Original Research Article

Methods of bilateral approximations for nonlinear eigenvalue problems

ABSTRACT

In this article, the research proposed by the author, the approach to the construction of methods and algorithms of bilateral approximations to the eigenvalues of nonlinear spectral problems, is continued. On the basis of Newton's method, some new algorithms of the bilateral approximations to their eigenvalues are constructed and substantiated.

Keywords: nonlinear eigenvalue problem, numerical algorithm, matrix determinant derivatives, bilateral approximations, alternating approximations, including approximation

1. INTRODUCTION

Nonlinear eigenvalue problems arise in many fields of natural sciences and engineering sciences. However, eigenvalue problems that are important to practice can very rarely be solved in a closed form and, as a rule, numerical methods need to be used to solve them. Most numerical methods simply provide approximation to their eigenvalues, but they do not allow to determine how far the calculated actual value from the exact. The class of self-adjoine eigenvalue problems is perhaps the most important class of the problems, because the numerous problems that arise in practice belong to this class. Since self-adjoint eigenvalue problems can have only real eigenvalues, the problem of obtaining approximation and the corresponding estimates of the accuracy of the approximation is equivalent to the definition (calculation) of the upper and lower bounds of eigenvalues.

As a rule, it is impossible to apply (generalize) those methods that exist for linear problems to find the upper and lower bounds of eigenvalues of nonlinear spectral problems. Namely: various variants of the method of intermediate problems (Weinstein's method) (see, for example [26], [2] – [4], [8], as well as a bibliography in [8], [11])), the Fichera method [7], as well as methods and algorithms based on inclusion theorems (see, for example, G. Temple [24], L. Collatz [6], and N. J. Lehmann [9], [10], H. Behnke [5], M. G. Marmorino [11]). Therefore, the concept and apparatus of interval analysis are used to construct methods of bilateral approximations (see, for example, [1], [12]).

This article is a continuation of the study proposed by the author of the approach to the construction of methods and algorithms of bilateral approximations to the eigenvalues of nonlinear with respect of spectral parameters the eigenvalue problems [17] – [20]. This approach does not use the concepts and apparatus of interval analysis.

The idea of the proposed approach is that for a continuous monotone in the neighborhood of a simple zero $\lambda^* \in [a,b]$ of some function $f:[a,b] \to R$ that describes the nonlinear equation, is constructed and explored some auxiliary function $g:[a,b] \to R$ that has the

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same zero as the function f and the necessary properties that allows to constract the iterative processes, which give monotone bilateral (alternate or inclused) approximations to the root of nonlinear equation [13] – [16].

In the framework of this approach, algorithms of the bilateral analogues of the Newton method for finding eigenvalues of nonlinear spectral problems are constructed and grounded. The conditions for the initial approximation are obtained, which ensure the alternate of approximations to the eigenvalue from both sides and guarantee the convergence of the iterative process.

2. STATEMENT OF THE PROBLEM AND SOME PRELIMINARY RESULTS

We consider the nonlinear eigenvalue problem

$$\mathbf{D}(\lambda)y = 0, \tag{2.1}$$

where $\mathbf{D}(\lambda)$ is a square matrix of order n, all elements of which are sufficiently smooth (at least twice continuously differentiable) functions of the parameter $\lambda \in R$, $y \in R^n$. The eigenvalues is sought as solutions of determinant equation

$$f(\lambda) \equiv \det \mathbf{D}(\lambda) = 0. \tag{2.2}$$

To determine the isolated eigenvalue of matrix $\mathbf{D}(\lambda)$ we proposed and justify the Newton-type iterative processes that give the alternate approximations to the root λ^* of the equation (2.2), ie

$$\lambda_0 < \lambda_2 < \ldots < \lambda_{2m} < \ldots < \lambda^* < \ldots < \lambda_{2m-1} < \ldots < \lambda_3 < \lambda_1$$

or (2.3)

$$\lambda_1 < \lambda_3 < \ldots < \lambda_{2m-1} < \ldots < \lambda^* < \ldots < \lambda_{2m} < \ldots < \lambda_2 < \lambda_0$$

and the included monotonous bilateral approximations to the root, i.e.

$$\mu_0 < \mu_1 < \ldots < \mu_{2m} < \ldots < \lambda^* < \ldots < \nu_{2m} < \ldots < \nu_1 < \nu_0 \tag{2.4}$$

without revealing in so doing the determinant $\det \mathbf{D}(\lambda)$. This means that the left hand side of equation (2.2) in explicit form is not set, but the algorithm of finding the functions $f(\lambda)$ and thein derivatives $f'(\lambda)$ and $f''(\lambda)$ at a fixed value of the parameter λ, using the *LU*-decomposition of the matrix $\mathbf{D}(\lambda)$ is proposed. This algorithm is based on the fact that the matrix $\mathbf{D}(\lambda)$ of the order n, in which at any given value $\lambda = \lambda_m$ the principal minors of all orders from 1 to $(n-1)$ differ from zero, by *LU* - decomposition can be written as

$$\mathbf{D}(\lambda) = \mathbf{L}(\lambda)\mathbf{U}(\lambda), \tag{2.5}$$

where $\mathbf{L}(\lambda)$ is the lower triangular matrix with single diagonal elements, and $\mathbf{U}(\lambda)$ is the upper triangular matrix. Then

$$f(\lambda) = \det \mathbf{L}(\lambda) \det \mathbf{U}(\lambda) = \prod_{i=1}^{n} u_{ii}(\lambda)$$

Since the elements of a square matrix $\mathbf{D}(\lambda)$ (and, therefore, the matrix $\mathbf{U}(\lambda)$) are differentiable function, with respect to λ, then for any λ we obtain that

$$f'(\lambda)=\sum_{k=1}^{n} v_{kk}(\lambda)\prod_{i=1,i\neq k}^{n} u_{ii}(\lambda),$$

$$f''(\lambda)=\sum_{k=1}^{n} w_{kk}(\lambda)\prod_{i=1,\, i\neq k}^{n} u_{ii}(\lambda)+\sum_{k=1}^{n} v_{kk}(\lambda)\left(\sum_{j=1,\, j\neq k}^{n} v_{jj}(\lambda)\prod_{i=1,\, i\neq k,\, i\neq j}^{n} u_{ii}(\lambda)\right) \tag{2.6}$$

where $v_{ii}(\lambda)=u'_{ii}(\lambda)$ and $w_{ii}(\lambda)=v'_{ii}(\lambda)$ are the elements of matrices $\mathbf{V}(\lambda)$ and $\mathbf{W}(\lambda)$ in such decompositions

$$\mathbf{D}'(\lambda)\equiv\mathbf{B}(\lambda)=\mathbf{M}(\lambda)\mathbf{U}(\lambda)+\mathbf{L}(\lambda)\mathbf{V}(\lambda),$$

$$\mathbf{D}''(\lambda)\equiv\mathbf{C}(\lambda)=\mathbf{N}(\lambda)\mathbf{U}(\lambda)+2\mathbf{M}(\lambda)\mathbf{V}(\lambda)+\mathbf{L}(\lambda)\mathbf{W}(\lambda).$$

In practice, the use of formulas (2.6) allows us to numerically calculate derivatives only for a given fixed parameter λ. Therefore, for calculation $f(\lambda_m)$, $f'(\lambda_m)$ and $f''(\lambda_m)$ it is necessary compute, for a fixed $\lambda=\lambda_m$, decompositions

$$\begin{aligned}&\mathbf{D}=\mathbf{LU},\\&\mathbf{B}=\mathbf{MU}+\mathbf{LV},\\&\mathbf{C}=\mathbf{NU}+\mathbf{2MV}+\mathbf{LW},\end{aligned} \tag{2.7}$$

whence we obtain

$$f(\lambda_m)=\prod_{i=1}^{n} u_{ii}, \quad f'(\lambda_m)=\sum_{k=1}^{n} v_{kk}\prod_{i=1,i\neq k}^{n} u_{ii},$$

$$f''(\lambda_m)=\sum_{k=1}^{n} w_{kk}\prod_{i=1,\, i\neq k}^{n} u_{ii}+\sum_{k=1}^{n} v_{kk}\left(\sum_{j=1,\, j\neq k}^{n} v_{jj}\prod_{i=1,\, i\neq k,\, i\neq j}^{n} u_{ii}\right). \tag{2.8}$$

The matrix elements in the decompositions (2.7) can be calculated using the the corresponding recurrence relations written in [21] (see also [17], [22], [23]).
Consequently, in order to calculate the derivatives in N points $\lambda=\lambda_m$, $m=1,2,..,N$ it is necessary to calculate N times the decomposition (2.7) and derivatives for each fixed $\lambda=\lambda_m$, $m=1,2,..,N$ using formulas (2.8).

So, not knowing the explicit dependence $f(\lambda)$ on λ, for any fixed λ we can find the value of $f(\lambda)$ and its derivatives. Therefore, for solving (2.2) we can use the methods that apply the derivatives, in particular, to construct the Newton-type methods, which give the bilateral approximation to the solution. This requires further study of the function $f(\lambda)$, which are realized later in the work.

3. AUXILIARY FUNCTION AND ITS PROPERTIES

Further, we demand $f(\lambda)$ to be a three times continuously differentiable function of real variable. By λ^* we denote an accurate simple root of equation (2.2) ($f(\lambda^*)=0$), in some neighborhood of which such behaviour of function $f(\lambda)$ is possible.

(A). Function $f(\lambda)$ is convex ($f''(\lambda)>0$) and its derivative is $f'(\lambda)<0$.
(B). Function $f(\lambda)$ is concave ($f''(\lambda)<0$) and its derivative is $f'(\lambda)<0$.

(C). Function $f(\lambda)$ is convex ($f''(\lambda)>0$) and its derivative is $f'(\lambda)>0$.
(D). Function $f(\lambda)$ is concave ($f''(\lambda)<0$) and its derivative is $f'(\lambda)>0$.

Along with $f(\lambda)$ we consider also a function

$$q(\lambda)=f(\lambda)/[f'(\lambda)]^2, \tag{3.1}$$

which obviously has the same zeros as the function $f(\lambda)$. It is easy to verify that $z(\lambda)$ is twice continuously differentiable at the point of λ^* for which the relation

$$q'(\lambda^*)=\frac{1}{f'(\lambda^*)} \qquad q''(\lambda^*)=-3\frac{f''(\lambda^*)}{[f'(\lambda^*)]^2} \tag{3.2}$$

is satisfied and which has the following properties.

Theorem 3.1. *Let* λ^* *be a simple real root of equation* (2.2) *in some neighborhood* U *of which for the function* $f(\lambda)$ *one of the conditions* (A) - (D) *is satisfied. Then there is a neighborhood of the root* $U_\varepsilon \subset U$ *, in which:*

1) when the condition (A) *is satisfied, the function* $q(\lambda)=f(\lambda)/[f'(\lambda)]^2$ *is a concave and monotonically decreasing function, its derivative* $q'(\lambda)<0$ *and it decreases monotonically;*

2) when the condition (B) *is satisfied, the function* $q(\lambda)=f(\lambda)/[f'(\lambda)]^2$ *is a convex and monotonically decreasing function, its derivative* $q'(\lambda)<0$ *and it increases monotonically.*

3) when the condition (C) *is satisfied, the function* $q(\lambda)=f(\lambda)/[f'(\lambda)]^2$ *is a concave and monotonically increasing function, its derivative* $q'(\lambda)>0$ *and it decreases monotonically;*

4) when the condition (D) *is satisfied, the function* $q(\lambda)=f(\lambda)/[f'(\lambda)]^2$ *is a convex and monotonically increasing function, its derivative* $q'(\lambda)>0$ *and it increases monotonically.*

Proof. Let $f(\lambda)$ be a decreasing and convex with respect to λ on U function, that is, $f'(\lambda)<0$ and $f''(\lambda)>0$ (the case (A)).

Since the function

$$s(\lambda)=\frac{2f(\lambda)f''(\lambda)}{(f'(\lambda))^2}$$

at the point $\lambda=\lambda^*$ is equal to zero, then because of continuity of $s(\lambda)$ there is such neighborhood of the root

$$U_\varepsilon(\lambda^*)=\left\{\lambda:\left|\lambda-\lambda^*\right|<\varepsilon\right\},$$

in which

$$|s(\lambda)|=\left|\frac{2f(\lambda)f''(\lambda)}{(f'(\lambda))^2}\right|\le t<1.$$

It follows that in the neighborhood $U_\varepsilon(\lambda^*)$ the function is $q'(\lambda)>0$, since

$$q'(\lambda)=\frac{1}{f'(\lambda)}\left[1-\frac{2f(\lambda)f''(\lambda)}{(f'(\lambda))^2}\right]. \tag{3.3}$$

Now from the mean value theorem, applied to differentiable functions $q(\lambda)$ on the interval $[\mu,\lambda]\in U_\varepsilon(\lambda^*)$ we obtain

$$q(\lambda)-q(\mu)=q'(\xi)(\lambda-\mu)\ , \qquad \xi\in[\mu,\lambda]\ ,$$

whence it follows that the function $z(\lambda)$ is a decreasing one.

Consider now the behavior of function $q'(\lambda)$ in the neighborhood $U_\varepsilon(\lambda^*)$, taking into account its image (3.3). For any $\lambda<\lambda^*$ and $\lambda>\lambda^*$ we obtain, respectively, the ratios

$$q'(\lambda^*)-q'(\lambda)=\frac{1}{f'(\lambda^*)}-\frac{1}{f'(\lambda)}+\frac{2f(\lambda)f''(\lambda)}{(f'(\lambda))^3}=\frac{f'(\lambda)-f'(\lambda^*)}{f'(\lambda)f'(\lambda^*)}+\frac{2f(\lambda)f''(\lambda)}{(f'(\lambda))^3}\ ,$$

$$q'(\lambda)-q'(\lambda^*)=\frac{1}{f'(\lambda)}-\frac{2f(\lambda)f''(\lambda)}{(f'(\lambda))^3}-\frac{1}{f'(\lambda^*)}=-\left[\frac{f'(\lambda)-f'(\lambda^*)}{f'(\lambda^*)}+\frac{2f(\lambda)f''(\lambda)}{(f'(\lambda))^3}\right]. \tag{3.4}$$

Since the first and second terms in (3.4) are positive, then from (3.4) it follows that in the neighborhood $U_\varepsilon\equiv U_\varepsilon(\lambda^*)$ the derivative $q'(\lambda)$ is decreasing, and therefore, the function $q(\lambda)$ is concave in this neighborhood of the root.

Similar statements about the function $q(\lambda)$ and its derivatives we obtain also for the cases (B), (C) and (D). But unlike the cases (A) and (C), in the cases (B) and (D) the function $q(\lambda)$ is convex. The theorem is proved.

Thus, Theorem 3.1 determines the properties of the function $z(\lambda)$, and Fig.1 illustrates its behavior depends on the properties of function $q(\lambda)$ in some neighborhood of the root λ^*.

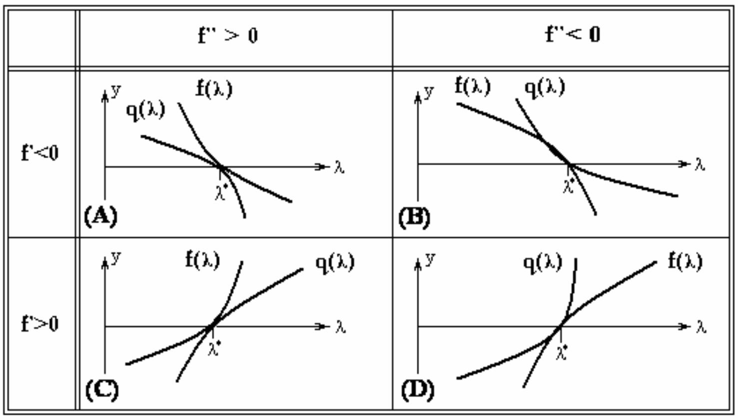

Fig.1. Behavior of the functions $f(\lambda)$ and $q(\lambda)$
in the neighborhood of a simple real root λ^* of functions $f(\lambda)$

Such character of the behavior of a function $q(\lambda)$ allows for us from the iterative formula

$$\lambda_{m+1}=\lambda_m-\frac{q(\lambda_m)}{q'(\lambda_m)},\qquad m=0,1,\ldots, \tag{3.5}$$

to get a monotone sequence of approximations to the root, moreover the iterative processes (3.5) and

$$\lambda_{m+1}=\lambda_m-\frac{f(\lambda_m)}{\operatorname{sgn} f'\cdot[f'(\lambda_m)]^2},\qquad m=0,1,\ldots, \tag{3.6}$$

have such monotonic properties.

Theorem 3.2. *If conditions (A) or (D) are satisfied in the neighborhood of the root* λ^**, then, starting with* $m=1$*, the sequence* $\{\lambda_m\}$*, defined by (3.5), monotonically decreases, and the sequence* $\{\lambda_m\}$*, defined by (3.6), monotonically increases*.

Theorem 3.3. *If conditions (B) or (C) are satisfied in the neighborhood of the root* λ^**, then, starting with* $m=1$*, the sequence* $\{\lambda_m\}$*, defined by (3.5), monotonically increases, and the sequence* $\{\lambda_m\}$*, defined by (3.6), monotonically decreases.*

The proofs of Theorems 3.2 and 3.3 are based on Theorem 3.1 and carried out by the method of mathematical induction according to a known scheme (see, for example, [6]).

4. BILATERAL ANALOGUES OF NEWTON METHOD

Using the properties of the function $q(\lambda)$, we construct a sequence of $\{\lambda_m\}$, which has the property (2.3).

For cases (A) and (D) we write the iterative process in the form

$$\begin{cases}\lambda_{2m+1}=\lambda_{2m}-\dfrac{f(\lambda_{2m})f'(\lambda_{2m})}{f'(\lambda_{2m})^2-2f(\lambda_{2m})f''(\lambda_{2m})},\\ \lambda_{2m+2}=\lambda_{2m+1}-\dfrac{f(\lambda_{2m+1})}{\operatorname{sgn} f'\cdot[f'(\lambda_{2m+1})]^2},\end{cases} \tag{4.1}$$

$$m=0,1,2,\ \ldots,\ \lambda_0\in(\lambda^*-\varepsilon,\lambda^*)$$

and for the cases (B) and (C) - in the form

$$\begin{cases}\lambda_{2m+1}=\lambda_{2m}-\dfrac{f(\lambda_{2m})}{\operatorname{sgn} f'\cdot[f'(\lambda_{2m})]^2},\\ \lambda_{2m+2}=\lambda_{2m+1}-\dfrac{f(\lambda_{2m+1})f'(\lambda_{2m+1})}{f'(\lambda_{2m+1})^2-2f(\lambda_{2m+1})f''(\lambda_{2m+1})},\end{cases} \tag{4.2}$$

$$m=0,1,2,\ \ldots,\ \lambda_0\in(\lambda^*-\varepsilon,\lambda^*).$$

Remarks 4.1. If the initial approximation is $\lambda_0\in(\lambda^*,\lambda^*+\varepsilon)$, then for the cases (A) and (D) iteration process (4.2) is required, and for cases (B) and (C) iteration process (4.1).

The following two theorems justify the bilateral convergence of iterative processes

Theorem 4.1. *Let* λ^* *is a simple real root of the equation (2.2) and let in some neighborhood of the root*

$$U_\varepsilon(\lambda^*) = \{\lambda : |\lambda - \lambda^*| < \varepsilon\},$$

in which

$$\left|\frac{2f(\lambda)f''(\lambda)}{[f'(\lambda)]^2}\right| < 1$$

for the three times continuously differentiable function $f(\lambda)$ *that describes equation (2.2), the condition (A) or (D) is fulfilled, and for the function* $q(\lambda) = f(\lambda)/\operatorname{sgn} f' \cdot [f'(\lambda)]^2$ *the conditions*

$$\left|\frac{q(\lambda_0)}{q'(\lambda_0)}\right| < \frac{2}{M_2} \qquad \text{for } \lambda_0 \in (\lambda^* - \varepsilon, \lambda^*), \tag{4.3}$$

$$\left|\frac{1}{q'(\lambda_1)}\right| > N \qquad \text{for } \lambda_1 \in (\lambda^*, \lambda^* + \varepsilon), \tag{4.4}$$

where

$$m_2 = \min_{\lambda \in U_\varepsilon(\lambda^*)} |q'(\lambda)|, \quad M_2 = \max_{\lambda \in U_\varepsilon(\lambda^*)} |q''(\lambda)|, \tag{4.5}$$

$$N = \max_{\lambda \in (\lambda^*, \lambda^* + \varepsilon)} \left|\frac{q(\lambda)q''(\lambda)}{q'(\lambda)^2}\right| .$$

is holds.
In addition, let the conditions to be met

$$t_0 = \frac{1}{2}\left(\frac{M_1 M_2^2}{m_1 m_2^2}\right)^{\frac{1}{3}} |\lambda_0 - \lambda^*| < 1, \quad t_1 = \frac{1}{2}\left(\frac{M_2 M_1^2}{m_2 m_1^2}\right)^{\frac{1}{3}} |\lambda_1 - \lambda^*| < 1, \tag{4.6}$$

where

$$m_1 = \min_{\lambda \in U_\varepsilon(\lambda^*)} |f'(\lambda)|, \quad M_1 = \max_{\lambda \in U_\varepsilon(\lambda^*)} |f''(\lambda)| . \tag{4.7}$$

Then the iterative process (4.1)*, starting with* $\lambda_0 \in (\lambda^* - \varepsilon, \lambda^*)$*, coincides to the* λ^* *on both sides*

$$\lambda_0 < \lambda_2 < \ldots < \lambda_{2m} < \lambda_{2m+2} < \ldots < \lambda^* < \ldots < \lambda_{2m+1} < \lambda_{2m-1} < \ldots < \lambda_3 < \lambda_1,$$

moreover, for the errors on the left hand and on the right hand from the root λ^* *the estimations*

$$|\lambda_{2m} - \lambda^*| < t_0^{4^m - 1} |\lambda_0 - \lambda^*| , \tag{4.8}$$

and

$$|\lambda_{2m-1} - \lambda^*| < t_1^{4^m - 1} |\lambda_1 - \lambda^*| . \tag{4.9}$$

are satisfied, respectively.

Proof. The application of Theorem 3.2 to the iterative process (4.1) guarantees placement of even approximations on the left of the root, and odd ones on the right of it. It is necessary to prove that the even approximations are monotonically increasing, while the odd ones are monotonically decreasing.. To do this, we first consider $\lambda_{2m+2} - \lambda_{2m}$. From (4.1) we obtain

$$\lambda_{2m+2} = \lambda_{2m} - \operatorname{sgn} f' \cdot q\left(\lambda_{2m} - \frac{q(\lambda_{2m})}{q'(\lambda_{2m})}\right) - \frac{q(\lambda_{2m})}{q'(\lambda_{2m})} \tag{4.10}$$

or

$$\lambda_{2m+2}-\lambda_{2m} = -\frac{q(\lambda_{2m})}{q'(\lambda_{2m})}\cdot\left[1+\operatorname{sgn} f'\cdot\frac{q''(\xi_{2m})}{2}\cdot\frac{q(\lambda_{2m})}{q'(\lambda_{2m})}\right], \tag{4.11}$$

$$\lambda_{2m}<\xi_{2m}<\lambda_{2m+1}.$$

We will prove that

$$\lambda_{2m+2}-\lambda_{2m}>0 \tag{4.12}$$

for any m by induction. For $m=0$ we have

$$\lambda_{2}-\lambda_{0} = -\frac{q(\lambda_{0})}{q'(\lambda_{0})}\cdot\left[1+\operatorname{sgn} f'\cdot\frac{q''(\xi_{0})}{2}\cdot\frac{q(\lambda_{0})}{q'(\lambda_{0})}\right], \tag{4.13}$$

$$\lambda_{0}<\xi_{0}<\lambda_{1}.$$

By the condition of the theorem $\lambda_0 \in (\lambda^* - \varepsilon, \lambda^*)$, therefore
<u>for case (**A**)</u> we have

$-\frac{q(\lambda_0)}{q'(\lambda_0)}>0$, $\operatorname{sgn} f' = -1$ and $q''(\xi_0)<0$ for any $\xi_0 \in U_\varepsilon$ (Theorem 3.1).

Now, taking into account that $|q''(\xi_0)|<M_2$, as well as the condition (4.3), we obtain

$$1+\operatorname{sgn} f'\frac{q''(\xi_0)}{2}\frac{q(\lambda_0)}{q'(\lambda_0)} = 1-\frac{q''(\xi_0)}{2}\frac{q(\lambda_0)}{q'(\lambda_0)} = 1-\frac{|q''(\xi_0)|}{2}\left(-\frac{q(\lambda_0)}{q'(\lambda_0)}\right) > 1-\frac{|q''(\xi_0)|}{M_2}>0.$$

Hence, from (4.13) it follows that $\lambda_2-\lambda_0>0$.

<u>In the case of **(D)** </u>we have

$-\frac{q(\lambda_0)}{q'(\lambda_0)}>0$, $\operatorname{sgn} f' = +1$ and $q''(\xi_0)>0$ for any $\xi_0 \in U_\varepsilon$ (Theorem 3.1).

and, taking into account that $q''(\xi_0)<M_2$, as well as the condition (4.3), we obtain similarly

$$1+\operatorname{sgn} f'\cdot\frac{q''(\xi_0)}{2}\frac{q(\lambda_0)}{q'(\lambda_0)} = 1+\frac{q''(\xi_0)}{2}\frac{q(\lambda_0)}{q'(\lambda_0)} > 1-\frac{q''(\xi_0)}{M_2}>0.$$

Hence, from (4.13) it follows that $\lambda_2-\lambda_0>0$.

Assume now that (4.11) is performed for $m=l-1>0$, that is, inequalities

$$\lambda_0<\lambda_2<\ldots<\lambda_{2l} \tag{4.14}$$

are satisfied, and we will prove that they are executed for $m=l$, i.e.

$$\lambda_{2l+2}-\lambda_{2l}>0. \tag{4.15}$$

For $m=l$, the expression (4.11) takes the form

$$\lambda_{2l+2}-\lambda_{2l}=-\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}\cdot\left[1+\operatorname{sgn} f'\cdot\frac{q'(\xi_{2l})}{2}\cdot\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}\right]. \tag{4.16}$$

From the fact that $\lambda_{2l}\in(\lambda^{*}-\varepsilon,\lambda^{*})$, the inequalities (4.14) is true , and also Theorem 3.1 is satisfied, we obtain that

$$-\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}>0,\qquad \left|\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}\right|<\left|\frac{q(\lambda_{0})}{q'(\lambda_{0})}\right| \quad \text{for any } \xi_{2l}\in U_{\varepsilon}. \tag{4.17}$$

<u>The case **(A)**.</u>
Taking into account that $f'<0$, and from the fact that $q'(\xi_{2l})<0$ for any $\xi_{2l}\in U_{\varepsilon}$ (Theorem 3.1) and $|q'(\xi_{2l})|<M_2$, as well as conditions (4.3) and (4.17), we obtain

$$1+\operatorname{sgn} f'\cdot\frac{q'(\xi_{2l})}{2}\cdot\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}=1-\frac{|q'(\xi_{2l})|}{2}\cdot\left|\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}\right|>1-\frac{|q'(\xi_{2l})|}{M_2}\geq 0.$$

Consequently, from (4.16) we obtain the inequality (4.15), which was necessary to prove for case (A).

<u>The case **(D)**.</u>
In this case, $f'>0$ and $q'(\xi_{2l})>0$ for any $\xi_{2l}\in U_{\varepsilon}$ (Theorem 3.1) therefore (4.16) can be written as

$$\lambda_{2l+2}-\lambda_{2l}=-\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}\cdot\left[1-\frac{q'(\xi_{2l})}{2}\left|\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}\right|\right]>-\frac{q(\lambda_{2l})}{q'(\lambda_{2l})}\cdot\left[1-\frac{q'(\xi_{2l})}{2}\left|\frac{q(\lambda_{0})}{q'(\lambda_{0})}\right|\right].$$

Now, taking into account that $q'(\lambda_{2l})<M_2$, as well as the condition (4.3), we obtain the inequality (4.15), which was necessary to prove for the case (D). Consequently, the even approximations for cases (A) and (D) increases monotonically.

Similarly, we prove that odd approximations are monotonically decreasing. To do this we consider $\lambda_{2m-1}-\lambda_{2m+1}$. From (4.1) we obtain

$$\lambda_{2m-1}-\lambda_{2m+1}=\operatorname{sgn} f'\cdot q(\lambda_{2m-1})+\frac{q(\lambda_{2m-1}-\operatorname{sgn} f'\cdot q(\lambda_{2m-1}))}{q'(\lambda_{2m-1}-\operatorname{sgn} f'\cdot q(\lambda_{2m-1}))}$$

or

$$\lambda_{2m-1}-\lambda_{2m+1}=\frac{q(\lambda_{2m-1})}{q'(\lambda_{2m-1})}+\operatorname{sgn} f'\cdot q(\lambda_{2m-1})\frac{q(\xi_{2m-1})q'(\xi_{2m-1})}{q'(\xi_{2m-1})^{2}}, \tag{4.18}$$

and by induction we will prove that

$$\lambda_{2m-1}-\lambda_{2m+1}>0\ . \tag{4.19}$$

For $m=1$ we get

$$\lambda_{1}-\lambda_{3}=\frac{q(\lambda_{1})}{q'(\lambda_{1})}+\operatorname{sgn} f'\cdot q(\lambda_{1})\frac{q(\xi_{1})q'(\xi_{1})}{q'(\xi_{1})^{2}}. \tag{4.20}$$

<u>The case **(A)**.</u>
In this case, $f' < 0$. Since $\lambda_1 \in (\lambda^*, \lambda^* + \varepsilon)$ (Theorem 3.2), taking into account the properties of $q(\lambda)$ and its derivatives (Theorem 3.1), we have

$$q(\lambda_1) < 0, \quad q'(\lambda) < 0, \quad q''(\lambda) < 0, \quad \forall \lambda \in U_\varepsilon(\lambda^*),$$

Then from (4.20) it follows that

$$\frac{q(\lambda_1)}{q'(\lambda_1)} - q(\lambda_1)\frac{q(\xi_1)q''(\xi_1)}{q'(\xi_1)^2} \geq 0 \qquad (4.21)$$

for any ξ_1 of the interval $\lambda^* < \xi_1 < \lambda_1$. It follows from this that $\lambda_1 - \lambda_3 > 0$ for any ξ_1 from the interval $\lambda^* < \xi_1 < \lambda_1$.

If ξ_1 belongs to interval $\lambda^* - \varepsilon < \xi_1 < \lambda^*$, then $q(\xi_1) \leq 0$. Now, taking into account the condition (4.4), the relation (4.20) can be given in the form

$$\left|\frac{q(\lambda_1)}{q'(\lambda_1)}\right| - |q(\lambda_1)|\left|\frac{q(\xi_1)q''(\xi_1)}{q'(\xi_1)^2}\right| = \left|\frac{1}{q'(\lambda_1)}\right| - \left|\frac{q(\xi_1)q''(\xi_1)}{q'(\xi_1)^2}\right| > \left|\frac{1}{q'(\lambda_1)}\right| - N \geq 0, \qquad (4.22)$$

from which it follows that $\lambda_1 - \lambda_3 > 0$ for ξ_1 from the interval $\lambda^* - \varepsilon < \xi_1 < \lambda^*$. Consequently, $\lambda_1 - \lambda_3 > 0$ on the entire interval $\lambda_0 < \xi_1 < \lambda_1$.

<u>The case (**D**).</u>
In this case, $f' > 0$. Since $\lambda_1 \in (\lambda^*, \lambda^* + \varepsilon)$ (Theorem 3.2), that taking into account the properties of $q(\lambda)$ and its derivatives (Theorem 3.1), we have

$$q(\lambda_1) < 0, \quad q'(\lambda) > 0, \quad q''(\lambda) > 0, \quad \forall \lambda \in U_\varepsilon(\lambda^*),$$

Then from (4.20) it follows that

$$\frac{q(\lambda_1)}{q'(\lambda_1)} + q(\lambda_1)\frac{q(\xi_1)q''(\xi_1)}{q'(\xi_1)^2} \geq 0 \qquad (4.23)$$

for any ξ_1 from the interval $\lambda^* < \xi_1 < \lambda_1$ on which $q(\xi_1) \geq 0$. From this it follows that $\lambda_1 - \lambda_3 > 0$ for any ξ_1 from the interval $\lambda^* < \xi_1 < \lambda_1$.

If ξ_1 belongs to the interval $\lambda^* - \varepsilon < \xi_1 < \lambda^*$, on which $q(\xi_1) \leq 0$, then, taking into account condition (4.4), the relation (4.20) can be given as (4.22), from which it follows that $\lambda_1 - \lambda_3 > 0$ and on the interval $\lambda^* - \varepsilon < \xi_1 < \lambda^*$. So, $\lambda_1 - \lambda_3 > 0$ over the entire interval $\lambda_0 < \xi_1 < \lambda_1$.

If ξ_1 belongs to the interval $\lambda^* - \varepsilon < \xi_1 < \lambda^*$, on which $q(\xi_1) \leq 0$, then, taking into account condition (4.4), the relation (4.20) can be given as (4.22), from which it follows that $\lambda_1 - \lambda_3 > 0$ and on the interval $\lambda^* - \varepsilon < \xi_1 < \lambda^*$. So, $\lambda_1 - \lambda_3 > 0$ over the entire interval $\lambda_0 < \xi_1 < \lambda_1$.

Suppose now that (4.18) is fulfilled for $m = l - 1 > 0$, that is, inequalities are satisfied

$$\lambda_{2m+1} < \lambda_{2m-1} < \ldots < \lambda_1 \qquad (4.24)$$

and we prove that it is satisfied for $m = l$, that is,

$$\lambda_{2l-1} - \lambda_{2l+1} > 0. \qquad (4.25)$$

For $m = l$, the expression (4.18) takes the form

$$\lambda_{2l-1} - \lambda_{2l+1} = \frac{q(\lambda_{2l-1})}{q'(\lambda_{2l-1})} + \operatorname{sgn} f' \cdot q(\lambda_{2l-1}) \frac{q(\xi_{2l-1})q''(\xi_{2l-1})}{q'(\xi_{2l-1})^2}.$$

$$\lambda_{2l} < \xi_{2l-1} < \lambda_{2l-1}$$

Since $\lambda_{2l+1} \in (\lambda^*, \lambda^* + \varepsilon)$ (Theorem 3.2), and also the expression (4.24) and Theorem 3.1 are satisfied, we obtain that

$$\left|\frac{1}{q'(\lambda_{2l+1})}\right| \geq \left|\frac{1}{q'(\lambda_{2l-1})}\right| > \ldots > \left|\frac{1}{q'(\lambda_1)}\right|$$

Again, since $\lambda_{2l-1} \in (\lambda^*, \lambda^* + \varepsilon)$ (Theorem 3.2), then, taking into account the properties of the function $q(\lambda)$ and its derivatives (Theorem 3.1), as well as the fulfilment of condition (4.4), the inequality (4.25) is satisfied for any $\xi_{2l-1} \in (\lambda_{2l}, \lambda_{2l-1})$. Consequently, odd approximations are monotonically decreasing.

Thus, we proved that the even approximations are monotonically increasing, while the odd ones are monotonically decreasing. It remains to prove that these approximations coincide to the root of both sides.

To do this, we again consider the relation (4.10). Note that (4.10) can be regarded as a partial case of a simple iteration method

$$\lambda_{2m+2} = \varphi(\lambda_{2m}), \qquad m = 0, 1, \ldots,$$

where

$$\varphi(\lambda) = \lambda - \frac{q(\lambda)}{q'(\lambda)} - q\left(\lambda - \frac{q(\lambda)}{q'(\lambda)}\right).$$

As you know, the iterative process is a process of n n-th order, if

$$\varphi'(\lambda^*) = 0, \ \varphi''(\lambda^*) = 0, \ \ldots, \ \varphi^{(k-1)}(\lambda^*) = 0, \ \varphi^{(k)}(\lambda^*) \neq 0.$$

Since $\varphi(\lambda^*) = \lambda^*$, then (4.10) we write in the form

$$\lambda_{2m+2} - \lambda^* = \varphi(\lambda_{2m}) - \varphi(\lambda^*)$$

and, using the Taylor formula, we obtain

$$\varphi(\lambda_{2m}) - \varphi(\lambda^*) = \varphi'(\lambda^*)(\lambda_{2m} - \lambda^*) + \frac{1}{2!}\varphi''(\lambda^*)(\lambda_{2m} - \lambda^*)^2 + \\ + \frac{1}{3!}\varphi'''(\lambda^*)(\lambda_{2m} - \lambda^*)^3 + \frac{1}{3!}\int_{\lambda^*}^{\lambda_{2m}} \varphi^{IV}(\lambda)(\lambda_{2m} - \lambda^*)^3 dx. \qquad (4.26)$$

It's easy to make sure that in our case

$$\varphi'(\lambda^*) = 0,\ \varphi''(\lambda^*) = 0,\ \varphi'''(\lambda^*) = 0,\ \varphi^{(IV)}(\lambda^*) \neq 0 .$$

Since the function $(\lambda - \lambda^*)$ does not change the sign on the segment of integrating, you can use the formula of the mean value and (4.26) write in the form

$$\left|\varphi(\lambda_{2m}) - \varphi(\lambda^*)\right| = \frac{1}{4!}\left|\varphi^{IV}(\xi_{2m})\right| \cdot \left|\lambda_{2m} - \lambda^*\right|^4. \tag{4.27}$$

On the other hand, the function $\varphi(\lambda)$ can be regarded as the iterated function [25], i.e.

$$\varphi(\lambda) = \varphi_1(\varphi_2(\lambda)),$$

where $\varphi_1(\lambda) = x - f(\lambda)/[f'(\lambda)]^2,\ \varphi_2(\lambda) = \lambda - q(\lambda)/q'(\lambda)$.

Since for Newton's method, taking into account (4.7) and (4.5) the inequalities

$$\left|\varphi_1(\lambda_m) - \varphi(\lambda^*)\right| \le \frac{M_1}{2m_1}\left|\lambda_m - \lambda^*\right|^2,$$

$$\left|\varphi_2(\lambda_m) - \varphi(\lambda^*)\right| \le \frac{M_2}{2m_2}\left|\lambda_m - \lambda^*\right|^2,$$

are valid, then for the iterated function $\varphi(\lambda)$ we get that

$$\left|\varphi(\lambda_m) - \varphi(\lambda^*)\right| \le \frac{M_1}{2m_1}\frac{M_2^2}{4m_2^2}\left|\lambda_m - \lambda^*\right|^4.$$

Consequently, (4.27) will be written in the form

$$\left|\lambda_{2m+2} - \lambda^*\right| = \left|\varphi(\lambda_{2m}) - \varphi(\lambda^*)\right| \le \frac{M_1 M_2^2}{8m_1 m_2^2} \cdot \left|\lambda_{2m} - \lambda^*\right|^4.$$

Now, when the first of the conditions (4.6) of Theorem is satisfied, we obtain the estimate (4.8), from which it follows the convergence from the left hand of the root. The estimate (4.8) is proved by the method of induction according to the known scheme [see, for example, [6]].

Similarly, an estimate (4.9) is established, from which it follows the convergence of the right hand of the root. Consequently, the Theorem is proved.

The Theorem on the convergence of the iterative process (4.2) for cases (B) and (C) is formulated as follows.

Theorem 4.2. *Let* λ^* *is a simple real root of the equation* (2.2) *and let in some neighborhood of the root*

$$U_\varepsilon(\lambda^*) = \{\lambda : |\lambda - \lambda^*| < \varepsilon\},$$

in which

$$\left|\frac{2f(\lambda)f''(\lambda)}{[f'(\lambda)]^2}\right| < 1$$

for the three times continuously differentiable function $f(\lambda)$ *that describes equation* (2.2), *the condition (A) or (D) is fulfilled, and for the function* $q(\lambda) = f(\lambda)/\operatorname{sgn} f' \cdot [f'(\lambda)]^2$ *the conditions*

$$\left|\frac{1}{q'(\lambda_0)}\right| > N_1 \qquad \textit{for } \lambda_0 \in (\lambda^* - \varepsilon, \lambda^*),$$

$$\left|\frac{q(\lambda_1)}{q'(\lambda_1)}\right| < \frac{2}{M_2} \qquad \textit{for } \lambda_1 \in (\lambda^*, \lambda^* + \varepsilon)\,,)$$

where

$$m_2 = \min_{\lambda \in U_\varepsilon(\lambda^*)} |q'(\lambda)|\,, \quad M_2 = \max_{\lambda \in U_\varepsilon(\lambda^*)} |q''(\lambda)|\ ,$$

$$N_1 = \max_{\lambda \in (\lambda^* - \varepsilon, \lambda^*)} \left|\frac{q(\lambda) q''(\lambda)}{q'(\lambda)^2}\right|.$$

is holds.
In addition, let the conditions to be met

$$t_0 = \frac{1}{2}\left(\frac{M_1 M_2^2}{m_1 m_2^2}\right)^{\frac{1}{3}} |\lambda_0 - \lambda^*| < 1\,, \quad t_1 = \frac{1}{2}\left(\frac{M_2 M_1^2}{m_2 m_1^2}\right)^{\frac{1}{3}} |\lambda_1 - \lambda^*| < 1\,,$$

where

$$m_1 = \min_{\lambda \in U_\varepsilon(\lambda^*)} |f'(\lambda)|\,, \quad M_1 = \max_{\lambda \in U_\varepsilon(\lambda^*)} |f''(\lambda)|\ .$$

Then the iterative process (4.2)*, starting with* $\lambda_0 \in (\lambda^* - \varepsilon, \lambda^*)$, *coincides to the* λ^* *on both sides*

$$\lambda_0 < \lambda_2 < \ldots < \lambda_{2m} < \lambda_{2m+2} < \ldots < \lambda^* < \ldots < \lambda_{2m+1} < \lambda_{2m-1} < \ldots < \lambda_3 < \lambda_1\,,$$

moreover, for the errors on the left hand and on the right hand from the root λ^* *the estimations*

$$|\lambda_{2m} - \lambda^*| < t_0^{4^m - 1} |\lambda_0 - \lambda^*|\,,$$

and

$$|\lambda_{2m-1} - \lambda^*| < t_1^{4^m - 1} |\lambda_1 - \lambda^*|\ .$$

are satisfied, respectively.

The scheme of proof of Theorem 4.2 is similar to the scheme of proof of Theorem 4.1.

Remark 4.2. Two different iterative processes (4.1) and (4.2) have been used above to justify alternate approximations, starting with $n = 0$, ideally, when the behaviour of a function $f(\lambda)$ is known or easily investigated.

In practice, one of them can be used for all cases (A) - (D) and regardless of which side (left or right of the root λ^*) is the initial approximation λ_0, but then the alternate approximations comes at least from $n = 1$.

For example, if, to the $f(\lambda)$ that satisfying the condition (A) or (D), apply the iterative process (4.2), we obtain an alternate approximations to the root λ^* in the form

$$\lambda_0 < \lambda_1 < \lambda_3 < \ldots < \lambda_{2m-1} < \lambda_{2m+1} < \ldots < \lambda^* < \ldots < \lambda_{2m+2} < \lambda_{2m} < \ldots < \lambda_4 < \lambda_2$$

provided that $\lambda_0 < \lambda^*$ and

$$\lambda_1 < \lambda_3 < \ldots < \lambda_{2m-1} < \lambda_{2m+1} < \ldots < \lambda^* < \ldots < \lambda_{2m+2} < \lambda_{2m} < \ldots < \lambda_4 < \lambda_2 < \lambda_0\,,$$

If $\lambda_0 > \lambda^*$.

5. ALGORITHMS AND DISCUSSION

Note that the iterative process, for example (4.1), which provides a bilateral approximation to its own value, taking into account the relations (2.8), will take the form

$$\lambda_{2m+1} = \lambda_{2m} - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}}\right) \Big/ \sum_{k=1}^{n} \left(\left(\frac{v_{kk}}{u_{kk}}\right)^2 - 2\frac{w_{kk}}{u_{kk}} + 2\frac{v_{kk}}{u_{kk}} \left(\sum_{i=1,\, i\neq k}^{n} \frac{v_{ii}}{u_{ii}} \right)\right),$$

$$\lambda_{2m+2} = \lambda_{2m+1} - \operatorname{sgn}\left(\sum_{k=1}^{n} v_{kk}^{0} \prod_{i=1, i\neq k} u_{ii}^{0}\right) \Big/ \prod_{i=1} \bar{u}_{ii} \left(\sum_{k=1}^{n} \frac{\bar{v}_{kk}}{\bar{u}_{kk}}\right)^2, \qquad (5.1)$$

$$m = 0,\ 1,\ 2,\ \ldots,$$

where u_{kk}, v_{kk}, w_{kk} are elements of the matrices $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ in the decompositions (2.7) for fixed $\lambda = \lambda_{2m}$, and $\bar{u}_{kk}, \bar{v}_{kk}$ and u_{kk}^{0}, v_{kk}^{0} are elements of the matrices of $\mathbf{U}, \mathbf{V}$ in the decompositions (2.7) for fixed $\lambda = \lambda_{2m+1}$ and $\lambda = \lambda_0$, respectively.

So, the algorithm can be written as follows:

Algorithm 1. Iterative process of alternating approximations

1. We set the initial approximation λ_0 to the s-th eigenvalue of the problem (2.2)
2. **for** $m = 0,1,2,\ldots$ until the accuracy is achieved **do**
3. if m is even
4. **than** calculate the values u_{kk}, v_{kk}, w_{kk} from the decomposition (2.7) for $\lambda = \lambda_{2m}$
5. calculate approximation to the eigenvalue λ_{2m+1} by the formula (5.1)
6. **else** calculate the values $\bar{u}_{kk}$, $\bar{v}_{kk}$ from the decomposition (2.7) for $\lambda = \lambda_{2m+1}$
7. calculate approximation to the eigenvalue λ_{2m+2} by the formula (5.1)
8. **end for** m.

From the algorithm it is seen that in order to obtain alternate approximations in each step of the algorithm it is necessary to refer to the algorithm of calculating the expansion (9).

In some cases, the algorithm constructed on the basis of iteration process of enclosing approximations is more optimum as to the number of accesses to the calculation of decomposition (9) [5]:

$$\mu_{m+1} = \mu_m - \frac{f(\mu_m) f'(\mu_m)}{f'(\mu_m)^2 - f(\mu_m) f''(\mu_m)},$$

$$\nu_{m+1} = \mu_m - \frac{f(\mu_m)}{f'(\mu_m)}, \qquad m = 0,\ 1,\ 2,\ \ldots, \qquad (5.2)$$

with the help of which we obtain the approximation of the enclosing approximations in the form

$$\lambda_0 = \mu_0 < \mu_1 < \mu_2 < \ldots < \mu_m < \ldots < \lambda_* < \ldots < \nu_m < \ldots < \nu_2 < \nu_1 \qquad (5.3)$$

or in the form

$$\lambda_0 = \mu_0 < \{\nu_1\} < \nu_2 < \ldots < \nu_m < \ldots < \lambda_* < \ldots < \mu_m < \ldots < \mu_2 < \mu_1,$$

using one initial approximation $\lambda_0 = \mu_0$ (in this case to the left hand of of the root λ_*).

If now again replace the values of the function and its derivatives at the desired points by the relations (2.8), then the iterative process (5.2) will look like

$$\mu_{m+1} = \mu_m - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}}\right) \Big/ \sum_{k=1}^{n} \left(\left(\frac{v_{kk}}{u_{kk}}\right)^2 - 2\frac{w_{kk}}{u_{kk}} + 2\frac{v_{kk}}{u_{kk}} \left(\sum_{i=1,\, i \neq k}^{n} \frac{v_{ii}}{u_{ii}} \right) \right)$$

$$\nu_{m+1} = \mu_m - \operatorname{sgn}\left(\sum_{k=1}^{n} v_{kk}^0 \prod_{i=1, i\neq k}^{n} u_{ii}^0 \right) \Big/ \prod_{i=1}^{n} u_{ii} \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}} \right)^2, \qquad (5.4)$$

$$m = 0, 1, 2, \dots,$$

where u_{kk}, v_{kk}, w_{kk} is the elements of the matrices $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ in the decompositions (2.7) at the fixed $\lambda = \lambda_{2m}$ and u_{kk}^0, v_{kk}^0 are elements of the matrices of $\mathbf{U}, \mathbf{V}$ in the decompositions (2.7) for fixed $\lambda = \lambda_0$.

So, it is proposed the following algorithm for finding the eigenvalues of a nonlinear spectral problem:

Algorithm 2. An iterative process of enclosed approximations

1. We set the initial approximation $\lambda_0 = \mu_0$ to the s-th eigenvalue of the problem (2.2)
2. **for** $m = 0, 1, 2, \dots$ until the accuracy is achieved **do**
3. calculate the values u_{kk}, v_{kk}, w_{kk} from the decomposition (2.7) at the $\lambda = \mu_m$
4. calculate approximation to the eigenvalue μ_{m+1} and ν_{m+1} by the formula (5.4)
5. **end for** m.

Consequently, we see that by algorithm 2, unlike algorithm 1, two approximations (from left hand of the root and from righ thand of the root) are calculated, for one access to the calculation of decomposition (2.8).

Now, we consider the application of the proposed algorithms to finding the generalized eigenvalues of a linear homogeneous integral equation, whose kernel analytically (nonlinear) depends on the spectral parameter [21]:

$$v(\xi, \lambda) = T(\lambda) v(\xi, \lambda) \equiv \int_{-1}^{1} \frac{F(\xi')}{f_0(\xi', \lambda)} K(\xi, \xi', \lambda) v(\xi', \lambda)\, d\xi',$$

where $F(\xi)$ is a continuous on the interval $[-1, +1]$ a real nonnegative function,

$$K(\xi, \xi', \lambda) = \frac{\sin \lambda (\xi - \xi')}{\pi(\xi - \xi')},$$

$$f_0(\xi, \lambda) = \int_{-1}^{1} F(\xi') K(\xi, \xi', \lambda)\, d\xi'.$$

The equation arises in finding the points of a possible branching of the connections of a nonlinear integral equation

$$f(\xi, \lambda) = \int_{-1}^{1} F(\xi') K(\xi, \xi', \lambda) \cdot \exp\{ i \arg f(\xi', \lambda) \}\, d\xi',$$

which is obtained as a result of variational formulation of the synthesis problems , in particular, of linear antennas for a given amplitude directivity pattern.

Having made not complicated transformations and applying the Quadrature Gaussian formula to integral operator $T(\lambda)$, we obtain a matrix self-adjoined eigenvalue problem [21]

$$D_n(\lambda)u \equiv T_n(\lambda)u - I_n u,$$

where I_n is a unit matrix in n-dimensional space.

Eigenvalues are sought as solutions to a determinant equation

$$f(\lambda) \equiv \det D_n(\lambda) = 0. \qquad (5.5)$$

To illustrate the operation of the algorithms 1 and 2 for a given function $F(\xi) = 1$, in Table. 1 shows the value of successive approximations of the parameter to the first root of the equation (5.5). For the function $F(\xi) = 1$, the first root of equation (5.5) can be calculated precisely and it is equal π, which allows us to compare the approximate solution with the exact one.

Table 1. Successive approximations to the first eigenvalue ($\lambda = \pi \approx 3.141593$)

(m)	Algorithm 1	Algorithm 2		
	$\lambda^{(m)}$	$\mu^{(m)}$	$\nu^{(m)}$	$\lambda^{(m)}$
0	2.0	2.0		
1	3.087993	3.087993	3.186991	3,137492
2	3.178098	3.137563	3.168124	3,152844
3	3.139249	3.141567	3.141618	3.141593
4	3.141601	3.141593	3.141593	3.141593
5	3.141593	-	-	-
6	3.141593			

From Tabl. 1, we see that algorithms 1 and 2 are effective both in the rate of convergence and in the generation of successive bilateral approximations to the eigenvalue.

6. CONCLUSION

Approbation of the constructed algorithms on model and physical problems, in particular on the one presented in the article, shows their reliability and efficiency, as well as the advantages compared with the usual method of Newton in the sense that at each step of the iterative process, we obtain two-sided estimates of the required solution, and therefore, at each step we get comfortable a posteriori estimates of errors.

The proposed approach can be applied also to the linear eigenvalue problems with respect to the spectral parameter, and if its is compared with existing approaches mentioned in the introduction for obtaining lower bounds of eigenvalues of self-adjoint spectral problems, then this approach has significant advantages which are mentioned in [19], namely: does not require construction an auxiliary operator with a known spectrum as in the method of intermediate operators, and also does not require knowledge of the lower bound of the next eigenvalue (assuming that the eigenvalues are arranged in ascending order) as in the algorithms, based on inclusion theorems.

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