Original Research Article

KREIGER GRAPHS AND FISCHER COVERS VS DYNAMICAL PROPERTIES

ABSTRACT. We will show that how the structure of the Kreiger graph and the Fischer cover of an irreducible subshift is related to the dynamical properties of that system. Also an extension of a half synchronized system called weak synchronized has been considered.

1. INTRODUCTION

One of the most studied dynamical systems is a subshift of finite type (SFT). An SFT is a system whose set of forbidden blocks is finite [8]; or equivalently, X is SFT iff there is $M \in \mathbb{N}$ such that any block of length greater than M is synchronizing. Recall that a block m is *synchronizing* if whenever v_1m and mv_2 are both blocks of X, then v_1mv_2 is a block of X as well. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system* and examples are *sofics* where they are factors of SFT's. Synchronized systems, has attracted much attention and extension of them has been of interest since that notion was introduced [5]. One was via *half synchronized systems*; that is, systems having *half synchronizing* blocks. In fact, if for a left transitive point such as rm and mv any block in X one has again $rmv \in X^- = \{x_- := \cdots x_{-1}x_0 : x = \cdots x_{-1}x_0x_1 \cdots \in X\}$, then m is called half synchronizing [5]. Clearly any synchronized system is half synchronized. Dyke (or Dyck!) subshifts and certain β-shifts are non-synchronized but half synchronized systems [9].

In 1992, Fiebigs in [5], as an extension to the Fischer cover of a synchronized system, introduced a unique component of the Kreiger graph as the Fischer cover of a half synchronized subshift. Here in Section 3, we will introduce the notion of a *regular weak synchronized* subshift and then in Section 5, we will show which of the components of the Kreiger graph of such a subshift could be a candidate to be suitable for a Fischer cover. Meanwhile in Section 4, we will show how the structure of the Kreiger graph of a subshift X and its dynamical properties are related. In Section 5, we collect results showing how the Fischer cover and dynamical properties of X are interrelated.

2. BACKGROUND AND DEFINITIONS

This section is devoted to the very basic definitions in symbolic dynamics. The notations has been taken from [8] and [5] for the relevant concepts.

First we present some elementary concept from [8]. Let $\mathcal{A}$ be an alphabet, that is a non-empty finite set. The full shift $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of

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all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *block* or *word* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* which is denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a block of length $j-i+1$ by $x_{[i,\,j]} = x_i x_{i+1} \dots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. By our topology, σ is a homeomorphism. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* or a *subshift* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks.

Let $\mathcal{B}_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $\mathcal{B}(X) = \cup_n \mathcal{B}_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks $u,\, v \in \mathcal{B}(X)$ there is a block $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$. It is *mixing* if for every ordered pair $u,\, v \in \mathcal{B}(X)$, there is an $N \in \mathbb{N}$ such that for each $n \geq N$ there is a block $w \in \mathcal{B}_n(X)$ such that $uwv \in \mathcal{B}(X)$. A shift space X is called a *shift of finite type* (SFT) if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. A shift of *sofic* is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sofic [8, Theorem 3.1.5], but the converse is not true [8, Page 67].

Let G be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* X_G is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ defined by

$$X_G = \left\{ \xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1}) \right\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathcal{G} = (G,\, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. For a path $\pi = \pi_0 \dots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \dots \mathcal{L}(\pi_k)$ is the label of π. By π_u we mean a path labeled u.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) :\ \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or *cover* of $X = \overline{X_{\mathcal{G}}}$. In particular, X is sofic if and only if $X = X_{\mathcal{G}}$ for a finite graph G [8, Proposition 3.2.10].

In this part we collect some information from [5]. Let X be a subshift and $x \in X$. Then, $x_+ = (x_i)_{i\in Z^+}$ (resp. $x_- = (x_i)_{i\leq 0}$) is called right (resp. left) infinite X-ray. Let $X^+ = \{x_+ : x \in X\}$ and $X^- = \{x_- : x \in X\}$. For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_- x_+ \in X\}$ and for $m \in \mathcal{B}(X)$ its follower set is $w_+(m) = \{x_+ \in X^+ : m x_+ \in X^+\}$. Analogously, we define predecessor sets $\omega_-(x_+) = \{x_- \in X^- : x_- x_+ \in X\}$ and $\omega_-(m) = \{x_- \in X^- : x_- m \in X^-\}$.

Consider the collection of all follower sets $\omega_+(x_-)$ as the set of vertices of a graph. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that $x_- a$ is an X-ray and $I_1 = \omega_+(x_-), I_2 = \omega_+(x_- a)$. This labeled graph is called the *Krieger graph* for X. A block $m \in \mathcal{B}(X)$ is *synchronizing* if whenever um and mv are in $\mathcal{B}(X)$, we have $umv \in \mathcal{B}(X)$. An irreducible shift space X is *synchronized system* if it has a synchronizing block. A block $m \in \mathcal{B}(X)$ is *half synchronizing* if there is a left transitive point $x \in X$ such that $x_{[-|m|+1,\,0]} = m$ and $\omega_+(x_{(-\infty,\,0]}) = \omega_+(m)$ which is called the *magic vertex* in the Krieger graph. If X

is a half synchronized system with half synchronizing m, the irreducible component of the Krieger graph containing the vertex $\omega_+(m)$ is denoted by X_0^+ and is called the *right Fischer cover* of X.

3. Weak synchronized systems

Definition 3.1. *Assume X is an irreducible subshift. Then, X is called* right *(resp.* left*)* weak synchronized system *if there is a block m of X and a point $x \in X$ such that $x_{[-|m|+1,\,0]} = m$ (resp. $x_{[0,\,|m|-1]} = m$) and $\omega_+(x_{(-\infty,\,0]}) = \omega_+(m)$ (resp. $\omega_-(x_{[0,\,\infty)}) = \omega_-(m)$) that we call m a* right weak synchronizing *(resp. left) block of X. Then, $\omega_+(m)$ is called a* weak synchronized vertex.

Note that if x was left (resp. right) transitive, then by definition, X would be right (resp. left) half synchronized system and so any half synchronized system is a weak synchronized system.

Here, whenever we say "weak synchronizing", we mean the right weak synchronizing. A similar observation as in half synchronizing appears here:

Proposition 3.2. *Suppose m is weak synchronizing with $x \in X$, $x_{[-|m|+1,\,0]} = m$ and $\omega_+(x_{(-\infty,\,0]}) = \omega_+(m)$ as in the definition 3.1. Then*

(1) $x_k \cdots x_{-|m|} x_{-|m|+1} \cdots x_0$ is a weak synchronizing block for $k \leq -|m| + 1$. That is, any left prolongation of m across x is also weak synchronizing.
(2) Any right prolongation of m is weak synchronizing.

Proof. (*1*) is trivial and for (*2*) let $m' = mu$ be any right prolongation of m. Let $v \in \omega_+(m')$ and $x'_- = x_- u$. Then, $uv \in \omega_+(m)$ which implies that $v \in \omega_+(x'_-)$. So $w_+(x'_{(-\infty,\,0]}) = w_+(m')$. □

Half synchronized systems are necessarily coded; this is not the case for weak synchronized systems. Our next result is concerned about non-coded systems.

Recall that a subshift X is *minimal* if for every $x \in X$ the orbit $\{\sigma^n x : n \in \mathbb{Z}\}$ is dense in X. A *trivial* minimal system is just the cycle of a point.

Also, an *spacing shift* X_S is a subshift on $\{0, 1\}$ such that if $x \in X_S$, then the distance between any two 1's appearing in x is in $S \subseteq \mathbb{N}$ [2]. There are rigid conditions for an spacing shift to be coded [2, Theorem 2.7].

Proposition 3.3. *(1) Any spacing shift is weak synchronized.*
(2) Non trivial minimal systems are not weak synchronized.

Proof. (*1*). Let X_S be a spacing subshift. Then, $\omega_+(0^\infty 1) = \omega_+(1)$ and so 1 is a weak synchronizing block for X_S.

(*2*). Let X be a non-trivial minimal subshift. If X is a weak synchronized, then there is a block m of X and a point $x \in X$ such that $x_{[-|m|+1,\,0]} = m$ and $w_+(x_-) = w_+(m)$. By the fact that any point in X is right and left transitive, so m is a half synchronizing block. This means X must be a non-trivial coded system that is absurd and so X is not weak synchronized. □

One of the below examples uses a non-synchronized beta shift and hence we recall some basic facts about these shifts.

Let β be a real number greater than 1. Set

$$1_\beta := a_1 a_2 a_3 \ldots \in \{0, 1, 2, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}},$$

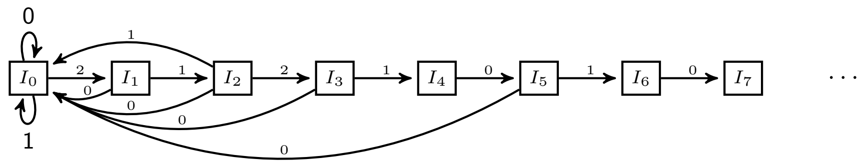

FIGURE 1. Cover for the β-shift with $1_\beta = 2121010\ldots$.

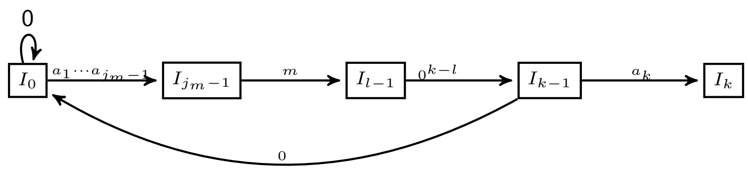

FIGURE 2. The subgraph H of $\mathcal{G}_\beta$ with $1_\beta = a_1a_2a_3\ldots$, where $l := j_m + |m|$.

where $a_1 = \lfloor \beta \rfloor$ and

$$a_i = \lfloor \beta^i(1 - a_1\beta^{-1} - a_2\beta^{-2} - \cdots - a_{i-1}\beta^{-i+1})\rfloor$$

where $i \geq 2$. The sequence 1_β is the expansion of 1 in the base β, that is, $1 = \sum_{i=1}^{\infty} a_i\beta^{-i}$. Let $\leq$ be the lexicographic ordering of $\{\mathbb{N} \cup \{0\}\}^{\mathbb{N}}$. The sequence 1_β has the property that

$$\sigma^k 1_\beta = a_{k+1}a_{k+2}\cdots \leq 1_\beta, \quad k \in \mathbb{N}. \tag{1}$$

Furthermore, it follows from (1) that

$$X_\beta = \{x \in \{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{Z}} : x_{[i,\infty)} < 1_\beta,\, i \in \mathbb{Z}\}$$

is a shift space with alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$, called the β-shift. Note that if any right infinite block $a_1a_2\cdots$ with $a_i \in \{0,\, 1, \ldots, \ell - 1\}$ satisfies (1), then there is a unique β where $a_1 = \lfloor\beta\rfloor$ and $1_\beta = a_1a_2\ldots$ [6, Theorem 2.3.2]. These shifts are symbolic spaces with rich structure whose all blocks are half synchronizing. For a more detailed treatment, see [6].

One may construct an infinite labeled graph $\mathcal{G}_\beta$ as a Fischer cover for a β-shift as follows. Take a countable infinite set of vertices I_0, $I_1, \ldots$. Let I_0 be the base point. Edges an defined as follows. First, for all $i \geq 0$ there is an edge labeled a_i with initial vertex I_i and terminal vertex I_{i+1}. Also, for every i and for every $0 \leq c < a_i$ there is an edge labeled c with initial vertex I_i and terminal vertex I_0.

For instance, let $a_1a_2a_3\cdots = 2121010\cdots$ be a sequence satisfying (1) and let $\beta > 1$ be the unique associated real number; thus $1_\beta = 2121010\cdots$. Figure 1 presents a part of the cover for this β-shift.

Example 3.4. *Now we present two sets of examples of coded weak synchronized systems which are not half synchronized and whose any of their blocks are weak*

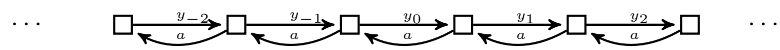

Figure 3. Cover for a weak synchronized system.

synchronizing. The former is a pretty known system and the latter an easy to construct example.

(1)- Our first example is X_β^{-1}, *where* $1 < \beta \in \mathbb{R}$ *is chosen so that* X_β *is not synchronized. Let* m^{-1} *be an arbitrary block in* $W(X_\beta^{-1})$. *First we show that* $0^\infty m^{-1} \in (X_\beta^{-1})^-$ *and* $m^{-1}0^\infty \in (X_\beta^{-1})^+$.

Since X_β *is not synchronized and* $m \in \mathcal{B}(X_\beta)$, m *is not a synchronizing block for* X_β *where then by* [9, Proposition 2.23], $m \subseteq 1_\beta = a_1a_2a_3\cdots$. *Assume* $m = a_{j_m}a_{j_m+1}\ldots a_{j_m+|m|-1}$ *(Figure 2) and set* $k := \min\{i > j_m + |m| - 1 : \ a_i > 0\}$. *Then, there is a finite path labeled* $m0^{k_m - j_m - |m| + 1}$ *with initial vertex* $I_{j_m - 1}$ *and terminal vertex* I_0. *Hence* $m0^\infty$ *is a right infinite* X_β*-ray and so* $0^\infty m^{-1}$ *is a left infinite* $(X_\beta)^{-1}$*-ray. Similar reasoning works for* $m^{-1}0^\infty \in (X_\beta^{-1})^+$ *and so we have* $w_+(0^\infty m^{-1}) = w_+(m^{-1})$ *and* $\omega_-(m^{-1}0^\infty) = \omega_-(m^{-1})$ *which that in turn shows that* m^{-1} *is a right and left weak synchronizing block for* X_β^{-1}. *But* m *was arbitrary and so we are done.*

(2)- Let Y *be a subshift on a finite alphabet* $\mathcal{A}$ *and* $y = \cdots y_{-1}y_0y_1y_2\cdots$ *a point in* Y. *Let* $a \notin \mathcal{A}$ *and* X *a subshift on* $\mathcal{A} \cup \{a\}$ *presented by the cover in the Figure 3. For any* $m \in \mathcal{B}(X)$, $a^\infty m$ *is a left infinite* X*-ray and any* π_m *(path labeled* m *in the cover) terminates at a vertex if and only if an infinite path* $\pi_{a^\infty m}$ *terminates at. So* $\omega_+(a^\infty m) = \omega_+(m)$ *and as a result, any block* m *is weak synchronizing.*

Furthurmore, if Y *is taken to be a non-trivial minimal system, then the cover is right-resolving and follower separated and the resulting space* X *is not half synchronized, but weak synchronized as stated above.*

Next example shows that there are coded systems which are not weak synchronized.

Example 3.5. *Let* $\mathcal{A} = \{a, b\}$ *and let*

$$\mathcal{W} = \{v_0 := a,\ v_1 := b^1ab^1,\ \ldots,\ v_n := b^nl_nb^n\}$$

be a generator for a subshift X *where* l_n *will be defined inductively. To do so, first for any* $n \in \mathbb{N}$ *and* $0 \le i_j < n$ *set* $v_{i_0i_1\cdots i_{n-1}} := v_{i_0}v_{i_1}\cdots v_{i_{n-1}}$ *and define* l_1 *and* l_2 *as*

$$l_1 = v_0 = a,\ l_2 = v_0^2v_0v_1v_1v_0v_1^2 = v_{00}v_{01}v_{10}v_{11} = v_{0^2}v_{01}v_{10}v_{1^2}.$$

So for defining l_2, *we have considered all possible* $v_{i_0i_1}$, $0 \le i_0,\ i_1 < 2$ *and then have concatenated the result according to lexicographic order of* i_0i_1 *in* $v_{i_0i_1}$. *Similarly when* v_{n-1} *and so* l_{n-1} *and all* $v_{i_0i_1\cdots i_{n-1}}$*'s are defined, define*

$$l_n = v_{0^n}v_{0^{n-1}1}\cdots v_{0^{n-1}(n-1)}\cdots v_{(n-1)^n}.$$

We claim that no blocks of X *can be weak synchronizing. Fix* $v \subseteq v_n$ *and write* $v_n = u'_nvu_n$ *for some* u_n *and* u'_n. *Then by the fact that both* v_n *and* $a = v_0$ *are elements of our generator,* $u_na^\infty \in \omega_+(u'_nv)$. *But* $v_n \subset l_p$ *for any* $p > n$ *and by the* *definition of l_p, v_n appears also in v_p and we may write $v_p = u'_p v_n u_p = u'_p u'_n v u_n u_p$ for some non-empty blocks u'_p and u_p. This means $u_n a^\infty \notin \omega_+(u'_p u'_n v)$.*

Now assume v is a weak synchronizing block and assume that x_-, provided by the definition of weak synchronizing, terminates at $v = x_{[-|v|+1,\, 0]}$. Since by Proposition 3.2 any right prolongation of x_- is again weak synchronizing block, we prolong x_- further to a $v \subset v_n$. Then by the above discussion $u_n a^\infty \notin \omega_+(x_-)$. However, $u_n a^\infty \in \omega_+(u'_n v) \subseteq \omega_+(v)$.

4. Structure of the Kreiger graph with respect to the dynamical properties

Recall that a graph G is irreducible if for every ordered pair of vertices I and J there is a path in G starting at I and terminating at J. Also a labeled graph is irreducible if its underlying graph is irreducible. Let $G_K = G_K(X)$ be the Kreiger graph for a subshift X. The subgraph H of G_K is called an *irreducible component* of G_K whenever H is irreducible and if H' is any irreducible subgraph G_K such that H is a subgraph of H', then $H = H'$ [5]. An irreducible component of G_K is called a *component cover* if it is a cover for X.

A directed graph such as G_K is *weakly connected* if there is a path between every two vertices in the underlying undirected graph. Similar to an irreducible component a *weakly connected component* is also defined.

We intend to study the connectedness issue of G_K for a coded system X. We already know that a shift is sofic X if and only if its Kreiger graph finite graph.

In the category of coded systems, a close concept to sofics is the class of *shifts with variable gap length* or SVGL. A subshift is SVGL if there is $M \in \mathbb{N}$ such that for all u, $v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$ and $|w| \leq M$ [7]. This is a non-mixing version of a *specified* system where for systems with specification property, $|w| \leq M$ is replaced with $|w| = M$. Any irreducible sofic is SVGL and all SVGL's are synchronized [7, Lemma 3.1]. Note that an SVGL was called *almost specified* in [7].

Lemma 4.1. *Assume X is an SVGL. Let x_- and y_+ be arbitrary left and right X-rays respectively. Then, there is $s_0 \in \mathcal{B}(X)$ such that $x_- s_0 y_+ \in X$.*

Proof. Let M be the constant provided by the definition of SVGL. Then, for any $n \in \mathbb{N}$, there is $s_n \in \mathcal{B}(X)$ such that $x_{[-n,\,0]} s_n y_{[1,\,n]} \in \mathcal{B}(X)$ and $|s_n| \leq M$. Now let S be the set of all $s'_n s$ and note that S is finite by the fact that blocks of length $\leq M$ are finite. Hence, there is $s_0 \in S$ such that $x_{[-n_k,\,0]} s_0 y_{[1,\,n_k]} \in \mathcal{B}(X)$ for infinitely many n_k and so $x_- s_0 y_+ \in X$ as required. □

Let X be a half synchronized system with $\mathcal{G} = (G, \mathcal{L})$ its Fischer cover. Then X is synchronized (resp. half synchronized) iff $\mathcal{L}_\infty(G)$ (resp. $\mathcal{L}^+_\infty(G)$) is residual in X (resp. X^+) [5].

Corollary 4.2. *Assume X is an SVGL and $\mathcal{G} = (G, \mathcal{L})$ its Fischer cover. Then, $\mathcal{L}^+_\infty(G) = X^+$.*

Proof. Let m be a synchronizing block and $x_- m$ a left ray for X. Let $y_+ \in X^+$ and by Lemma 4.1 choose s_0 so that $x_- m s_0 y_+ \in X$. In particular, there will be a right infinite path in G labeled y_+ and so $\mathcal{L}^+_\infty$ is onto. □

The converse of the above corollary is not correct. For instance if X is an S-gap, then $\mathcal{L}^+_\infty(G) = X^+$. However, not all the S-gaps are SVGL. In fact, if $S = \{s_k\}$

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is an increasing sequence of natural numbers such that $\{s_{k+1}-s_k:\ k\in\mathbb{N}\}$ is not bounded, then $X=X(S)$ is not SVGL [1].

In the next proposition, we will see that how dynamical properties of subshifts and the connectedness of the Kreiger graph are related.

Proposition 4.3. *Let X be an irreducible subshift and G_K its Kreiger graph.*

(1) If X is an SFT, then $G=G_K$ where $\mathcal{G}=(G,\mathcal{L})$ is the Fischer cover of X.
(2) If X is SVGL, then G_K is weakly connected. In particular, X is irreducible sofic iff G_K is finite and weakly connected.
(3) There is an example of a synchronized system with a non weakly connected G_K.
(4) If X is half synchronized, then all weak synchronized vertices are in a unique weakly connected component of G_K.
(5) There is an example of a weak synchronized subshift with uncountably many weakly connected components having weak synchronized vertices.

Proof. (1) Let x be an arbitrary point in X and recall that $\omega_+(x_-)\in\mathcal{V}(G_K)$ and $\mathcal{V}(G)\subseteq\mathcal{V}(G_K)$. By [8, Theorem 2.1.8], there is $M\geq 0$ such that $m:=x_{[-M+1,\,0]}$ is a synchronizing block and so $\omega_+(x_-)=\omega_+(m)$. But any synchronizing block such as $\omega_+(m)$ is in a unique irreducible component of G_K [5, page 146] and so $\omega_+(x_-)\in\mathcal{V}(G)$. Hence $\mathcal{V}(G)=\mathcal{V}(G_K)$ or equivalently $G=G_K$.

(2) Let x_- and x'_- be two different left X-rays. We will show that there is a vertex α in G_K such that $\omega_+(x_-u)=\omega_+(x'_-u')=\alpha$ for some $u,\,u'\in\mathcal{B}(X)$. By [7, Lemma 3.1], X is a synchronized system and let $m\in\mathcal{B}(X)$ be a synchronizing block of X. Pick $z_+\in X^+$ with initial segment m. By Lemma 4.1, there are $s,\,s'\in\mathcal{B}(X)$ such that $x_-sz_+,\,x'_-s'z_+\in X$. Thus $\omega_+(x_-sm)=\omega_+(x'_-s'm)$ and so G_K is weakly connected.

Now recall that X is sofic iff it has finite follower sets or equivalently iff $|\mathcal{V}(G_K)|<\infty$. Thus the end result of this part follows from the fact that any irreducible sofic is SVGL.

(3) Let X be a subshift on $\{a,\,b\}$ generated by $\{a^nb^na^n:\ \ n\in\mathbb{N}\}$. Then, for any n, $a^nb^na^n$ is a synchronizing block and so X is synchronized. (In fact, since no subblock of ab^na is synchronizing, X has infinitely many synchronizing blocks whose any subblock, except itself, is not synchronizing.) Note that if ever $b^\infty x_{(i,\,+\infty)}\in X$ (resp. $x_{(-\infty,\,i)}b^\infty\in X$), then

$$x_{(i,\,+\infty)}\in\{a^\infty,\,b^\infty\}\text{ (resp. }x_{(-\infty,\,i)}\in\{a^\infty,\,b^\infty\}).\tag{2}$$

Also note that if $ab^n\subset x_-$, then there is at least one x_+ containing aba such that $x_-x_+\in X$. On the other hand, if for $k\geq 0$, $x_-\neq b^\infty a^k$, then this x_- must contain some ab^n and if $\omega_+(x_-)=\omega_+(b^\infty a^k)$, then $a^kx_+\in\omega_+(b^\infty)$ violating 2. This implies that $\omega_+(b^\infty)$ and $\omega_+((aba)^\infty)$ are in different weakly connected components as required. For this example, G_K consists of two weakly connected components depicted in Figure 4.

(4) The main ingredient of the proof of this part is from [5, page 146] where there it is used to prove that half synchronizing vertices are in a unique irreducible component of G_K. So let m and m' be weak and half synchronizing blocks of a half synchronized system X respectively. There are $x_-,\,x'_-\in X^-$ with terminal segments m and m' respectively such that $\omega_+(x_-)=\omega_+(m)$ and $\omega_+(x'_-)=\omega_+(m')$ and x' is a left transitive point in X. Pick u such that mum' is the terminal segment of x'. We need to show that $\omega_+(x_-um')=\omega_+(x'_-)$ which this in turn

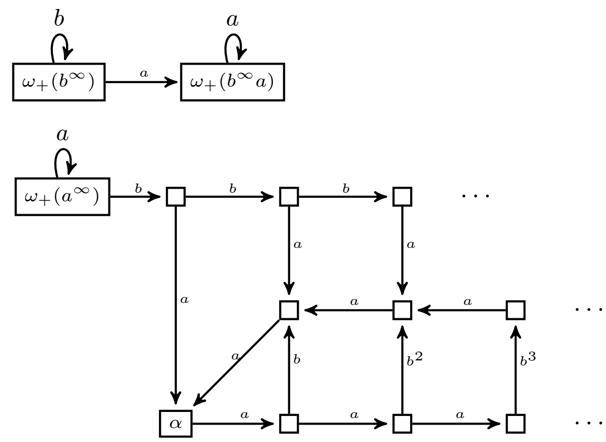

FIGURE 4. Krieger graph G_K consisting of two separate weakly connected components (Proposition 4.3 (2)). The irreducible component containing the vertex $\alpha := \omega_+((aba)^\infty)$ is the Fischer cover.

implies that there is a finite path in G_K with initial vertex $\omega_+(x_-)$ and terminal vertex $\omega_+(x'_-)$. We have $\omega_+(x_-um') \subseteq \omega_+(m') = \omega_+(x'_-)$ and the other because for any $x_+ \in \omega_+(x'_-)$ we have $x_+ \in \omega_+(mum')$. Thus $um'x_+ \in \omega_+(m)$ and so $x_+ \in \omega_+(mum')$.

(5) For this part we exploit the second example in Example 3.4 and we let Y be a non-trivial minimal substitution system. Since the asymptotic orbits for such systems are finite [4], there are uncountably many $y \in Y$ such that if $y_-z^+ \in Y$ (resp. $z_-y^+ \in Y$) then $z^+ = y^+$ (resp. $z_- = y_-$). In particular, if $y_{(-\infty,0]}y_i = y_{(-\infty,0]}y_j$, then $y_i = y_j$.

Now consider the graph depicted in Figure 3 constructed by such a y. We claim that graph is actually an irreducible maximal component of G_K. For let $\omega_+(y_-)$ be the vertex in G_K. Thus if π_u is any path in the figure starting at $t(\pi_{y_0})$, then π_u is a path in G_K starting at $\omega_+(y_-)$ as well. Now if there is a non-empty $v = v_0 \cdots v_k$ such that $i(\pi_v) = \omega_+(y_-)$ but $i(\pi_v) \neq t(\pi_{y_0})$, then without loss of generality we may assume that there is not a π_{v_0} starting at $t(\pi_{y_0})$ in our figure. Then by the fact that G_K is right resolving $v_0 \neq a$ and if $v_0 = y_i$ for some $y_i \neq y_1$, then there must be a point $y' \in Y$ with $y'_{(-\infty,1]} = y_{(-\infty,0]}y_i$ which is impossible. $\square$

For the part (3) of the above proposition, one may choose a synchronized system on alphabet $\{a, b_1, \dots, b_k\}$ and generator $\cup_{i=1}^k \{a^n b_i^n a^n : n \in \mathbb{N}\}$ to give an example of a Kreiger graph with exactly $k+1$ weakly connected components.

5. Fischer covers vs dynamical properties

Let X be a weak synchronized system and let $\mathrm{WH}(X)$ denote the set of weak synchronizing blocks for X. For $m \in \mathrm{WH}(X)$, denote by $(X_m)_0^+$ the maximal irreducible component of the Krieger graph X containing the vertex $\omega_+(m)$. Note that irreducible components are countable labeled graphs and so $\overline{\mathcal{L}((X_m)_0^+)}$ is a coded system which is a subsystem of X. Examples for such covers are Fischer covers for half synchronized systems where for them $X = \overline{\mathcal{L}((X_m)_0^+)}$ [5].

Definition 5.1. *If there is $m \in WH(X)$ such that $\overline{\mathcal{L}((X_m)_0^+)} = X$, then X is called* regular weak synchronized system *and $(X_m)_0^+$ the* weak Fischer cover *of X.*

Next proposition gives sufficient condition for a weak synchronizing block being half synchronizing and in particular it shows that any half synchronized system is a regular weak synchronized system.

Proposition 5.2. *Let $m \in WH(X)$. Then, m is a half synchronizing block if and only if $\overline{\mathcal{L}((X_m)_0^+)} = X$ and there is a finite path π_m in $(X_m)_0^+$ labeled m such that $\omega_+(m) = t(\pi_m)$.*

Proof. Let m be a half synchronizing block for X and let $x \in X$ be left transitive with $m = x_{[-|m|+1,\,0]}$ and $\omega_+(x_-) = \omega_+(m)$. Since x is left transitive, we can choose $u \in \mathcal{B}(X)$ such that mum is a terminal segment of x_- and so $\omega_+(x_-) = \omega_+(x_-um)$. Thus there is a cycle in the Fischer cover $X_0^+ = (X_m)_0^+$ meeting $\omega_+(x_-)$ and labeled um which this in turn shows that $\omega_+(m) = t(\pi_m)$.

Conversely, let π_m be a finite path in $(X_m)_0^+$ such that $\omega_+(m) = t(\pi_m) := \alpha$. Take $E = \{\pi_1,\, \pi_2,\, \ldots\}$ to be the set of all finite paths in $(X_m)_0^+$. By irreducibility of $(X_m)_0^+$, there are finite paths $\pi_1',\, \pi_2',\, \ldots$ in $(X_m)_0^+$ such that $\pi_- := \cdots \pi_2\pi_2'\pi_1\pi_1'\pi_m$ is an infinite path and $t(\pi_-) = \alpha = \omega_+(m)$. Set $x_-' := \mathcal{L}^-(\pi_-)$. Then, x' is a left transitive point for $\overline{\mathcal{L}((X_m)_0^+)} = X$ and $m = x'_{[-|m|+1,\,0]}$. Also $\omega_+(m) = \omega_+(\alpha)$ [5, Page 146] and so $\omega_+(x_-') = \omega_+(m)$. Thus m is a half synchronizing block. $\square$

The next corollary shows that there are regular weak synchronized systems which are not half synchronized. We can obtain it by omitting condition π_m in $(X_m)_0^+$.

Corollary 5.3. *Let $\mathcal{G}$ be a weak Fischer cover of X. Then, there is a finite path π_m in $\mathcal{G}$ labeled m such that $\omega_+(m) = t(\pi_m)$ if and only if $\mathcal{L}_\infty^+(\mathcal{G})$ is residual in X^+.*

The Following example shows that we may have $m \in \mathrm{WH}(X)$ and a finite path π_m in $(X_m)_0^+$ labeled m such that $\omega_+(m) = t(\pi_m)$. But $\overline{\mathcal{L}((X_m)_0^+)} \neq X$.

Example 5.4. *Consider Example 3.4(2) and choose Y to be a non-trivial subshift having at least one non-fixed point a^∞. Also, let $y = \cdots y_{-1}y_0y_1 \cdots \in Y$ defining the cover in that example being different from a^∞ and let X be the associated space.*

Let $x = a^\infty$ and $m = a$ and recall that any block of X is in $WH(X)$. We will show that $(X_m)_0^+$ is an irreducible component of the Kreiger graph consisting of a vertex $\alpha = \omega_+(x_-)$ and an edge labeled a initiating and terminating at α, or equivalently a loop whose edge is labeled a.

To see this, suppose there is $u \in \mathcal{B}(X)$ such that $u = u_1u_2 \cdots u_n \not\subseteq a^\infty$ and $\omega_+(a^\infty u) = \omega_+(a^\infty)$. Set $i_0 := \max\{1 \le i \le n : \ u_i \neq a\}$ and so

$$u = u_1 \cdots u_{i_0-1}u_{i_0}a^{n-i_0}.$$

Now pick $v = v_1 \cdots v_{n-i_0}v_{n-i_0+1} \subseteq y$ such that $v_{n-i_0+1} \neq u_{i_0}$. Then, $u_{i_0}a^{n-i_0}av \notin \mathcal{B}(X)$ and so $av \notin \omega_+(a^\infty u)$; however, $av \in \omega_+(a^\infty)$. This shows that the hypothesis of Proposition 5.2 can not be weakened to weak synchronizing block.

Proposition 5.5. *Let $m \in WH(X)$ and $\overline{\mathcal{L}((X_m)_0^+)} = X$. Assume there is $m' \in WH(X)$ so that x_-' defining m' terminates at mum'. Then m' is half synchronizing.*

Proof. Let x_- be the left ray terminating at m and $\omega_+(x_-) = \omega_+(m)$. First note that if in the proof of part 4 of Proposition 4.3, we let m' being weak synchronizing instead of half synchronizing and having u such that x'_- terminates at mum', again that conclusion holds. That is, $\omega_+(x_-um') = \omega_+(x'_-)$. Now using the fact that $\overline{\mathcal{L}((X_m)_0^+)} = X$ and similar to the proof of Proposition 5.2, pick a left transitive path π_- terminating at $\omega_+(m) = \omega_+(\alpha)$. Thus $\pi_-\pi_{um'}$ is a left transitive path terminating at $\pi_{um'}$ whose follower set is the same as the follower set of $\pi_{m'}$. This means m' is half synchronizing as required. □

Corollary 5.6. *Let X, m be as in the hypotheis of Proposition 5.5 and so that there is $v \in \mathcal{B}(X)$ such that x_- terminates at mvm. Then, X is a half synchronized system. In particular, if x_- is periodic, then X is a half synchronized.*

Proof. By Proposition 3.2 (1), $m' := vm$ is weak synchronizing defined by $x'_- := x_-vm$. Now apply Proposition 5.5. □

The converse to the last part of the above corollary is true as well. That is

Corollary 5.7. *Let m be half synchronizing. Then, there is a periodic left ray x'_- terminating at m and so that $\omega_+(x'_-) = \omega_+(m)$.*

Proof. Let x_- be the left transitive ray terminating at m with $\alpha := \omega_+(x_-) = \omega_+(m)$. Choose u so that x_- terminates at mum. Then, $\alpha = \omega_+(x_-um) = \omega_+(x_-) = \omega_+(m)$ which this in turn means that π_{um} is a cycle starting and terminating at α in our Kreiger graph. Now set x'_- to be the periodic left ray $(um)^\infty = \cdots umum$. □

Remark 5.8. *Let X be an irreducible subshift and G_K its Kreiger graph and X^+ the associated one sided shift. Then,*

(1) X is sofic iff X_0^+ is a finite graph. In this case for any point $x \in X$, there is a unique bi-infinite path π_x in X_0^+ such that $\mathcal{L}(\pi_x) = x$ [5].
(2) X is synchronized iff G_K has an irreducible component cover H so that $\mathcal{L} : H \to X$ is residual on the set of bi-infinite paths [5, Remark 2.13]. *Moreover, such H is unique and is isomorphic to the Fischer cover and contains all synchronizing blocks.*
(3) X is right half synchronized iff G_K has an irreducible component cover H such that $\mathcal{L} : H \to X^+$ is residual on the set of right infinite paths [5, Theorem 1.4]. *Moreover, such H is unique and is isomorphic to the Fischer cover and contains all right half synchronizing blocks* [5, Theorem 2.12].

Note that uniqueness in (*2*) arises from the fact that a cover must have at least a path π_m labeled m for any synchronizing block m. But then all π_m's for a synchronizing m must be just in one component. This is not the case for a half synchronizing m. For instance it is not hard to show that for the Dyke system which is half synchronized, but not synchronized, there are infinitely many such irreducible component covers in Krieger graph.

From (*3*) in the above remark we have that if m and m' are two half synchronizing blocks, then $(X_m)_0^+ = (X_{m'})_0^+$ [5]; however, this is not true when m and m' are two weak synchronizing blocks:

Example 5.9. *Pick $1_\beta = a_1a_2\ldots$ such that X_β is not synchronized and $a_i \in \{0, 1\}$ for $i \in \mathbb{N}$. The first part of Example 3.4 shows that 0 and 1 are two weak synchronizing blocks. We claim that $(X_0)_0^+ \neq (X_1)_0^+$.*

Let $G_K(X_\beta^{-1})$ *be the Kreiger graph for* X_β^{-1} *and* $a_1a_2\dots a_n = 1^n$ *such that* $a_{n+1} = 0$. *Now suppose there is a path in* $G_K(X_\beta^{-1})$ *labeled* $u^{-1} := b_k \dots b_2 b_1$ *from* $\omega_+(0^\infty 1)$ *to* $\omega_+(0^\infty)$. *Then,*

$$\omega_+(0^\infty 1 u^{-1}) = \omega_+(0^\infty). \tag{3}$$

Let $b_1 = 1$. *Since* $0^\infty 1^n$ *is a left infinite* X_β^{-1}*-ray, so* $1^n \in \omega_+(0^\infty)$ *and so by* (3), $1^n \in \omega_+(0^\infty 1 u^{-1})$. *Thus* $0^\infty 1 u^{-1} 1^n$ *is a left infinite* X_β^{-1}*-ray and so* $1^n 1 b_2 \dots b_k 1 = 1^n u 1 \in \mathcal{B}(X_\beta)$. *Thus* $1^{n+1} \in \mathcal{B}(X_\beta)$ *that is absurd and so there is* $n_1 \in \mathbb{N}$ *such that* $u^{-1} = b_k b_{k-1} \dots b_{n_1+2} 1 0^{n_1}$ *and so by* (3),

$$\omega_+(0^\infty 1 b_k b_{k-1} \dots b_{n_1+2} 1 0^{n_1}) = \omega_+(0^\infty). \tag{4}$$

But $01 \in \omega_+(0^\infty)$ *and by* (4), $01 \in \omega_+(0^\infty 1 b_k b_{k-1} \dots b_{n_1+2} 1 0^{n_1})$. *Thus* $10^{1+n_1}1 \in \mathcal{B}(X_\beta)$. *Since* X_β *is not synchronized, so* $10^{1+n_1}1 \subseteq 1_\beta$ [6, Proposition 2.4.4]. *Set*

$$k' := \min\{i \in \mathbb{N} : \ a_{[i,\, n_1+i+2]} = 10^{1+n_1}1\}.$$

Then, $a_{[1,\, k'+n_1+2]} = a_1 a_2 \dots a_{k'-1} 10^{n_1+1}1$ *and so*

$$a_1 a_2 \dots a_{k'-1} 10^{n_1} 1 \notin \mathcal{B}(X_\beta). \tag{5}$$

But $1 a_{k'-1} \dots a_2 a_1 \in \omega_+(0^\infty)$ *and so by* (4),

$$1 a_{k'-1} \dots a_2 a_1 \in \omega_+(0^\infty 1 b_k b_{k-1} \dots b_{n_1+2} 1 0^{n_1}) \tag{6}$$

Hence $0^\infty 1 b_k b_{k-1} \dots b_{n_1+2} 1 0^{n_1} 1 a_{k'-1} \dots a_2 a_1$ *is a left infinite* X_β^{-1}*-ray. Thus*

$$a_1 a_2 \dots a_{k'-1} 10^{n_1} 1 \in \mathcal{B}(X_\beta)$$

that is absurd by (5). *Thus there is no path in* X_β^{-1} *from* $\omega_+(0^\infty 1)$ *to* $\omega_+(0^\infty)$ *and so* $(X_0)_0^+ \neq (X_1)_0^+$.

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