Original Research Article

Bayesian Estimation of 3-Component Mixture of the Inverted Exponential Distributions

Abstract

This paper is about studying a 3-component mixture of the Inverted Exponential distributions under Bayesian view point. The type-I right censored sampling scheme is considered because of its extensive use in reliability theory and survival analysis. The expressions for the Bayes estimators and their posterior risks are derived under different loss scenarios. In case, no or little prior information is available, elicitation of hyper parameters is given. In order to study numerically, the execution of the Bayes estimators under different loss functions, their statistical properties have been simulated for different sample sizes and test termination times. A real life data example is given to illustrate the study. Graphical representation of the simulation analysis results is also given to study the properties of the Bayes estimators.

Keywords: Bayes Estimators, Censoring, Informative prior, Loss Functions, Posterior Risks.

1. Introduction

The exponential distribution is most commonly used in reliability studies but its suitability is restricted to its constant hazard rates. When the failure rate is monotonically increasing or decreasing, the two parameter weibull and the Gamma distributions are appropriate for analyzing the life time data. Recently two new distributions have been introduced the Generalized exponential(two parameter) and the Inverted exponential(one parameter) distributions. When skewed distributions is needed, then the Generalized exponential distribution can be used more effectively. Gupta(1999) described several properties of the two parameter Generalized exponential distribution. Dey (2007) investigated the Inverted exponential as a lifetime model from a Bayesian viewpoint. Prakash (2012) examined the properties of Bayes estimators of the parameters, reliability function and hazard rate under the symmetric and asymmetric loss functions for the Inverted exponential distribution.

Mixtures models play an important role in many applicable fields such as medicine, psychology, cluster analysis, life testing and reliability analysis. A finite mixture of some suitable probability distribution is recommended to study a population that is supposed to comprise a number of subpopulations mixing in an unknown proportion. However, several researchers are interested with different parameters of mixture distributions. The analysis of mixture models under Bayesian framework has developed a significant interest among statisticians. Majeed (2012) described the Bayesian anlysis of 2-component mixture of Inverted exponential distribution under quadratic loss function. Ali (2015) described the 2-component mixture of the inverse Rayleigh distributions under Bayesian framework. Sultana and [;p,Aslam (2016) presented 3-component mixture of Inverse Rayleigh distributions, properties and estimation under the Bayesian framework.

Several types of data are encountered in everyday life, regarding simple data, grouped data, truncated data, censored data and progressively censored data. Censoring is an inevitable part of the lifetime data. A valuable account of censoring is given in Gijbles (2010) and Kalbfleisch and Prentice (2011). There are different sorts of censoring schemes, including right, left and interval censoring, single or multiple censoring and type-1 and type-II censoring.

Inspired by above mentioned applications of mixture models, we intend to study Bayesian analysis of a 3-component mixture of the Inverted Exponential distributions with unknown mixing proportions. The parameters of component distributions are assumed to be unknown. Three different priors and three different loss functions are used for the Bayesian analysis. Moreover, an ordinary type-I right censored sampling scheme is used.

The rest of the paper is organized as follows. In section 2, 3-Component mixture of Inverted Exponential(IE) distribution is presented. The likelihood function of the mixture model is defined in section 3. Posterior distributions using the uniform prior (UP), the Jeffreys' prior (JP) and the inverse Gamma prior (IGP) are derived in section 4. The BEs and PRs are derived using the UP, the JP and the IGP under squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF) are presented in section 5, 6 and 7, respectively. The limiting expressions are discussed in section 8. The simulation study for the mixture model is given in section 9. A real life data application is given in section 10. This article concludes with a brief discussion in section 11.

2. 3-Component mixture of the Inverted Exponential (IE) Distributions

The probability density function (p.d.f) and the cumulative distribution function (c.d.f) of the IE distribution for a random variable X are given by:

$$f_m\left(x;\theta_m\right)=\frac{1}{x^2\theta_m}\exp\left[-\left(\frac{1}{x\theta_m}\right)\right],\ x>0,\theta_m>0,m=1,2,3. \tag{2.1}$$

$$F_m\left(x\right)=\exp\left[-\left(\frac{1}{x\theta_m}\right)\right],\ m=1,2,3. \tag{2.2}$$

A finite 3-component mixture model with the unknown mixing proportions $p_1$and p_2 is :

$$f(x)=p_1f_1(x)+p_2f_2(x)+(1-p_1-p_2)f_3(x),p_1,p_2\geq 0,p_1+p_2\leq 1 \tag{2.3}$$

(2.4)

$$f\left(x,\theta_1,\theta_2,\theta_3,p_1,p_2\right)=p_1\left(\frac{1}{x^2\theta_1}\right)\exp\left[-\left(\frac{1}{x\theta_1}\right)\right]+p_2\left(\frac{1}{x^2\theta_2}\right)\exp\left[-\left(\frac{1}{x\theta_2}\right)\right]$$
$$+\left(1-p_1-p_2\right)\left(\frac{1}{x^2\theta_3}\right)\exp\left[-\left(\frac{1}{x\theta_3}\right)\right];p_1,p_2\geq 0,p_1+p_2\leq 1$$

While the c.d.f of 3-component mixture model is:

(2.5) $$F\left(x\right)=p_1F_1\left(x\right)+p_2F_2\left(x\right)+\left(1-p_1-p_2\right)F_3\left(x\right)$$

(2.6)
$$F\left(x\right)=p_1\exp\left[-\left(\frac{1}{x\theta_1}\right)\right]+p_2\exp\left[-\left(\frac{1}{x\theta_2}\right)\right]+\left(1-p_1-p_2\right)\exp\left[-\left(\frac{1}{x\theta_3}\right)\right]$$

3. The Likelihood Function

Suppose 'n' units from the 3-component mixture of Inverted Exponential distributions are used in a life testing experiment with fixed test termination time t. Let 'r' units out of 'n' units failed until fixed test termination time't' and the remaining (n-r) units are still working. According to Mendenhall and Hader (1958), there are many practical situations in which the failed objects can be pointed out easily as subset of subpopulation-I, subpopulation-II or subpopulation-III. Out of 'r' units, supposer_1, r_2 and $r_3$units belong to subpopulation-I, subpopulation-II or subpopulation-III respectively and such that$r = r_1 + r_2 + r_3$. Now we definex_{lk},$0 < x_{lk} < t$ be the failure time of k^{th}unit belonging to the l^{th} subpopulation, where $l = 1, 2, 3$ and $k = 1, 2, ..., r_l$. For a 3-component mixture model, the likelihood function can be written as

(3.1) $$L\left(\phi\mid\mathbf{x}\right)\propto\left\{\prod_{k=1}^{r_1}p_1f_1\left(x_{1k}\right)\right\}\left\{\prod_{k=1}^{r_2}p_2f_2\left(x_{2k}\right)\right\}\left\{\prod_{k=1}^{r_3}\left(1-p_1-p_2\right)f_3\left(x_{3k}\right)\right\}\times\left[1-F\left(t\right)\right]^{n-r}$$

After simplification, the likelihood function of 3-component mixture of IE distributions is given:

(3.2)
$$L\left(\phi|\mathbf{x}\right)\propto\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}\left(-1\right)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}(\frac{1}{\theta_1})^{r_1}(\frac{1}{\theta_2})^{r_2}(\frac{1}{\theta_3})^{r_3}$$
$$\times\exp\left\{-\frac{1}{\theta_1}\left(\sum_{k=1}^{r_1}x_{1k}^{-1}+\frac{i-j}{t}\right)\right\}\exp\left\{-\frac{1}{\theta_2}\left(\sum_{k=1}^{r_2}x_{2k}^{-1}+\frac{j-l}{t}\right)\right\}$$
$$\times\exp\left\{-\frac{1}{\theta_3}\left(\sum_{k=1}^{r_3}x_{3k}^{-1}+\frac{l}{t}\right)\right\}p_1^{i-j+r_1}p_2^{j-l+r_2}\left(1-p_1-p_2\right)^{l+r_3}$$

4. The posterior distribution using the non-informative and the informative priors

In this section, posterior distributions of parameters given data, say $\mathbf{x}$, are derived using the non-informative (Uniform and Jeffreys') and the informative (Inverse Gamma) priors.

4.1. The Posterior Distribution using the Uniform Prior (UP). When elicitation of hyper parameters is difficult or little prior information is given, then usually the non-informative prior is assumed to be the UP. Ups over the intervals $(0,\infty)$ and $(0,1)$ are taken for the parameters $(\theta_1,\theta_2\&\theta_3)$ of IE distribution and for the mixing proportions (p_1,p_2), respectively. With these settings, joint prior distribution of parameters$(\theta_1,\theta_2,\theta_3,p_1,p_2)$, as defined by Saleem (2010), is given by:

$$\pi_1(\phi) \propto 1;\ \theta_1,\theta_2,\theta_3>0, p_1,p_2\geq 0, p_1+p_2\leq 1 \tag{4.1}$$

The joint posterior distribution of parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 given data $\mathbf{x}$ assuming the UP is:

$$\begin{aligned} g_1(\phi|\mathbf{x}) =& \Lambda_1^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l} \\ & \times \theta_1^{-(A_{11}+1)}\theta_2^{-(A_{21}+1)}\theta_3^{-(A_{31}+1)}\exp\left(-\frac{B_{11}}{\theta_1}\right)\exp\left(-\frac{B_{21}}{\theta_2}\right) \\ & \times \exp\left(-\frac{B_{31}}{\theta_3}\right)p_1^{A_{01}-1}p_2^{B_{01}-1}(1-p_1-p_2)^{C_{01}-1} \end{aligned} \tag{4.2}$$

where $A_{11}=r_1-1, A_{21}=r_2-1, A_{31}=r_3-1, B_{11}=\sum_{k=1}^{r_1}x_{1k}^{-1}+\frac{i-j}{t}$, $B_{21}=\sum_{k=1}^{r_2}x_{2k}^{-1}+\frac{j-l}{t}, B_{31}=\sum_{k=1}^{r_3}x_{3k}^{-1}+\frac{l}{t}, A_{01}=i-j+r_1+1$, $B_{01}=j-l+r_2+1, C_{01}=l+r_3+1$

$$\begin{aligned} \Lambda_1 =& \sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}B(A_{01},C_{01}) \\ & \times B(B_{01},A_{01}+C_{01})\frac{\Gamma(A_{11})}{B_{11}^{A_{11}}}\frac{\Gamma(A_{21})}{B_{21}^{A_{21}}}\frac{\Gamma(A_{31})}{B_{31}^{A_{31}}} \end{aligned} \tag{4.3}$$

4.2. The posterior distribution using the Jeffreys' prior (JP). According to Jeffreys' (1946, 1998), the JP is defined as $p(\theta_m)\propto\sqrt{|I(\theta_m)|}, m=1,2,3,$ where $I(\theta_m)=-E\left[\frac{\partial^2 f\langle x|\theta_m\rangle}{\partial\theta_m^2}\right]$ is the Fisher's information matrix. The prior distributions of the mixing proportions p_1 and p_2 are again taken to be the uniform over the interval$(0,1)$. Under the assumption of independence of all parameters, the joint prior distribution of $(\theta_1,\theta_2,\theta_3,p_1,p_2)$ is:

$$\pi_2(\phi)\propto\frac{1}{\theta_1\theta_2\theta_3},\ \theta_1,\theta_2,\theta_3\geq 0, p_1,p_2\geq 0, p_1+p_2\leq 1 \tag{4.4}$$

The joint posterior distribution of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 given data $\mathbf{x}$ assuming the JP is:

$$
\begin{aligned}
g_2\left(\phi|\mathbf{x}\right) =& \Lambda_2^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \\
& \times \theta_1^{-(A_{12}+1)} \theta_2^{-(A_{22}+1)} \theta_3^{-(A_{32}+1)} \exp\left(-\frac{B_{12}}{\theta_1}\right) \exp\left(-\frac{B_{22}}{\theta_2}\right) \\
& \times \exp\left(-\frac{B_{32}}{\theta_3}\right) p_1^{A_{02}-1} p_2^{B_{02}-1} \left(1-p_1-p_2\right)^{C_{02}-1}
\end{aligned} \tag{4.5}
$$

where $A_{12} = r_1, A_{22} = r_2, A_{32} = r_3, B_{12} = \sum_{k=1}^{r_1} x_{1k}^{-1} + \frac{i-j}{t}, B_{22} = \sum_{k=1}^{r_2} x_{2k}^{-1} + \frac{j-l}{t}$, $B_{32} = \sum_{k=1}^{r_3} x_{3k}^{-1} + \frac{l}{t}, A_{02} = i - j + r_1 + 1, B_{02} = j - l + r_2 + 1, C_{02} = l + r_3 + 1$, and

$$
\begin{aligned}
\Lambda_2 =& \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} B\left(A_{02}, C_{02}\right) \\
& \times B\left(B_{02}, A_{02} + C_{02}\right) \frac{\Gamma\left(A_{12}\right)}{B_{12}^{A_{12}}} \frac{\Gamma\left(A_{22}\right)}{B_{22}^{A_{22}}} \frac{\Gamma\left(A_{32}\right)}{B_{32}^{A_{32}}}
\end{aligned} \tag{4.6}
$$

4.3. The Posterior Distribution using Inverse Gamma Prior (IGP). Let us assume that the prior distributions of θ_1, θ_2 and θ_3 are IGP with hyperparameters $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3), respectively and Bivariate Beta prior for proportion parameters p_1, p_2 with hyperparameters (a, b, c). Again assuming independence of all parameters, the joint prior distribution of $(\theta_1, \theta_2, \theta_3, p_1, p_2)$ is given by:

$$
\begin{aligned}
\pi_3\left(\phi\right) \propto & \ \theta_1^{-(a_1+1)} \exp\left(-\frac{b_1}{\theta_1}\right) \theta_2^{-(a_2+1)} \exp\left(-\frac{b_2}{\theta_2}\right) \theta_3^{-(a_3+1)} \exp\left(-\frac{b_3}{\theta_3}\right) \\
& \times p_1^{a-1} p_2^{b-1} \left(1-p_1-p_2\right)^{c-1}
\end{aligned} \tag{4.7}
$$

The joint posterior distribution of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 given data $\mathbf{x}$ is:

$$
\begin{aligned}
g_3\left(\phi|\mathbf{x}\right) =& \Lambda_3^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \\
& \times \theta_1^{-(A_{13}+1)} \theta_2^{-(A_{23}+1)} \theta_3^{-(A_{33}+1)} \exp\left(-\frac{B_{13}}{\theta_1}\right) \exp\left(-\frac{B_{23}}{\theta_2}\right) \\
& \times \exp\left(-\frac{B_{33}}{\theta_3}\right) p_1^{A_{03}-1} p_2^{B_{03}-1} \left(1-p_1-p_2\right)^{C_{03}-1}
\end{aligned} \tag{4.8}
$$

where $A_{13} = r_1 + a_1, A_{23} = r_2 + a_2, A_{33} = r_3 + a_3, M_{13} = \sum_{k=1}^{r_1} x_{1k}^{-}1 + \frac{i-j}{t} + b_1$, $B_{23} = \sum_{k=1}^{r_2} x_{2k}^{-}1 + \frac{j-l}{t} + b_2, B_{33} = \sum_{k=1}^{r_3} x_{3k}^{-}1 + \frac{l}{t} + b_3, A_{03} = i - j + r_1 + a$, $B_{03} = j - l + r_2 + b, C_{03} = l + r_3 + c$,and

$$
\begin{aligned}
\Lambda_3 =& \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} B\left(A_{03}, C_{03}\right) \\
& \times B\left(B_{03}, A_{03} + C_{03}\right) \frac{\Gamma\left(A_{13}\right)}{B_{13}^{A_{13}}} \frac{\Gamma\left(A_{23}\right)}{B_{23}^{A_{23}}} \frac{\Gamma\left(A_{33}\right)}{B_{33}^{A_{33}}}
\end{aligned} \tag{4.9}
$$

5. The Bayes estimators and posterior risks using the UP, the JP and IGP under SELF

If $\hat{d}$ is a Bayes estimator then $\rho\left(\hat{d}\right)$ is called posterior risk. Our purpose, in this study, is to look for efficient Bayes estimators of the different parameters. The SELF, defined as $L\left(\theta,d\right)=\left(\theta-d\right)^2$,was introduced by Legendre to develop the least squares theory. For a given prior, the Bayes estimator and posterior risk under SELF are calculated as: $\hat{d}=E_{\theta|x}\left(\theta\right)$ and $\rho\left(\hat{d}\right)=E_{\theta|x}\left(\theta^2\right)-\left\{E_{\theta|x}\left(\theta\right)\right\}^2$, respectively. The Bayes estimators and posterior risks using the UP, the JP and IGP for parameters $\theta_1,\theta_2,\theta_3,p_1$ and $p_2$under SELF are obtained with their respective marginal posterior distributions are given below:

$$\begin{aligned}\hat{\theta}_{1v}=&\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}-1\right)}{B_{1v}^{A_{1v}-1}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\\&\times\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\end{aligned}\tag{5.1}$$

$$\begin{aligned}\hat{\theta}_{2v}=&\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}-1\right)}{B_{2v}^{A_{2v}-1}}\\&\times\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\end{aligned}\tag{5.2}$$

$$\begin{aligned}\hat{\theta}_{3v}=&\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\\&\times\frac{\Gamma\left(A_{3v}-1\right)}{B_{3v}^{A_{3v}-1}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\end{aligned}\tag{5.3}$$

$$\begin{aligned}\hat{p}_{1v}=&\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}\\&\times B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+1,B_{0v}+C_{0v}\right)\end{aligned}\tag{5.4}$$

$$\begin{aligned}\hat{p}_{2v}=&\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}\\&\times B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)\end{aligned}\tag{5.5}$$

$$\begin{aligned}\rho\left(\hat{\theta}_{1v}\right)=&\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}-2\right)}{B_{1v}^{A_{1v}-2}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\\&\times\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)-\left(\hat{\theta}_{1v}\right)^2\end{aligned}\tag{5.6}$$

$$\rho\left(\hat{\theta}_{2v}\right) = \Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v}-2)}{B_{2v}^{A_{2v}-2}} \times\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})-\left(\hat{\theta}_{2v}\right)^2 \tag{5.7}$$

$$\rho\left(\hat{\theta}_{3v}\right) = \Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \times\frac{\Gamma(A_{3v}-2)}{B_{3v}^{A_{3v}-2}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})-\left(\hat{\theta}_{3v}\right)^2 \tag{5.8}$$

$$\rho(\hat{p}_{1v}) = \Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \times\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})-(\hat{p}_{1v})^2 \tag{5.9}$$

$$\rho(\hat{p}_{2v}) = \Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \times\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})-(\hat{p}_{2v})^2 \tag{5.10}$$

where $v = 1$ for the UP, $v = 2$ for the JP and $v = 3$ for the IGP.

6. The Bayes estimators and posterior risks using the UP, the JP and IGP under PLF

Norstrom discussed an asymmetric PLF and also introduced a special case of general class of PLFs, which is defined as$L(\theta, d) = \frac{(\theta-d)^2}{d}$. The Bayes estimator and posterior risk are:
$\hat{d} = \left\{E_{\theta|x}\left(\theta^2\right)\right\}^{\frac{1}{2}}, \rho\left(\hat{d}\right) = 2\left\{E_{\theta|x}\left(\theta^2\right)\right\}^{\frac{1}{2}} - 2E_{\theta|x}(\theta)$, respectively. The respective marginal posterior distribution yields the Bayes estimators and posterior risk using the UP, the JP and the IGP for parameters $\theta_1, \theta_2, \theta_3, p_1$ and $p_2$under PLF as:

$$\hat{\theta}_{1v} = \left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v}-2)}{B_{1v}^{A_{1v}-2}} \frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}} \tag{6.1}$$

$$\hat{\theta}_{2v} = \left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}} \frac{\Gamma(A_{2v}-2)}{B_{2v}^{A_{2v}-2}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{\frac{1}{2}} \tag{6.2}$$

$$\hat{\theta}_{3v} = \left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}} \frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v}-2)}{B_{3v}^{A_{3v}-2}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\}^{\frac{1}{2}} \tag{6.3}$$

$$\hat{p}_{1v} = \left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}} \frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}} B(B_{0v}, C_{0v}) B(A_{0v}+2, B_{0v}+C_{0v}) \right\}^{\frac{1}{2}} \tag{6.4}$$

$$\hat{p}_{2v} = \left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}} \frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}+2, A_{0v}+C_{0v}) \right\}^{\frac{1}{2}} \tag{6.5}$$

$$\begin{aligned} \rho\left(\hat{\theta}_{1v}\right) = 2&\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v}-2)}{B_{1v}^{A_{1v}-2}} \frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\}^{\frac{1}{2}} \\ -2&\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v}-1)}{B_{1v}^{A_{1v}-1}} \frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}} \frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\} \end{aligned} \tag{6.6}$$

$$\begin{aligned} \rho\left(\hat{\theta}_{2v}\right) = 2&\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}} \frac{\Gamma(A_{2v}-2)}{B_{2v}^{A_{2v}-2}} \frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\}^{\frac{1}{2}} \\ -2&\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}} \frac{\Gamma(A_{2v}-1)}{B_{2v}^{A_{2v}-1}} \frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \right\} \end{aligned} \tag{6.7}$$

$$\rho\left(\hat{\theta}_{3v}\right) = 2\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\right.$$
$$\left.\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}-2\right)}{B_{3v}^{A_{3v}-2}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\right\}^{\frac{1}{2}}$$
$$(6.8)\quad -2\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\right.$$
$$\left.\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}-1\right)}{B_{3v}^{A_{3v}-1}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v},A_{0v}+C_{0v}\right)\right\}$$

$$\rho\left(\hat{p}_{1v}\right) = 2\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\right.$$
$$\left.\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+2,B_{0v}+C_{0v}\right)\right\}^{\frac{1}{2}}$$
$$(6.9)\quad -2\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\right.$$
$$\left.\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+1,B_{0v}+C_{0v}\right)\right\}$$

$$\rho\left(\hat{p}_{2v}\right) = 2\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\right.$$
$$\left.\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+2,A_{0v}+C_{0v}\right)\right\}^{\frac{1}{2}}$$
$$(6.10)\quad -2\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\right.$$
$$\left.\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)\right\}$$

7. The Bayes estimators and posterior risks using the UP, the JP and IGP under DLF

DeGroot (2005) introduced the asymmetric loss function, $L\left(\theta\right)=\left(\frac{\theta-d}{d}\right)^2$known as DLF. The Bayes estimator and its posterior risk under DLF are: $\hat{d}=\frac{E_{\theta|x}\left(\theta^2\right)}{E_{\theta|x}\left(\theta\right)}and\rho\left(\hat{d}\right)=1-\frac{\left\{E_{\theta|x}\left(\theta\right)\right\}^2}{E_{\theta|x}\left(\theta^2\right)}$, respectively. The Bayes estimators and posterior risks using the

UP, the JP and the IGP for parameters $\theta_1, \theta_2, \theta_3, p_1$and $p_2$under DLF are:

$$
\hat{\theta}_1 = \frac{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}-2\right)}{B_{1v}^{A_{1v}-2}} \frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}} \frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \right\}}{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}-1\right)}{B_{1v}^{A_{1v}-1}} \frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}} \frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \right\}} \tag{7.1}
$$

$$
\hat{\theta}_2 = \frac{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}} \frac{\Gamma\left(A_{2v}-2\right)}{B_{2v}^{A_{2v}-2}} \frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \right\}}{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}} \frac{\Gamma\left(A_{2v}-1\right)}{B_{2v}^{A_{2v}-1}} \frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \right\}} \tag{7.2}
$$

$$
\hat{\theta}_3 = \frac{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}} \frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}} \frac{\Gamma\left(A_{3v}-2\right)}{B_{3v}^{A_{3v}}-2} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \right\}}{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}} \frac{\Gamma\left(A_{2v}-1\right)}{B_{2v}^{A_{2v}}} \frac{\Gamma\left(A_{3v}-1\right)}{B_{3v}^{A_{3v}-1}} B\left(A_{0v}, C_{0v}\right) B\left(B_{0v}, A_{0v}+C_{0v}\right) \right\}} \tag{7.3}
$$

$$
\hat{p}_1 = \frac{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}} \frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}} \frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}} B\left(B_{0v}, C_{0v}\right) B\left(A_{0v}+2, B_{0v}+C_{0v}\right) \right\}}{\left\{ \Lambda_v^{-1} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \sum_{l=0}^{j} (-1)^i \binom{n-r}{i} \binom{i}{j} \binom{j}{l} \frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}} \frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}} \frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}} B\left(B_{0v}, C_{0v}\right) B\left(A_{0v}+1, B_{0v}+C_{0v}\right) \right\}} \tag{7.4}
$$

$$\hat{p}_2 = \frac{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})\,B(B_{0v}+2,A_{0v}+C_{0v})\right\}}{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})\,B(B_{0v}+1,A_{0v}+C_{0v})\right\}} \tag{7.5}$$

$$\rho\left(\hat{\theta}_1\right) = 1-\frac{\left\{\lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v}-1)}{B_{1v}^{A_{1v}-1}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})\,B(B_{0v},A_{0v}+C_{0v})\right\}^2}{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v}-2)}{B_{1v}^{A_{1v}-2}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})\,B(B_{0v},A_{0v}+C_{0v})\right\}} \tag{7.6}$$

$$\rho\left(\hat{\theta}_2\right) = 1-\frac{\left\{\lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v}-1)}{B_{2v}^{A_{2v}-1}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})\,B(B_{0v},A_{0v}+C_{0v})\right\}^2}{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v}-2)}{B_{2v}^{A_{2v}-2}}\frac{\Gamma(A_{3v})}{B_{3v}^{A_{3v}}}B(A_{0v},C_{0v})\,B(B_{0v},A_{0v}+C_{0v})\right\}} \tag{7.7}$$

$$\rho\left(\hat{\theta}_3\right) = 1-\frac{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}}\frac{\Gamma(A_{3v}-1)}{B_{3v}^{A_{3v}-1}}B(A_{0v},C_{0v})\,B(B_{0v},A_{0v}+C_{0v})\right\}^2}{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma(A_{1v})}{B_{1v}^{A_{1v}}}\frac{\Gamma(A_{2v})}{B_{2v}^{A_{2v}}}\frac{\Gamma(A_{3v}-2)}{B_{3v}^{A_{3v}-2}}B(A_{0v},C_{0v})\,B(B_{0v},A_{0v}+C_{0v})\right\}} \tag{7.8}$$

$$\rho\left(\hat{p}_1\right)=1-\frac{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(B_{0v},C_{0v}\right)B\left(A_{0v}+1,B_{0v}+C_{0v}\right)\right\}^2}{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+2,A_{0v}+C_{0v}\right)\right\}} \tag{7.9}$$

$$\rho\left(\hat{p}_2\right)=1-\frac{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+1,A_{0v}+C_{0v}\right)\right\}^2}{\left\{\Lambda_v^{-1}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\sum_{l=0}^{j}(-1)^i\binom{n-r}{i}\binom{i}{j}\binom{j}{l}\frac{\Gamma\left(A_{1v}\right)}{B_{1v}^{A_{1v}}}\frac{\Gamma\left(A_{2v}\right)}{B_{2v}^{A_{2v}}}\frac{\Gamma\left(A_{3v}\right)}{B_{3v}^{A_{3v}}}B\left(A_{0v},C_{0v}\right)B\left(B_{0v}+2,A_{0v}+C_{0v}\right)\right\}} \tag{7.10}$$

8. Limiting Expressions

Letting $t \to \infty$, all the observations that are incorporated in our analysis are uncensored and therefore r tends n, $r_1$tends to the unknown n_1, r_2 tends to the unknown n_2 and r_3 tends to the unknown n_3. As a result, the amount of information contained in the sample expands, which results in the depletion of the variance of the estimates.

9. Simulation Study

Simulation study is conducted in order to investigate the role of our derived Bayes estimators in terms of three different loss functions. Different set of the parametric values $(\theta_1, \theta_2, \theta_3, p_1, p_2) = (2,\ 3,\ 4,\ 0.30,\ 0.50)$, $(4,\ 3,\ 2,\ 0.50,\ 0.30)$, $(3,\ 3,\ 3,\ 0.40,\ 0.40)$. For fixed sample size, test termination time and set of parameters, the simulation is repeated 1000 times and the results are then averaged. Sample of sizes $p_1 n, p_2 n$ and $(1-p_1-p_2)\,n$ are chosen randomly from first component density$f_1\left(x;\theta_1\right)$, second component density $f_2\left(x;\theta_2\right)$ and third component density$f_3\left(x;\theta_3\right)$, respectively. The observations which are greater than a fixed t are declared as censored observations. For each t only failures have been examined either as a member of subpopulation-I or subpopulation-II or subpopulation-III. On the basis of each sample size, the BEs and PRs are computed using the informative and non-informative priors under SELF, PLF and DLF.To obtain BEs under informative priors, hypeparameters are chosen in such a way that prior mean become the expected value of the corresponding parameter. In order to evaluate the impact of test termination time on Bayes estimators, the Type-I right censoring scheme is used for fixed test termination time t=15 and 20. For each of the 1000 samples, the Bayes estimators and Posterior risks were calculated using a routine in Mathematica 10.0 and the results are presented in Tables 2-10. The simulation study gives us some interesting characteristics of the BEs. The properties have been foregrounded in terms of sample sizes, size of mixing proportion parameters, different loss functions and censoring rates. It is noticed that because of censoring, the posterior risks of all the parameters are reduced with an increase in sample size. The same results examined in graphs (given in Appendix Fig.1-10) that are based on simulation analysis tables corresponding to the different prior distributions and various loss functions. In Fig.1-5, the UP, the JP and the IGP are represented by (red, yellow and blue) colors while in Fig.6-10, SELF, PLF and DLF are represented by (red, yellow and blue) colors respectively.

It is noticed from these results that Bayes estimates perform well under all priors with slight variation. When using IGP, underestimation is observed in BEs for all parametric values considered. Underestimation increases for SELF, but underestimation for the gained BEs improves with increasing the sample size.

10. A Real Life Data Application

Davis (1952) reported the real mixture data on lifetimes of many components used in aircraft sets.To illustrate the proposed methodology, we take the data on three components namely, transmitter tube, combination of transformers and combination of relays. Tahir (2015) used this data for 3-Component mixture of the exponential distributions. We used this data for 3-Component mixture of the inverted exponential distributions by using the inverse transformation. To have a type-I right censored data, we fix t=0.029. The sample statistics required to evaluate the proposed estimates are as follows:

$n = 702, r_1 = 310, r_2 = 148, r_3 = 181, r = 639, n - r = 63,$
$\sum_{k=1}^{r_1} x_{1k}^{-1} = 5.6958, \sum_{k=1}^{r_2} x_{2k}^{-1} = 2.1722, \sum_{k=1}^{r_3} x_{3k}^{-1} = 3.5284$

The BEs and PRs using the UP, the JP and the IGP under SELF, PLF and DLF are presented in the table 1 .

From the above table, it is noticed that results obtained through real data are compatible with simulation results.

11. Conclusion

In this paper, we have considered the Bayesian estimation of 3-component mixture of Inverted Exponential distributions using the non-informative (Uniform and Jeffreys') and the informative (Inverse Gamma) priors under SELF, PLF and DLF. The purpose of this paper is to disclose the appropriate combinations of prior distributions and loss functions to estimate the parameters of the 3-component mixture of the Inverted Exponential distributions. We conducted a extensive simulation study to regulate the relative performance of the Bayes estimators. From simulated results, we observed that an increase in the sample size and test termination time provides better Bayes estimators. Furthermore, as sample size increases (decreases) the posterior risks of Bayes estimators decreases (increases) for a fixed

Table 1. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of inverted exponential distributions using the UP, the JP, and the IGP under SELF, PLF and DLF with Davis(1952) mixture data

Prior	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
UP	SELF	BE	0.01849	0.01488	0.01971	0.48442	0.23209
		PR	**0.000001**	**0.000002**	**0.000002**	**0.000388**	**0.000277**
	PLF	BE	0.01852	0.01493	0.01977	0.48482	0.23268
		PR	**0.000060**	**0.000102**	**0.000111**	**0.000801**	**0.001193**
	DLF	BE	0.01855	0.01498	0.01982	0.48523	0.23328
		PR	**0.003247**	**0.006849**	**0.005587**	**0.001652**	**0.005119**
JP	SELF	BE	0.01843	0.01478	0.01960	0.48442	0.23209
		PR	**0.000001**	**0.000001**	**0.000002**	**0.000388**	**0.000277**
	PLF	BE	0.01846	0.01483	0.01966	0.48482	0.23268
		PR	**0.000060**	**0.000101**	**0.000109**	**0.000801**	**0.001193**
	DLF	BE	0.01849	0.01488	0.01971	0.48522	0.23328
		PR	**0.003236**	**0.006803**	**0.005556**	**0.001652**	**0.005119**
IGP	SELF	BE	0.000005	0.000009	0.000007	0.00011	0.00005
		PR	**0.000001**	**0.0000004**	**0.000002**	**0.000052**	**0.000012**
	PLF	BE	0.02487	0.04156	0.03063	0.48468	0.23303
		PR	**0.000080**	**0.000279**	**0.000169**	**0.000799**	**0.001188**
	DLF	BE	0.02490	0.04170	0.03071	0.48508	0.23363
		PR	**0.003226**	**0.006711**	**0.005525**	**0.001648**	**0.005092**

test termination time. Also, the DLF is observed as a suitable choice for estimating component parameters and SELF is preferable for estimating the proportion parameters. Finally, we conclude that the IGP is suitable prior in order to estimate the component parameters. When SELF is used, the IGP is an appropriate prior for proportion parameters. The same pattern is observed for the JP when non-informativse priors are considered.

In case of non-informative priors,overestimation is found when uniform prior is used. But the problem of overestimation exists only for small samples. PRs using Jeffreys prior are smaller than PRs obtained under uniform prior. So, the performance of Jeffreys prior can be concluded to be better as it produces elegant BEs and the differences among PRs is negligible. It is also examined that PRs is higher for higher parametric values and smaller for smaller values of parameters. In general,Posterior risk(DLF)<Posterior risk(PLF)<Posterior risk(SELF) for the component parameters.For the proportional weights,Posterior risk(SELF)<Posterior risk(PLF)<Posterior risk(DLF). The same interpretation is obtained in the graphs (Fig.1-10) of the simulation results.

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APPENDIX

Table 2. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of IE distributions using the UP under SELF, PLF and DLF with $\theta_1 = 2, \theta_2 = 3, \theta_3 = 4, p_1 = 0.30, p_2 = 0.50, t = 15, 20$.

t	n	Loss Functions		UP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	2.30473	3.26468	5.02734	0.30172	0.49063
			PR	**0.486162**	**0.511918**	**4.101040**	**0.004011**	**0.004740**
		PLF	BE	2.38750	3.33737	5.40387	0.30913	0.49460
			PR	**0.192394**	**0.149270**	**0.715334**	**0.013129**	**0.009626**
		DLF	BE	2.50720	3.39844	5.79342	0.31514	0.49981
			PR	**0.078937**	**0.044135**	**0.127311**	**0.042136**	**0.019325**
	75	SELF	BE	2.20507	3.21201	4.65225	0.29452	0.50014
			PR	**0.275254**	**0.306698**	**1.96829**	**0.002714**	**0.003254**
		PLF	BE	2.28400	3.21788	4.82796	0.29933	0.50287
			PR	**0.118697**	**0.091377**	**0.385128**	**0.009105**	**0.006460**
		DLF	BE	2.32829	3.26200	4.99760	0.30386	0.50640
			PR	**0.051370**	**0.028188**	**0.078359**	**0.030288**	**0.012825**
	100	SELF	BE	2.15853	3.12599	4.42057	0.30105	0.49510
			PR	**0.182901**	**0.214900**	**1.238030**	**0.002080**	**0.002452**
		PLF	BE	2.16805	3.15019	4.58305	0.304675	0.49743
			PR	**0.079096**	**0.068035**	**0.261830**	**0.006895**	**0.005315**
		DLF	BE	2.23658	3.21305	4.66155	0.30734	0.49946
			PR	**0.024120**	**0.023036**	**0.045403**	**0.093218**	**0.007226**
20	50	SELF	BE	2.33389	3.23921	5.06514	0.30185	0.49052
			PR	**0.495791**	**0.502523**	**4.121830**	**0.003976**	**0.004703**
		PLF	BE	2.39776	3.30414	5.26498	0.30850	0.49487
			PR	**0.192228**	**0.147123**	**0.691113**	**0.013052**	**0.009565**
		DLF	BE	2.51612	3.45315	5.80993	0.31488	0.50034
			PR	**0.078516**	**0.043918**	**0.127028**	**0.041910**	**0.019171**
	75	SELF	BE	2.22047	3.15760	4.59187	0.29460	0.50000
			PR	**0.277366**	**0.294962**	**1.900790**	**0.002683**	**0.003219**
		PLF	BE	2.23706	3.21676	4.74236	0.29891	0.50348
			PR	**0.115746**	**0.090866**	**0.377350**	**0.009047**	**0.006417**
		DLF	BE	2.33600	3.29112	5.00497	0.30406	0.50630
			PR	**0.050938**	**0.028084**	**0.078031**	**0.030004**	**0.012741**
	100	SELF	BE	2.13964	3.10194	4.46672	0.30092	0.49539
			PR	**0.179242**	**0.211105**	**1.254900**	**0.002068**	**0.002445**
		PLF	BE	2.17625	3.12386	4.59888	0.30451	0.49744
			PR	**0.079210**	**0.066020**	**0.261692**	**0.006846**	**0.004735**
		DLF	BE	2.23731	3.17610	4.73764	0.30763	0.50012
			PR	**0.036351**	**0.021038**	**0.056230**	**0.022318**	**0.009882**

Table 3. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of inverted Exponential distributions using the JP under SELF, PLF and DLF with $\theta_1 = 4, \theta_2 = 3, \theta_3 = 2, p_1 = 0.50, p_2 = 0.30, t = 15, 20$.

t	n	Loss Functions		JP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	2.18631	3.11620	4.42623	0.30173	0.49017
			PR	**0.404880**	**0.447676**	**2.75740**	**0.004004**	**0.004732**
		PLF	BE	2.24153	3.16639	4.74810	0.30824	0.49551
			PR	**0.167592**	**0.135265**	**0.554694**	**0.013143**	**0.009607**
		DLF	BE	2.30241	3.29099	5.10348	0.31548	0.49992
			PR	**0.073137**	**0.042281**	**0.113398**	**0.042130**	**0.019336**
	75	SELF	BE	2.07082	3.07166	4.32192	0.29500	0.49964
			PR	**0.230697**	**0.274372**	**1.567970**	**0.002706**	**0.003311**
		PLF	BE	2.15999	3.12273	4.48751	0.29888	0.50345
			PR	**0.106901**	**0.086048**	**0.331799**	**0.009110**	**0.006456**
		DLF	BE	2.22415	3.15293	4.65857	0.30430	0.50605
			PR	**0.049156**	**0.027551**	**0.072852**	**0.030144**	**0.012869**
	100	SELF	BE	2.07998	3.07492	4.22341	0.30038	0.49638
			PR	**0.079070**	**0.095679**	**0.870514**	**0.001660**	**0.000480**
		PLF	BE	2.10444	3.09391	4.35272	0.30451	0.49746
			PR	**0.074865**	**0.064449**	**0.236623**	**0.006804**	**0.005162**
		DLF	BE	2.15824	3.11868	4.42167	0.30785	0.50074
			PR	**0.034987**	**0.021035**	**0.053989**	**0.022552**	**0.011038**
20	50	SELF	BE	2.16436	3.15823	4.40441	0.30211	0.49031
			PR	**0.391031**	**0.457151**	**2.711980**	**0.003980**	**0.004705**
		PLF	BE	2.25185	3.17899	4.71005	0.30792	0.49547
			PR	**0.167393**	**0.135260**	**0.546365**	**0.013050**	**0.009544**
		DLF	BE	2.35281	3.24969	5.06273	0.31509	0.49966
			PR	**0.072729**	**0.042139**	**0.112395**	**0.041874**	**0.019228**
	75	SELF	BE	2.10857	3.07733	4.33154	0.29526	0.49959
			PR	**0.236758**	**0.272878**	**1.566030**	**0.002687**	**0.003217**
		PLF	BE	2.14042	3.12245	4.45803	0.29954	0.50281
			PR	**0.105061**	**0.085936**	**0.328503**	**0.009045**	**0.006429**
		DLF	BE	2.22460	3.19774	4.58012	0.30388	0.50646
			PR	**0.048459**	**0.027287**	**0.072379**	**0.029995**	**0.012718**
	100	SELF	BE	2.06359	3.07043	4018396	0.30089	0.49523
			PR	**0.160686**	**0.201598**	**1.033550**	**0.002067**	**0.002442**
		PLF	BE	2.09415	3.10090	4.34849	0.30447	0.49727
			PR	**0.074168**	**0.064284**	**0.234930**	**0.006833**	**0.004877**
		DLF	BE	2.14839	3.13542	4.48785	0.30730	0.49815
			PR	**0.035911**	**0.021429**	**0.053778**	**0.021952**	**0.010628**

Table 4. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of inverted Exponential distributions using the UP under SELF, PLF and DLF with $\theta_1 = 4, \theta_2 = 3, \theta_3 = 2, p_1 = 0.50, p_2 = 0.30, t = 15, 20$.

t	**n**	**Loss Functions**		**UP**				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	4.38238	3.47520	2.49519	0.48998	0.30195
			PR	**0.923645**	**1.101720**	**1.025600**	**0.004717**	**0.003985**
		PLF	BE	4.47292	3.65565	2.67793	0.49578	0.30808
			PR	**0.198822**	**0.293591**	**0.360394**	**0.009557**	**0.013053**
		DLF	BE	4.58748	3.75099	2.90140	0.49934	0.31572
			PR	**0.044090**	**0.078281**	**0.129488**	**0.019299**	**0.041794**
	75	SELF	BE	4.19306	3.30769	2.33219	0.49989	0.29507
			PR	**0.521025**	**0.611841**	**0.502947**	**0.003224**	**0.002686**
		PLF	BE	4.28791	3.40419	2.42282	0.50305	0.29965
			PR	**0.121315**	**0.175373**	**0.196136**	**0.006431**	**0.009034**
		DLF	BE	4.34293	3.49233	2.52575	0.50678	0.30387
			PR	**0.028075**	**0.050987**	**0.079478**	**0.012748**	**0.030038**
	100	SELF	BE	4.17144	3.20032	2.24246	0.49537	0.30141
			PR	**0.367439**	**0.408163**	**0.317356**	**0.002376**	**0.001875**
		PLF	BE	4.17290	3.26067	2.27852	0.49754	0.30430
			PR	**0.087136**	**0.114878**	**0.129900**	**0.005363**	**0.006950**
		DLF	BE	4.29290	3.32267	2.32531	0.50033	0.31125
			PR	**0.021879**	**0.004246**	**0.058109**	**0.010611**	**0.023697**
20	50	SELF	BE	4.40298	3.46283	2.52788	0.49030	0.30207
			PR	**0.927865**	**1.083750**	**1.036230**	**0.004696**	**0.003966**
		PLF	BE	4.43894	3.60390	2.72042	0.49535	0.30831
			PR	**0.196854**	**0.287160**	**0.361226**	**0.009516**	**0.012978**
		DLF	BE	4.54317	3.77487	2.88951	0.49974	0.31546
			PR	**0.043923**	**0.077985**	**0.128555**	**0.019179**	**0.041634**
	75	SELF	BE	4.24636	3.31725	2.33620	0.50019	0.29470
			PR	**0.532535**	**0.613627**	**0.497866**	**0.003209**	**0.002671**
		PLF	BE	4.28561	3.37518	2.39919	0.50310	0.29941
			PR	0.120931	**0.173373**	**0.192759**	**0.006402**	**0.008996**
		DLF	BE	4.32337	3.49628	2.52298	0.50627	0.30413
			PR	**0.028126**	**0.050731**	**0.078730**	**0.012561**	**0.029636**
	100	SELF	BE	4.17202	3.22280	2.21762	0.49503	0.30100
			PR	**0.380990**	**0.403205**	**0.311112**	**0.002440**	**0.002056**
		PLF	BE	4.21956	3.28422	2.30262	0.49748	0.30445
			PR	**0.089141**	**0.119902**	**0.132525**	**0.004914**	**0.006788**
		DLF	BE	4.24956	3.34781	2.34302	0.50040	0.30787
			PR	**0.000975**	**0.032929**	**0.057014**	**0.009473**	**0.115847**

Table 5. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of inverted Exponential distributions using the JP under SELF, PLF and DLF with $\theta_1 = 4, \theta_2 = 3, \theta_3 = 2, p_1 = 0.50, p_2 = 0.30, t = 15, 20$.

t	**n**	**Loss Functions**		**JP**				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	4.17041	3.23700	2.24754	0.49023	0.30188
			PR	**0.799329**	**0.884883**	**0.720849**	**0.004716**	**0.003983**
		PLF	BE	4.26232	3.31171	2.37828	0.49459	0.30880
			PR	**0.181966**	**0.245281**	**0.281168**	**0.009583**	**0.013049**
		DLF	BE	4.36635	3.46432	2.54263	0.49960	0.31498
			PR	**0.042214**	**0.072857**	**0.114394**	**0.019289**	**0.041980**
	75	SELF	BE	4.10033	3.13598	2.17021	0.49976	0.29465
			PR	**0.484920**	**0.525176**	**0.398717**	**0.003224**	**0.002686**
		PLF	BE	4.15034	3.24652	2.22961	0.50313	0.29932
			PR	**0.114269**	**0.159350**	**0.166906**	**0.006440**	**0.009048**
		DLF	BE	4.24005	3.31874	2.33978	0.50593	0.30427
			PR	**0.027356**	**0.048416**	**0.073393**	**0.012778**	**0.029973**
	100	SELF	BE	4.07259	3.08975	2.09345	0.49455	0.30163
			PR	**0.336734**	**0.318795**	**0.245456**	**0.002634**	**0.001797**
		PLF	BE	4.15437	3.16760	2.12551	0.49761	0.30410
			PR	**0.086133**	**0.112135**	**0.116298**	**0.004934**	**0.006825s**
		DLF	BE	4.15864	3.25939	2.22241	0.49866	0.30775
			PR	**0.017564**	**0.035955**	**0.054035**	**0.010660**	**0.023969**
20	50	SELF	BE	4.15509	3.21406	2.21595	0.49039	0.30194
			PR	**0.791323**	**0.862090**	**0.698123**	**0.004691**	**0.003961**
		PLF	BE	4.26312	3.37954	2.38041	0.49469	0.30891
			PR	**0.181329**	**0.248895**	**0.279223**	**0.009529**	**0.012970**
		DLF	BE	4.30490	3.44151	2.51106	0.49981	0.31456
			PR	**0.042053**	**0.072465**	**0.113470**	**0.019164**	**0.041728**
	75	SELF	BE	4.09315	3.17988	2.16313	0.49976	0.29496
			PR	**0.482854**	**0.536829**	**0.392604**	**0.003212**	**0.002674**
		PLF	BE	4.15703	3.26277	2.20069	0.50336	0.29926
			PR	**0.114042**	**0.159457**	**0.163784**	**0.006401**	**0.009001**
		DLF	BE	4.23709	3.29438	2.28920	0.50590	0.30384
			PR	**0.027289**	**0.048301**	**0.072790**	**0.012721**	**0.029886**
	100	SELF	BE	4.09935	3.09145	2.10000	0.49513	0.30084
			PR	**0.360104**	**0.357825**	**0.263036**	**0.002440**	**0.002056**
		PLF	BE	4.15293	3.16231	2.18127	0.49736	0.30412
			PR	**0.085964**	**0.111564**	**0.118400**	**0.004900**	**0.006787**
		DLF	BE	4.17844	3.23497	2.24954	0.50005	0.30800
			PR	**0.020673**	**0.035027**	**0.053890**	**0.009814**	**0.022185**

Table 6. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of inverted Exponential distributions using the UP under SELF, PLF and DLF with $\theta_1 = 3, \theta_2 = 3, \theta_3 = 3, p_1 = 0.40, p_2 = 0.40, t = 15, 20$.

t	n	Loss Functions		UP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	3.36701	3.32722	3.74018	0.39602	0.39629
			PR	**0.715015**	**0.697159**	**2.26778**	**0.004520**	**0.004521**
		PLF	BE	3.42987	3.45525	4.00984	0.40185	0.40171
			PR	**0.196700**	**0.198130**	**0.533432**	**0.011338**	**0.011340**
		DLF	BE	3.51377	3.56232	4.26898	0.40786	0.40758
			PR	**0.056463**	**0.056505**	**0.128780**	**0.028015**	**0.028044**
	75	SELF	BE	3.20857	3.24437	3.47953	0.397007	0.397973
			PR	**0.400637**	**0.409637**	**1.10276**	**0.003099**	**0.003101**
		PLF	BE	3.27509	3.28301	3.62248	0.40151	0.40124
			PR	**0.119936**	**0.120260**	**0.291913**	**0.007754**	**0.007763**
		DLF	BE	3.34300	3.33303	3.75681	0.40478	0.40495
			PR	**0.035311**	**0.036645**	**0.075744**	**0.006872**	**0.018702**
	100	SELF	BE	3.17666	3.16778	3.37678	0.39794	0.39786
			PR	**0.283748**	**0.285586**	**0.715058**	**0.002374**	**0.002427**
		PLF	BE	3.18946	3.20665	3.44271	0.40129	0.40147
			PR	**0.084990**	**0.074857**	**0.198044**	**0.005336**	**0.006645**
		DLF	BE	3.26551	3.25649	3.57785	0.40373	0.40400
			PR	**0.026831**	**0.026402**	**0.056608**	**0.015281**	**0.014504**
20	50	SELF	BE	3.31207	3.34749	3.73894	0.39592	0.39619
			PR	**0.685450**	**0.701290**	**2.262500**	**0.004498**	**0.004498**
		PLF	BE	3.47000	3.47456	4.08618	0.40186	0.40175
			PR	**0.197960**	**0.198278**	**0.539323**	**0.011268**	**0.011270**
		DLF	BE	3.56412	3.55242	4.28960	0.40767	0.40762
			PR	**0.056222**	**0.056237**	**0.127709**	**0.027863**	**0.027873**
	75	SELF	BE	3.22203	3.20107	3.46011	0.39754	0.39725
			PR	**0.402475**	**0.397444**	**1.083870**	**0.003077**	**0.003076**
		PLF	BE	3.29931	3.29242	3.56203	0.40177	0.40098
			PR	**0.120203**	**0.120232**	**0.285269**	**0.007710**	**0.007721**
		DLF	BE	3.38521	3.35540	3.80310	0.40505	0.40508
			PR	**0.036129**	**0.036110**	**0.078204**	**0.019129**	**0.019142**
	100	SELF	BE	3.14473	3.15489	3.31537	0.39807	0.39808
			PR	**0.276978**	**0.278331**	**0.690130**	**0.002340**	**0.002342**
		PLF	BE	3.19722	3.20978	3.43478	0.40111	0.40094
			PR	**0.085638**	**0.086033**	**0.196980**	**0.005862**	**0.005864**
		DLF	BE	3.24961	3.23003	3.54049	0.40407	0.40378
			PR	**0.026595**	**0.026637**	**0.056548**	**0.014602**	**0.014528**

Table 7. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of IE distribution using the JP under SELF, PLF and DLF with $\theta_1 = 3, \theta_2 = 3, \theta_3 = 3, p_1 = 0.40, p_2 = 0.40, t = 15, 20$.

t	n	Loss Functions		JP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	3.14623	3.15008	3.38857	0.39587	0.39639
			PR	**0.588463**	**0.586696**	**1.627790**	**0.004517**	**0.004520**
		PLF	BE	3.22340	3.27657	3.49151	0.40157	0.40168
			PR	**0.174877**	**0.177651**	**0.408597**	**0.011337**	**0.011335**
		DLF	BE	3.30679	3.26877	3.80942	0.40788	0.40750
			PR	**0.053412**	**0.053480**	**0.113921**	**0.027997**	**0.028047**
	75	SELF	BE	3.07233	3.09435	3.20727	0.39692	0.39751
			PR	**0.355164**	**0.358824**	**0.864405**	**0.003093**	**0.003095**
		PLF	BE	3.15974	3.15650	3.38515	0.40086	0.40114
			PR	**0.111757**	**0.111554**	**0.251236**	**0.007761**	**0.007758**
		DLF	BE	3.18872	3.22265	3.48227	0.40508	0.40549
			PR	**0.035041**	**0.034996**	**0.073156**	**0.019278**	**0.019241**
	100	SELF	BE	3.06949	3.07278	3.10762	0.39809	0.39778
			PR	**0.258215**	**0.259099**	**0.576701**	**0.002355**	**0.002358**
		PLF	BE	3.11870	3.11432	3.24477	0.40094	0.40092
			PR	**0.080591**	**0.082374**	**0.178224**	**0.005877**	**0.006396**
		DLF	BE	3.14853	3.15180	3.36353	0.40377	0.40437
			PR	**0.025969**	**0.026122**	**0.054093**	**0.014860**	**0.014311**
20	50	SELF	BE	3.14433	3.11726	3.35716	0.39589	0.39596
			PR	**0.584706**	**0.574549**	**1.580120**	**0.004499**	**0.004499**
		PLF	BE	3.26716	3.23539	3.53025	0.40170	0.40129
			PR	**0.176642**	**0.175078**	**0.410260**	**0.011287**	**0.011296**
		DLF	BE	3.33531	3.33345	3.80173	0.40744	0.40754
			PR	**0.053244**	**0.053228**	**0.113027**	**0.027878**	**0.027866**
	75	SELF	BE	3.11465	3.14302	3.21287	0.39760	0.39707
			PR	**0.362450**	**0.369878**	**0.865536**	**0.003078**	**0.003077**
		PLF	BE	3.15291	3.18036	3.34394	0.40102	0.40110
			PR	**0.110992**	**0.111910**	**0.246836**	**0.007713**	**0.007711**
		DLF	BE	3.20734	3.22340	3.42018	0.40506	0.40512
			PR	**0.034889**	**0.034881**	**0.072569**	**0.019154**	**0.019148**
	100	SELF	BE	3.06338	3.06178	3.17449	0.39791	0.39821
			PR	**0.256050**	**0.255497**	**0.599099**	**0.002337**	**0.002343**
		PLF	BE	3.10539	3.10577	3.23082	0.40106	0.40080
			PR	**0.081026**	**0.081077**	**0.174977**	**0.005860**	**0.005862**
		DLF	BE	3.15036	3.15543	3.33287	0.40391	0.40381
			PR	**0.025852**	**0.025951**	**0.053360**	**0.014462**	**0.014612**

Table 8. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of IE distributions using the IGP under SELF, PLF and DLF with $\theta_1 = 2, \theta_2 = 3, \theta_3 = 4, a_1 = 1, a_2 = 2, a_3 = 1, b_1 = 2, b_2 = 4, b_3 = 2, a = 2.0, b = 1.75, c = 1.50, p_1 = 0.50, p_2 = 0.30, t = 15, 20$.

t	n	Loss Functions		IGP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	1.99495	2.83175	4.00133	0.28734	0.44973
			PR	**0.641867**	**0.969988**	**3.18203**	**0.009643**	**0.018969**
		PLF	BE	2.19963	3.11098	4.47665	0.31354	0.48837
			PR	**0.152831**	**0.122457**	**0.465614**	**0.012487**	**0.009327**
		DLF	BE	2.29486	3.15685	4.70003	0.31964	0.49369
			PR	**0.068522**	**0.038981**	**0.101649**	**0.039675**	**0.019051**
	75	SELF	BE	1.94388	2.82985	3.85846	0.27885	0.46246
			PR	**0.487274**	**0.821489**	**2.237250**	**0.008005**	**0.018030**
		PLF	BE	2.15531	3.06887	4.29523	0.30367	0.49830
			PR	**0.101281**	**0.080213**	**0.296184**	**0.008714**	**0.006066**
		DLF	BE	2.18224	3.07041	4.38390	0.30800	0.50162
			PR	**0.046417**	**0.025959**	**0.067795**	**0.028288**	**0.011859**
	100	SELF	BE	1.94851	2.82658	3.85915	0.28333	0.45364
			PR	**0.417046**	**0.739193**	**1.87566**	**0.007428**	**0.015397**
		PLF	BE	2.10944	3.04672	4.23543	0.30563	0.49463
			PR	**0.072578**	**0.060970**	**0.218273**	**0.007727**	**0.026620**
		DLF	BE	2.17680	3.08901	4.31760	0.30796	0.50030
			PR	**0.034145**	**0.019853**	**0.050794**	**0.029956**	**0.013056**
20	50	SELF	BE	2.00453	2.81192	3.96029	0.28650	0.45079
			PR	**0.642816**	**0.961382**	**3.141060**	**0.009643**	**0.019163**
		PLF	BE	2.23813	3.10010	4.41131	0.31372	0.48848
			PR	**0.154321**	**0.121546**	**0.457478**	**0.012396**	**0.009268**
		DLF	BE	2.31585	3.13628	4.58665	0.32002	0.49299
			PR	**0.067891**	**0.038874**	**0.100994**	**0.039252**	**0.018941**
	75	SELF	BE	1.95380	2.84207	3.86976	0.27906	0.46198
			PR	**0.487808**	**0.826740**	**2.234400**	**0.007994**	**0.018101**
		PLF	BE	2.14124	3.08109	4.36703	0.30364	0.49805
			PR	**0.100099**	**0.080342**	**0.299620**	**0.008735**	**0.006310**
		DLF	BE	2.18751	3.13514	4.44932	0.30818	0.50102
			PR	**0.046165**	**0.025915**	**0.067408**	**0.028568**	**0.012640**
	100	SELF	BE	1.91667	2.81492	3.83777	0.28411	0.45810
			PR	**0.404391**	**0.738354**	**1.857140**	**0.007536**	**0.017303**
		PLF	BE	2.09662	3.03557	4.22915	0.30737	0.49393
			PR	**0.071565**	**0.060377**	**0.216566**	**0.006594**	**0.004390**
		DLF	BE	2.13346	3.05979	4.28705	0.31058	0.49199
			PR	**0.033907**	**0.019794**	**0.050595**	**0.023579**	**0.008590**

Table 9. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of IE distribution using the IGP under SELF, PLF and DLF with $\theta_1 = 4, \theta_2 = 3, \theta_3 = 2, a_1 = 1, a_2 = 2, a_3 = 1, b_1 = 2, b_2 = 4, b_3 = 2, a = 2.0, b = 1.75, c = 1.50, p_1 = 0.50, p_2 = 0.30, t = 15, 20$.

t	**n**	**Loss Functions**		**IGP**				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	3.55887	2.67661	1.93765	0.42581	0.26458
			PR	**2.535680**	**1.689320**	**1.039050**	**0.0301982**	**0.013481**
		PLF	BE	4.18011	3.14375	2.33585	0.49296	0.30971
			PR	**0.170924**	**0.202807**	**0.246968**	**0.009221**	**0.012506**
		DLF	BE	4.29945	3.20248	2.43490	0.49801	0.31574
			PR	**0.040418**	**0.063515**	**0.102812**	**0.018585**	**0.040049**
	75	SELF	BE	3.54666	2.68289	1.89392	0.43695	0.25919
			PR	**2.208550**	**1.450640**	**0.796053**	**0.029416**	**0.011646**
		PLF	BE	4.11135	3.15003	2.20534	0.50125	0.30059
			PR	**0.110197**	**0.140640**	**0.153657**	**0.006351**	**0.009059**
		DLF	BE	4.13600	3.18340	2.28891	0.50418	0.30439
			PR	**0.026615**	**0.044173**	**0.068316**	**0.013182**	**0.031001**
	100	SELF	BE	3.53555	2.66520	1.85726	0.43255	0.26384
			PR	**2.068120**	**1.291320**	**0.682325**	**0.028100**	**0.011642**
		PLF	BE	4.12100	3.08866	2.16556	0.49495	0.30571
			PR	**0.083893**	**0.101990**	**0.112353**	**0.005688**	**0.013843**
		DLF	BE	4.13600	3.18340	2.28891	0.50418	0.30439
			PR	**0.026615**	**0.044173**	**0.068316**	**0.013182**	**0.031001**
20	50	SELF	BE	3.54399	2.65565	1.92498	0.42552	0.26419
			PR	**2.537860**	**1.675840**	**1.027530**	**0.030403**	**0.013525**
		PLF	BE	4.15429	3.20963	2.35515	0.49286	0.30950
			PR	**0.169413**	**0.206243**	**0.246556**	**0.009176**	**0.012437**
		DLF	BE	4.26315	3.22369	2.43934	0.49782	0.31551
			PR	**0.040339**	**0.063340**	**0.101958**	**0.018523**	**0.039912**
	75	SELF	BE	3.57815	2.65846	1.90501	0.43704	0.25950
			PR	**2.238300**	**1.411890**	**0.801257**	**0.029379**	**0.011685**
		PLF	BE	4.13568	3.09271	2.19249	0.50140	0.30057
			PR	**0.110123**	**0.137747**	**0.151294**	**0.006008**	**0.008850**
		DLF	BE	4.20609	3.18553	2.29484	0.50459	0.30562
			PR	**0.026542**	**0.043993**	**0.068037**	**0.012760**	**0.029737**
	100	SELF	BE	3.56725	2.67341	1.87305	0.43365	0.26453
			PR	**2.097070**	**1.301940**	**0.689093**	**0.028069**	**0.011418**
		PLF	BE	4.08364	3.09561	2.16819	0.49582	0.30497
			PR	**0.082767**	**0.102013**	**0.111851**	**0.005183**	**0.009375**
		DLF	BE	4.16574	3.14452	2.21929	0.49898	0.30782
			PR	**0.020159**	**0.032677**	**0.050975**	**0.009238**	**0.017656**

Table 10. Bayes estimates (BEs) and posterior risks (PRs) of 3-component mixture of IE distributions using the IGPP under SELF, PLF and DLF with $\theta_1 = 3, \theta_2 = 3, \theta_3 = 3, a_1 = 1, a_2 = 2, a_3 = 1, b_1 = 2, b_2 = 4, b_3 = 2, a = 2.0, b = 1.75, c = 1.50, p_1 = 0.40, p_2 = 0.40, t = 15, 20$.

t	n	Loss Functions		IGP				
				$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
15	50	SELF	BE	2.84224	2.79167	2.94841	0.36273	0.35812
			PR	**1.310390**	**1.242680**	**1.95655**	**0.016736**	**0.016516**
		PLF	BE	3.21094	3.12517	3.38967	0.40310	0.39907
			PR	**0.165281**	**0.152838**	**0.354793**	**0.010846**	**0.010920**
		DLF	BE	3.24173	3.18462	3.63541	0.40869	0.40486
			PR	**0.050799**	**0.048257**	**0.102066**	**0.026745**	**0.027172s**
	75	SELF	BE	2.82307	2.79119	2.88343	0.36423	0.36156
			PR	**1.087850**	**1.052720**	**1.443710**	**0.015226**	**0.015031**
		PLF	BE	3.13559	3.06402	3.24686	0.40276	0.39901
			PR	**0.106878**	**0.101164**	**0.224895**	**0.007516**	**0.007551**
		DLF	BE	3.16150	3.11366	3.45679	0.40576	0.40321
			PR	**0.033834**	**0.032753**	**0.067997**	**0.019605**	**0.019966**
	100	SELF	BE	2.77716	2.79218	2.80717	0.36236	0.35728
			PR	**0.969081**	**0.975957**	**1.193650**	**0.014657**	**0.015772**
		PLF	BE	3.09315	3.09373	3.18715	0.40063	0.40022
			PR	**0.078884**	**0.076942**	**0.164690**	**0.005777**	**0.012811**
		DLF	BE	3.14764	3.09021	3.26260	0.40448	0.40325
			PR	**0.025379**	**0.024725**	**0.051054**	**0.018270**	**0.014750**
20	50	SELF	BE	2.82708	2.77359	2.94867	0.36283	0.35816
			PR	**1.303780**	**1.219900**	**1.949500**	**0.016734**	**0.016376**
		PLF	BE	3.14637	3.12264	3.27083	0.40334	0.39925
			PR	**0.161127**	**0.152028**	**0.341318**	**0.010785**	**0.010857**
		DLF	BE	3.29164	3.21842	3.50938	0.40877	0.40468
			PR	**0.050581**	**0.048096**	**0.101544**	**0.026586**	**0.027042**
	75	SELF	BE	2.83239	2.78367	2.84294	0.36353	0.36112
			PR	**1.105990**	**1.057920**	**1.408600**	**0.015313**	**0.015152**
		PLF	BE	3.12554	3.07564	3.24891	0.40228	0.39936
			PR	**0.106238**	**0.101060**	**0.223489**	**0.007480**	**0.007496**
		DLF	BE	3.17857	3.13180	3.34545	0.40591	0.40360
			PR	**0.033727**	**0.032557**	**0.067733**	**0.018537**	**0.018715**
	100	SELF	BE	2.80008	2.78098	2.84476	0.36450	0.36051
			PR	**0.973048**	**0.955440**	**1.214180**	**0.014384**	**0.013736**
		PLF	BE	3.07553	3.08378	3.19921	0.40127	0.39984
			PR	**0.078187**	**0.076467**	**0.164301**	**0.006126**	**0.011433**
		DLF	BE	3.12538	3.12050	3.28668	0.40481	0.39936
			PR	**0.025276**	**0.024668**	**0.050658**	**0.015469**	**0.012630**

Graphical Representation of the Simulation Results

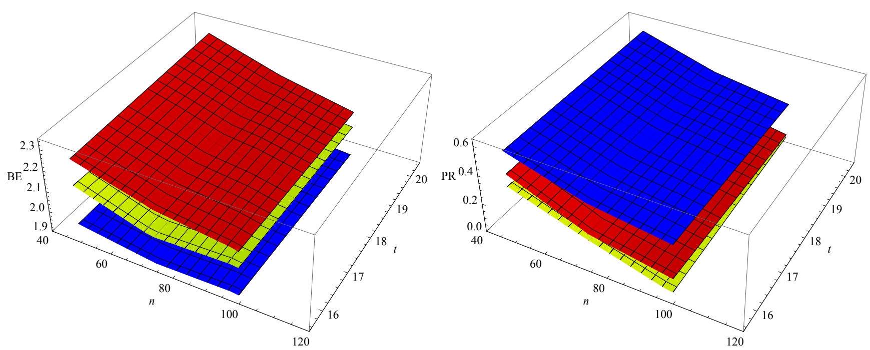

Figure 1. Graphs of BEs and BPRs θ_1 under SELF

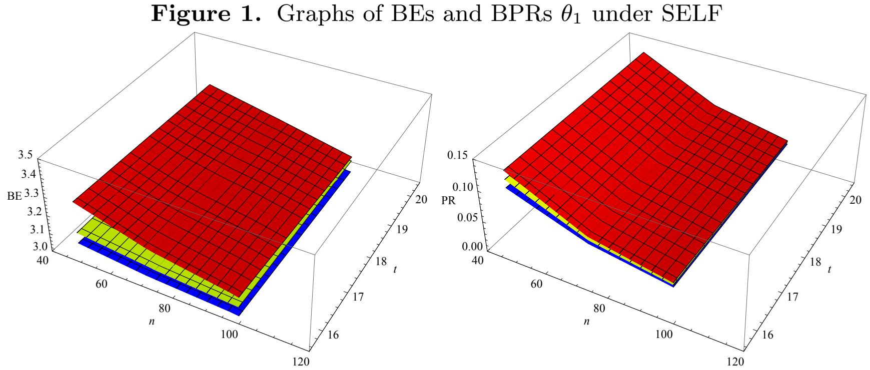

Figure 2. Graphs of BEs and BPRs of θ_2 under PLF

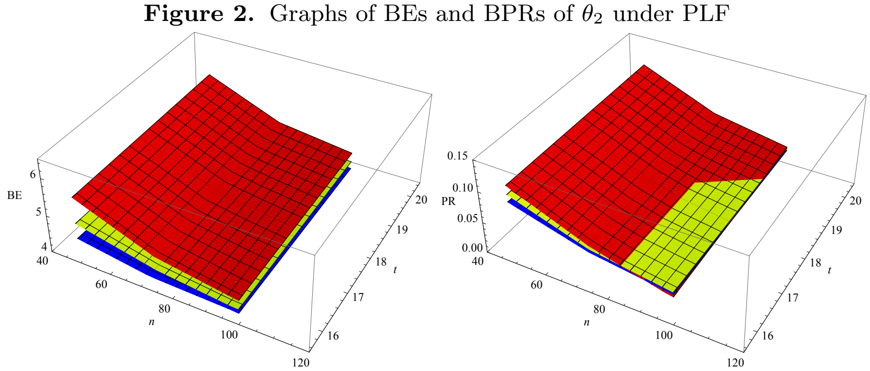

Figure 3. Graphs of BEs and BPRs of θ_3 under DLF

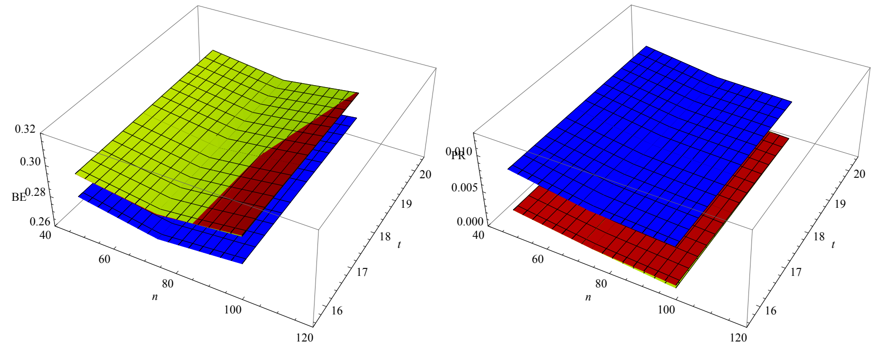

Figure 4. Graphs of BEs and BPRs of p_1 under SELF

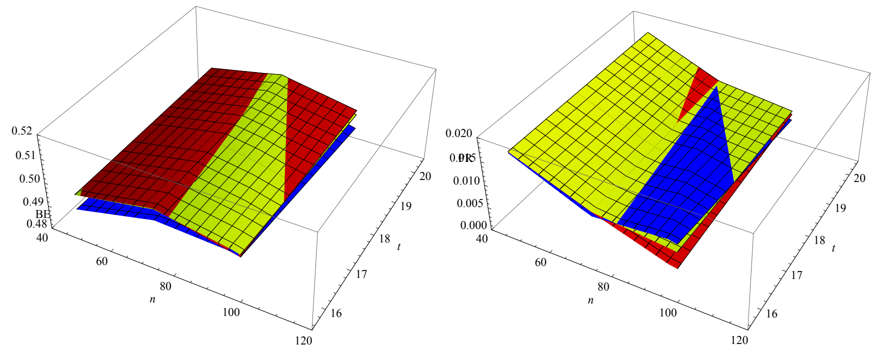

Figure 5. Graphs of BEs and BPRs of p_2 under DLF

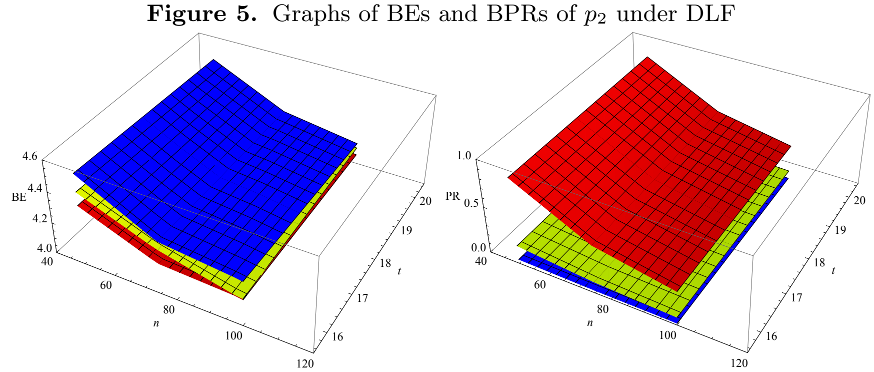

Figure 6. Graphs of BEs and BPRs of θ_1 using UP

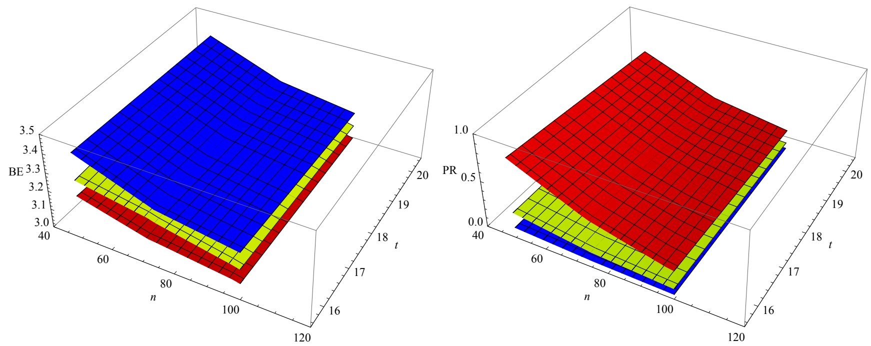

Figure 7. Graphs of BEs and BPRs of θ_2 using JP

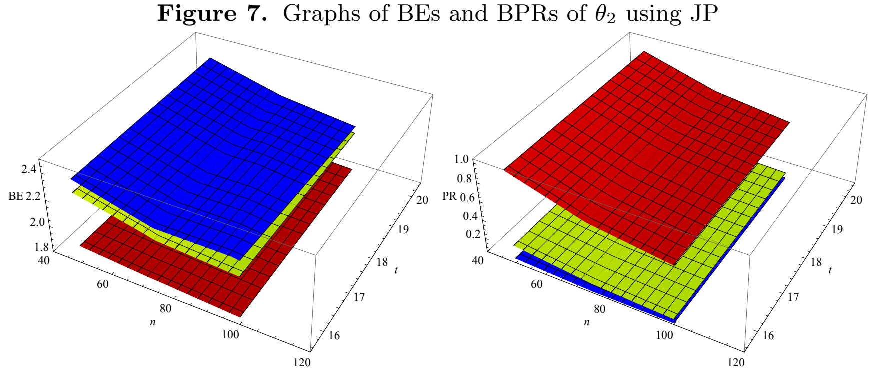

Figure 8. Graphs of BEs and BPRs of θ_3 using IGP

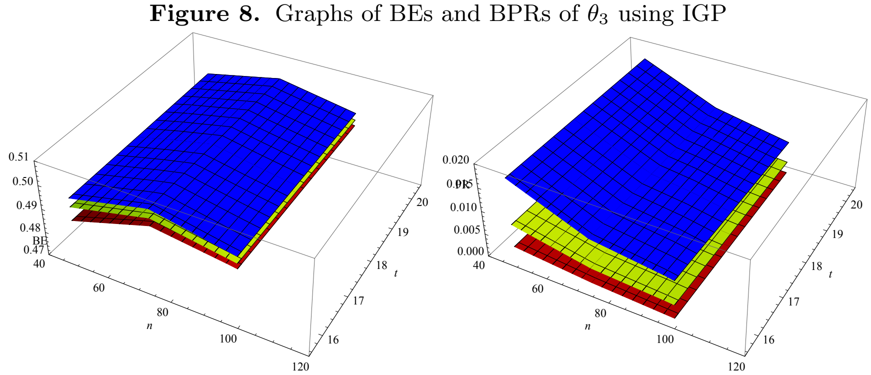

Figure 9. Graphs of BEs and BPRs of p_1 using UP

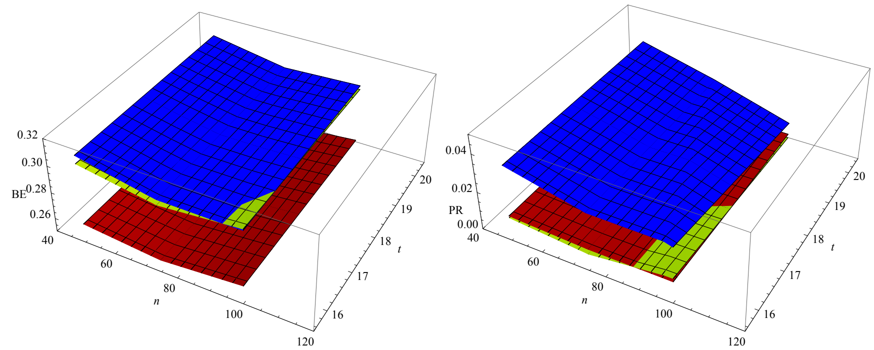

Figure 10. Graphs of BEs and BPRs of p_2 using IGP