computer packages, including ones allowing symbolic manipulations like Maple and Mathematica. A major advantage they have is their parsimonious generality, and their ability to give explicit answers to problems. It is hoped that this paper has made the case for their potential in quantitative economics.

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APPENDIX : SPECIAL NOTATIONAL AND FUNCTIONS

≡ : identity; when variables or functions are equivalent for all defined values of the parameters and the arguments.

= : equality; when two expressions are not equivalent, but have equal principal values or are equal for a certain range of parameter or argument values.

~ : distributed as.

C , N , R, Z : the sets of complex, natural, real, and integer numbers, respectively.

pdf: probability density function.

cdf: cumulative distribution function.

$\boldsymbol{i} = \sqrt{-\mathbf{1}}$: the imaginary unit.

|z| : modulus (or absolute value) of z.

B(x, y) = Γ(x)Γ(y)/Γ(x + y) : Beta function.

Γ(v) : gamma function.

$\binom{v}{j} = \Gamma(v+1) / [\Gamma(v+1-j)\, j!]$: Binomial Coefficients.

$(\mathbf{v})_{\mathbf{j}} \equiv v(v+1) \ldots\ldots\ldots\ldots (v+j-1) = \Gamma(v+j)/\Gamma(v)$: Pochhammer's symbol.

γ(v , z), Γ(v, z) : incomplete gamma functions.

${}_p\boldsymbol{F}_q(\boldsymbol{a}_1 \ldots . \boldsymbol{a}_p;\ \boldsymbol{c}_1, \ldots\ldots, \boldsymbol{c}_q;\ \boldsymbol{z})$: generalized hypergeometric series.

${}_2\boldsymbol{F}_1(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c};\ \boldsymbol{z})\ \boldsymbol{or}\ \boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{c};\ \boldsymbol{z})$: Gauss hypergeometric series (the hypergeometric function).

${}_1\boldsymbol{F}_1(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{z})\ \boldsymbol{or}\ \boldsymbol{M}(\boldsymbol{a}; \boldsymbol{b}; \boldsymbol{z})$: Kummer's function (confluent/degenerate hypergeometric function).

φ(z) , Φ(z) : standard Normal pdf and cdf respectively.

int(.) : integer part of the argument.

$\boldsymbol{I}_v(\boldsymbol{z})$: modified Bessel function of the first kind of order v.

sgn(z) : signum (sign) function of z; returning ±1 for z ∈ R±, or 0 for z = 0.

UNDER PEER REVIEW