Geometrical Properties and exact solutions of Three (3+1)-dimensional nonlinear evolution equations in Mathematical Physics using different expansion methods

Abstract: In this article, A Variation of $(\boldsymbol{G'/G})$-Expansion Method and $\left(\boldsymbol{G'/G^2}\right)$-Expansion Method have been applied to find the traveling wave solutions of the (3+1)-dimensional Zakhrov-Kuznetsov (ZK) equation, the (3+1)-dimensional Potential-YTSF Equation and the (3+1)-dimensional generalized Shallow water equation. The efficiency of these methods for finding the exact solutions have been demonstrated. As a result, some new exact traveling wave solutions are obtained which include solitary wave solutions. It is shown that the methods are effective and can be used for many other nonlinear evolution equations (NLEEs) in mathematical physics.
Keywords: Nonlinear partial differential equations, A Variation of (G'/G)-expansion method, (G'/G^2)-expansion method, travelling wave solutions, the (3+1)-dimensional Zakhrov-Kuznetsov equation , the (3+1)-dimensional Potential-YTSF Equation and the (3+1)-dimensional generalized Shallow water equation ,Gaussian curvature, Mean curvature .

1.Introduction:

Nowadays NLEEs have been the subject of all-embracing studies in various branches of nonlinear sciences. A special class of analytical solutions named traveling wave solutions for NLEEs have a lot of importance, because most of the phenomena that arise in mathematical physics and engineering fields can be described by NLEEs. NLEEs are frequently used to describe many problems of chemically reactive materials, in physics the heat flow and the wave propagation phenomena, quantum mechanics, fluid mechanics, plasma physics, propagation of shallow water waves, optical fibers, biology, solid state physics, chemical kinematics, geochemistry, meteorology, electricity etc. Therefore investigation traveling wave solutions are becoming more and more attractive in nonlinear sciences day by day.there are different methods for solving these equations such as the inverse scattering transform method [1], the exp-function method [2-4], the Hirota's bilinear operators [5], the Weierstrass elliptic function method [6], the Jacobi elliptic function method [7, 8], the homogeneous balance method [9], the variation of (G'/G)-Expansion Method [10].

Zayed [11,12] proposed an alternative approach of the (G'/G)-expansion method, A. R. Shehata[13]used the modified (G'/G)-expansion method.

Guo and Zhou [14] presented the extended the (G'/G)-expansion method . Liu and Niuj [15] A generalized (G'/G)-expansion method .Zhang [16] proposed the modified (G'/G)-expansion method . Recently we have considered the (2+1)-Dimensional Broer-Kaup-Kuperschmidt Equation and have obtained several new exact solutions using an extension of (G'/G)-expansion method[17]. There is (G'/G^2)-expansion method [18] that has been recently proposed, this can be applied to various nonlinear equations and this also gives a few new kinds of solutions.

In this paper, by using a variation of the(G'/G)-expansion method and (G'/G^2)-expansion method, we applied them on some nonlinear partial differential equations, namely the (3+1)-dimensional Zakhrov-Kuznetsov equation , the (3+1)-dimensional Potential-YTSF Equation and the (3+1)-dimensional generalized Shallow water equation and find out the exact travelling wave solutions then we study its geometrical properties.

2. Analysis for the variation of (G'/G)-expansion method:

Suppose we have the following nonlinear partial differential equation:

$$F\left(u\,,u_t\,,u_x\,,u_{tt}\,,u_{xt}\,,u_{xx}\,,u_{xy}\,,u_{yy}\,,u_{yt}\,,u_{zz}\,,u_{zt}\,,u_{zx}\,,u_{zy}\,,\ldots\right)=0, \qquad (2.1)$$

where $u = u(x, y, z, t)$ is an unknown function, F is polynomial in $u = u(x, y, z, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The method is given in the following steps.

Step 1. The travelling wave variable :

$$u(x, y, z, t) = u(\xi), \quad \xi = x + y + z - Vt\,, \qquad (2.2)$$

where V is a constant represents the speed of the traveling wave transformation to be determined later, the traveling wave transformation permits us reducing Eq. (2.1) into an ordinary differential equation in the form:

$$P(u\,, u', u'', u''', \ldots) = 0. \qquad (2.3)$$

Where prime stands for ordinary derivative with respect to ξ and P is a polynomial in

$u = u(\xi)$ and its derivatives.

Step 2. For simplicity , if it is possible we integrate Eq.(2.3) term by term one or more times yields constant(s) of integration.

Step 3. Assume that the solution of Eq.(2.3) can be expressed in the following form:

$$u(\xi) = \sum_{i=0}^{m} a_i (G'/G)^i + \sum_{i=1}^{m} b_i (G'/G)^{i-1}(F'/F), \qquad (2.4)$$

where $G = G(\xi)$ and $F = F(\xi)$ expresses the solution of the coupled Riccati equation,

$$G'(\xi) = -G(\xi).F(\xi), \qquad (2.5)$$

$$F'(\xi) = 1 - F(\xi)^2\ , \qquad (2.6)$$

where prime denotes derivative with respect to ξ , $a_i(i=0,1,\dots,m), b_i(i=1,2,\dots,m)$ are constants to be determined later.

These governing equations lead us two types of general solutions:

$$G(\xi)=\pm\operatorname{sech}(\xi),\ \ F(\xi)=\tanh(\xi), \tag{2.7}$$

$$G(\xi)=\pm\operatorname{csch}(\xi),\ \ F(\xi)=\coth(\xi). \tag{2.8}$$

Step 4. By considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(2.3) we can find the positive integer m as follows:
If $D[u(\xi)]=m$, then $D\left[u^r\left(\frac{d^q u}{d\xi^q}\right)^s\right]=mr+s(q+m)$, where D denotes the degree of the expression.

Step 5. Substituting Eq.(2.4) into Eq.(2.3) and using Eq.(2.5) and Eq.(2.6), collecting all terms with the same order of (G'/G) or (F) together, left-hand side of Eq.(2.3) is converted into another polynomial in (G'/G) or (F). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_i(i=0,1,\dots,m), b_i(i=1,2,\dots,m)$, and V .

Step 6. Determining the constants $a_i(i=0,1,\dots,m), b_i(i=1,2,\dots,m)$ and V by solving the algebraic equations in step 5. As the general solutions of Eq.(2.5) and Eq.(2.6) are already known to us ,then substituting $a_i(i=0,1,\dots,m), b_i(i=1,2,\dots,m)$, V and the general solutions of Eq.(2.5) and Eq.(2.6), we obtain the travelling wave solutions of Eq.(2.1).

3. Analysis for the $\left(\frac{G'}{G^2}\right)$-expansion method:

Suppose we have the following nonlinear partial differential equation:

$$F\left(u,u_t,u_x,u_{tt},u_{xt},u_{xx},u_{xy},u_{yy},u_{yt},u_{zz},u_{zt},u_{zx},u_{zy},\dots\right)=0, \tag{3.1}$$

where $u=u(x,y,z,t)$ is an unknown function, F is polynomial in $u=u(x,y,z,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The method is given in the following steps.

Step 1. The travelling wave variable :

$$u(x,y,z,t)=u(\xi),\quad \xi=x+y+z-Vt, \tag{3.2}$$

where V is a constant represents the speed of the traveling wave transformation to be determined later, permits us reducing Eq. (2.1) into an ordinary differential equation in the form:

$$P(u,u',u'',u''',\dots)=0. \tag{3.3}$$

Where prime stands for ordinary derivative with respect to ξ and P is a polynomial in

$u=u(\xi)$ and its derivatives.

Step 2. For simplicity , if it is possible we integrate Eq.(3.3) term by term one or more times yields constants of integration.

Step 3. The formal solution of ODE can be written as follows:

$$u(\boldsymbol{\xi}) = \boldsymbol{a_0} + \sum_{n=1}^{N} \boldsymbol{a_n} \left(\frac{G'}{G^2}\right)^{\boldsymbol{n}} + \boldsymbol{b_n} \left(\frac{G'}{G^2}\right)^{-\boldsymbol{n}}, \tag{3.4}$$

$$\left(\frac{G'}{G^2}\right)' = \mu + \lambda \left(\frac{G'}{G^2}\right)^2 \tag{3.5}$$

In Eq. (3.4), $\lambda \neq 0, \mu \neq 1$ are integers and $a_0, a_n, b_n, (n = 1,2, \dots, N)$ are constants to be determined.

The value of positive integer N is easy to find by balancing the highest order derivative and nonlinear terms appearing in Eq.(3.3).

step 4. substituting Eq. (3.4) and use Eq. (3.5) into Eq.(3.3),collect the coefficients with the same order of $\left(\frac{G'}{G^2}\right)^i$, $(i = 0, \pm 1, \pm 2, \dots)$ and set the coefficients to zero, nonlinear all powers algebraic equations are acquired. Solutions to the resulting algebraic system are derived by using the $\left(\frac{\mathbf{G'}}{\mathbf{G^2}}\right)$-expansion method with the aid of Maple.

step 5. On the basis of the general solutions to Eq.(3.5), the ratio $\left(\frac{\mathbf{G'}}{\mathbf{G^2}}\right)$ can be divided into three cases, i.e.

$$\frac{\mathbf{G'}}{\mathbf{G^2}} = \sqrt{\frac{\mu}{\lambda}} \left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D \sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C \sin\sqrt{\mu\lambda}\xi}\right), \mu\lambda > 0, \tag{3.6}$$

$$\frac{\mathbf{G'}}{\mathbf{G^2}} = -\frac{\sqrt{|\mu\lambda|}}{\lambda} \left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C \cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C \cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) - D}\right), \mu\lambda < 0, \tag{3.7}$$

$$\frac{\mathbf{G'}}{\mathbf{G^2}} = -\frac{C}{\lambda(C\xi + D)}, \mu = 0, \lambda \neq 0 \tag{3.8}$$

In the above expressions C and D are nonzero constants. Three types of solution for Eq. (3.1) can be obtained by putting the values of $a_0, a_n, b_n, (n = 1,2, \dots, N)$ and the ratios (3.6)-(3.8) into Eq.(3.4).

4. Applications of the methods:

Here we use the above two methods respectively

4.1 Example 1: The (3+1)-dimensional Zakhrov-Kuznetsov equation:

Here, we study the (3+1)-dimensional Zakhrov-Kuznetsov equation in the form:

$$u_t + auu_x + u_{xx} + u_{yy} + u_{zz} = 0\,, \tag{4.1.1}$$

where a is a positive constant.

The traveling wave transformation equation $u(\xi) = u(x, y, z, t), \xi = x + y + z - Vt$ transform Eq.(4.1.1) to the following ordinary differential equation:

$$-Vu' + auu' + 3u'' = 0. \tag{4.1.2}$$

Now integrating Eq.(4.1.2) with respect to ξ once, we have

$$c - Vu + \frac{1}{2}au^2 + 3u' = 0, \qquad (4.1.3)$$

Where c is a constant of integration. Balancing the highest-order derivative u' and the nonlinear term u^2, from Eq.(4.1.3), yields $2m = m + 1$ which gives $m = 1$.

Hence for $m = 1$ Eq.(2.4) reduces to

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + b_1\left(\frac{F'}{F}\right),$$

$$= a_0 - a_1 F + b_1(F^{-1} - F), \qquad (4.1.4)$$

Substituting Eq. (4.1.4) into Eq. (4.1.3) , collecting the coefficients of $(F)^i (i = 0, \pm1, \pm2)$,and letting it be zero, yields a set of simultaneous algebraic equations for a_0, a_1, b_1, V and c

To solve this set of algebraic equations for a_0, a_1, b_1, V and c by using of Maple, we get,

Case 1:

$$c = \frac{1}{2}\frac{a^2a_0^2 - 36}{a}, \; a_1 = -\frac{6}{a}, \; b_1 = 0, \; V = aa_0, \qquad (4.1.5)$$

where a_0 is arbitrary.

Case 2:

$$c = \frac{1}{2}\frac{a^2a_0^2 - 144}{a}, \; a_1 = -\frac{12}{a}, \; b_1 = \frac{6}{a}, \; V = aa_0, \qquad (4.1.6)$$

where a_0 is arbitrary.

Substituting Eqs.(4.1.5),(4.1.6) into Eq.(4.1.4) we get two types of the travelling wave solutions of Eqs.(4.1.1) as follows:

According to case 1.

Type 1:

Class I:

$$u_{11}(x,t) = a_0 + \frac{6}{a}\tanh(x + y + z - aa_0 t) \qquad (4.1.7)$$

Class II:

$$u_{12}(x,t) = a_0 + \frac{6}{a}\coth(x + y + z - aa_0 t), \qquad (4.1.8)$$

According to case 2.

Type 2:

Class I:

$$u_{21}(x,t) = a_0 + \frac{6}{a}\tanh(x + y + z - aa_0 t) + \frac{6}{a}\coth(x + y + z - aa_0 t) \qquad (4.1.9)$$

Class II:

$$u_{22}(x,t) = a_0 + \frac{6}{a}\coth(x + y + z - aa_0 t) + \frac{6}{a}\tanh(x + y + z - aa_0 t) \qquad (4.1.10)$$

The solutions for $\frac{G'}{G^2}-$ expansion method can be expressed as follows:

$$u(\boldsymbol{\xi})=\boldsymbol{a_0}+\boldsymbol{a_1}\left(\frac{\boldsymbol{G'}}{\boldsymbol{G^2}}\right)+\boldsymbol{b_1}\left(\frac{\boldsymbol{G'}}{\boldsymbol{G^2}}\right)^{-1}, \tag{4.1.11}$$

Where a_0, a_1, b_1 are unknown constants. We substitute Eq.(4.1.11) into (4.1.3) along with Eq.(3.5) to collect all the coefficients with the same power of $\left(\frac{G'}{G^2}\right)^i$, $(i=0,\pm1,\pm2,\dots)$. From Eq.(4.1.11) each coefficient of $\left(\frac{G'}{G^2}\right)^i$ is set to zero, and system of algebraic equations about a_0, a_1, b_1 is attained as follows:

The following results are obtained upon solving the above system of algebraic equations using Maple

Case 1:

$$\boldsymbol{c}=\frac{1}{2}\frac{\boldsymbol{a^2a_0{}^2+36\mu\lambda}}{\boldsymbol{a}}, \boldsymbol{V}=\boldsymbol{aa_0},\ \boldsymbol{a_0}=\boldsymbol{a_0},\ \boldsymbol{a_1}=\boldsymbol{0}, \boldsymbol{b_1}=\frac{\boldsymbol{6\mu}}{\boldsymbol{a}}$$

Case 2:

$$\boldsymbol{c}=\frac{1}{2}\frac{\boldsymbol{a^2a_0{}^2+36\mu\lambda}}{\boldsymbol{a}}, \boldsymbol{V}=\boldsymbol{aa_0},\ \boldsymbol{a_0}=\boldsymbol{a_0},\ \boldsymbol{a_1}=-\frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}, \boldsymbol{b_1}=\boldsymbol{0}$$

Case 3:

$$\boldsymbol{c}=\frac{1}{2}\frac{\boldsymbol{a^2a_0{}^2+144\mu\lambda}}{\boldsymbol{a}}, \boldsymbol{V}=\boldsymbol{aa_0},\ \boldsymbol{a_0}=\boldsymbol{a_0},\ \boldsymbol{a_1}=-\frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}, \boldsymbol{b_1}=\frac{\boldsymbol{6\mu}}{\boldsymbol{a}}$$

In Eq.(4.1.11) we substitute the above cases along with ratios (3.6)-(3.8), and three groups of solutions for Eq.(3.1) exist.

Solution 1: When $\boldsymbol{\mu\lambda}>0$, the trigonometric solution corresponding to case 1 can be expressed as

$$u_{11}=a_0+\frac{\boldsymbol{6\mu}}{\boldsymbol{a}}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi+D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi-C\sin\sqrt{\mu\lambda}\xi}\right)\right)^{-1} \tag{4.1.12}$$

When $\boldsymbol{\mu\lambda}<0$ the hyperbolic solution corresponding to case 1 can be expressed as

$$u_{12}=a_0+\frac{\boldsymbol{6\mu}}{\boldsymbol{a}}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right)+C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right)+D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right)+C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right)-D}\right)\right)^{-1} \tag{4.1.13}$$

When $\mu=0$, $\lambda\neq0$, the rational solution corresponding to case 1 can be expressed as

$$u_{13}=a_0 \tag{4.1.14}$$

Where $\xi=x+y+z-aa_0t$

Solution 2:

When $\boldsymbol{\mu\lambda} > 0$, the trigonometric solution corresponding to case 2 can be expressed as

$$u_{21} = a_0 - \frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right) \quad (4.1.15)$$

When $\boldsymbol{\mu\lambda} < 0$ the hyperbolic solution corresponding to case 2 can be expressed as

$$u_{22} = a_0 - \frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) - D}\right)\right) \quad (4.1.16)$$

When $\mu = 0$, $\lambda \neq 0$, the rational solution corresponding to case 2 can be expressed as

$$u_{23} = a_0 - \frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}\left(-\frac{C}{\lambda(C\xi + D)}\right) \quad (4.1.17)$$

Where $\xi = x + y + z - aa_0t$

Solution 3:

When $\boldsymbol{\mu\lambda} > 0$, the trigonometric solution corresponding to case 3 can be expressed as

$$u_{31} = a_0 - \frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right) + \frac{\boldsymbol{6\mu}}{\boldsymbol{a}}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right)^{-1} \quad (4.1.18)$$

When $\boldsymbol{\mu\lambda} < 0$ the hyperbolic solution corresponding to case 3 can be expressed as

$$u_{22} = a_0 - \frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) - D}\right)\right) +$$

$$\frac{\boldsymbol{6\mu}}{\boldsymbol{a}}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) - D}\right)\right)^{-1}, \quad (4.1.19)$$

when $\mu = 0$, $\lambda \neq 0$, the rational solution corresponding to case 3 can be expressed as

$$u_{23} = a_0 - \frac{\boldsymbol{6\lambda}}{\boldsymbol{a}}\left(-\frac{C}{\lambda(C\xi + D)}\right), \quad (4.1.20)$$

where $\xi = x + y + z - aa_0t$

4.2 Example 2: The (3+1)-dimensional Potential-YTSF Equation

We start the (3+1)-dimensional Potential-YTSF Equation in the following form

$$-4\boldsymbol{u_{xt}} + \boldsymbol{u_{xxxz}} + \boldsymbol{4u_xu_{xz}} + \boldsymbol{2u_{xx}u_z} + \boldsymbol{3u_{yy}} = \boldsymbol{0}. \quad (4.2.1)$$

This equation was called the Potential-YTSF Equation and it was developed by using the strong symmetry. The traveling wave variable (2.2) permits us converting Eq.(4.2.1) into the following ODE. After integrating once, we have the following form:

$$\boldsymbol{c + 4Vu' + u''' + 3u'^2 + 3u' = 0}, \quad (4.2.2)$$

where $\boldsymbol{c}$ is a constant of integration . Now by considering the homogeneous balance between the order of $\boldsymbol{u'''}$ and $\boldsymbol{u'^2}$ in Eq.(4.2.2), we obtain $\boldsymbol{m = 1}$.

By using step 3 the solution of Eq. (4.2.2), can be written as,

$$\boldsymbol{U(\xi) = a_0 + a_1(G'/G) + b_1(F'/F)}$$

$$= \boldsymbol{a_0 - a_1 F + b_1\left(F^{-1} - F\right)}, \tag{4.2.3}$$

Substituting Eq. (4.2.3) into Eq. (4.2.2) , collecting the coefficients of $(\boldsymbol{F})^{\boldsymbol{i}}(\boldsymbol{i} = \boldsymbol{0}, \pm\boldsymbol{2}, \pm\boldsymbol{4})$,and letting it be zero, yields a set of simultaneous algebraic equations for $\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{b_1}$, $\boldsymbol{V}$ and $\boldsymbol{c}$

After solving these algebraic equations for $\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{b_1}, \boldsymbol{V}$ and $\boldsymbol{c}$ with the help of software Maple, yields the following results.

Case 1:

$$\mathbf{c} = \mathbf{0}\,,\ \ \boldsymbol{V} = \frac{\mathbf{-7}}{\mathbf{4}}\,,\ \ \boldsymbol{a_1} = -\mathbf{2}\,,\ \ \boldsymbol{b_1} = \mathbf{2}\,, \tag{4.2.4}$$

where $\boldsymbol{a_0}$ is arbitrary.

Case 2:

$$\boldsymbol{c} = \mathbf{0}\,,\ \ \boldsymbol{V} = \frac{\mathbf{-19}}{\mathbf{4}}\,,\ \ \boldsymbol{a_1} = -\mathbf{4}\,,\ \ \boldsymbol{b_1} = \mathbf{2}\ , \tag{4.2.5}$$

where $\boldsymbol{a_0}$ is arbitrary.

Substituting Eqs.(4.2.4),(4.2.5) into Eq.(4.2.3) we get two types of the exact solutions of Eq.(4.2.1) as follows:

According to case 1.

Type 1:

Class I: $$\boldsymbol{u_{11}(x,t) = a_0 + 2\ coth\left(x + y + z + \frac{7}{4}t\right)}. \tag{4.2.6}$$

Class II: $$\boldsymbol{u_{12}(x,t) = a_0 + 2\,tanh\left(x + y + z + \frac{7}{4}t\right)}. \tag{4.2.7}$$

According to case 2.

Type 2:

Class I: $$\boldsymbol{u_{21}(x,t) = a_0 + 2\,tanh\left(x + y + z + \frac{19}{4}t\right) + 2\,coth\left(x + y + z + \frac{19}{4}t\right)}. \tag{4.2.8}$$

Class II: $$\boldsymbol{u_{22}(x,t) = a_0 + 2\,coth\left(x + y + z + \frac{19}{4}t\right) + 2\,tanh\left(x + y + z + \frac{19}{4}t\right)}. \tag{4.2.9}$$

The solutions can be expressed as follows:

$$u(\boldsymbol{\xi}) = \boldsymbol{a_0} + \boldsymbol{a_1}\left(\frac{\boldsymbol{G'}}{\boldsymbol{G^2}}\right) + \boldsymbol{b_1}\left(\frac{\boldsymbol{G'}}{\boldsymbol{G^2}}\right)^{\mathbf{-1}}, \tag{4.2.10}$$

Where a_0, a_1, b_1 are unknown constants. We substitute Eq.(4.2.10) into (4.2.2) along with Eq.(3.5) to collect all the coefficients with the same power of $\left(\frac{G'}{G^2}\right)^i$, $(i = 0, \pm 1, \pm 2, \dots)$. From Eq.(4.2.10) each coefficient of $\left(\frac{G'}{G^2}\right)^i$ is set to zero, and system of algebraic equations about a_0, a_1, b_1 is attained as follows:

The following results are obtained upon solving the above system of algebraic equations using Maple

Case 1:

$$\boldsymbol{c=0\,, V=\lambda\mu-\frac{3}{4}\,,\; a_0=a_0\,,\; a_1=0\,, b_1=2\mu}$$

Case 2:

$$\boldsymbol{c=0\,, V=\lambda\mu-\frac{3}{4}\,,\; a_0=a_0\,, a_1=-2\lambda\,, b_1=0}$$

Case 3:

$$\boldsymbol{c=0\,, V=4\lambda\mu-\frac{3}{4}\,,\; a_0=a_0\,, a_1=-2\lambda\,, b_1=2\mu}$$

In Eq.(4.2.10) we substitute the above cases along with ratios (3.6)-(3.8), and three groups of solutions for Eq.(4.2.1) exist.

Solution 1: When $\boldsymbol{\mu\lambda}>0$, the trigonometric solution corresponding to case 1 can be expressed as

$$u_{11}=a_0+2\mu\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\,\cos\sqrt{\mu\lambda}\xi+D\sin\sqrt{\mu\lambda}\xi}{D\,\cos\sqrt{\mu\lambda}\xi-C\sin\sqrt{\mu\lambda}\xi}\right)\right)^{-1} \qquad (4.2.11)$$

When $\boldsymbol{\mu\lambda}<0$ the hyperbolic solution corresponding to case 1 can be expressed as

$$u_{12}=a_0+2\mu\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\,\sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right)+C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right)+D}{C\,\sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right)+C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right)-D}\right)\right)^{-1} \qquad (4.2.12)$$

When $\mu=0\,,\lambda\neq0$, the rational solution corresponding to case 1 can be expressed as

$$u_{13}=a_0 \qquad (4.2.13)$$

Where $\xi=x+y+z-\left(\lambda\mu-\frac{3}{4}\right)t$

Solution 2:

When $\boldsymbol{\mu\lambda}>0$, the trigonometric solution corresponding to case 2 can be expressed as

$$u_{21}=a_0-2\lambda\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\,\cos\sqrt{\mu\lambda}\xi+D\sin\sqrt{\mu\lambda}\xi}{D\,\cos\sqrt{\mu\lambda}\xi-C\sin\sqrt{\mu\lambda}\xi}\right)\right) \qquad (4.2.14)$$

When $\boldsymbol{\mu\lambda}<0$ the hyperbolic solution corresponding to case 2 can be expressed as

$$u_{22}=a_0-2\lambda\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\,\sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right)+C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right)+D}{C\,\sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right)+C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right)-D}\right)\right) \qquad (4.2.15)$$

When $\mu=0\,,\lambda\neq0$, the rational solution corresponding to case 2 can be expressed as

$$u_{23}=a_0-2\lambda\left(-\frac{C}{\lambda(C\xi+D)}\right) \qquad (4.2.16)$$

Where $\xi = x + y + z - \left(\lambda\mu - \frac{3}{4}\right)t$

Solution 3:

When $\boldsymbol{\mu\lambda} > 0$, the trigonometric solution corresponding to case 3 can be expressed as

$$u_{31} = a_0 - 2\lambda\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right) + 2\mu\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right)^{-1} \quad (4.2.17)$$

When $\boldsymbol{\mu\lambda} < 0$ the hyperbolic solution corresponding to case 3 can be expressed as

$$u_{22} = a_0 - 2\lambda\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) - D}\right)\right) +$$

$$2\mu\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\ \xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\ \xi\right) - D}\right)\right)^{-1} \quad (4.2.18)$$

When $\mu = 0$, $\lambda \neq 0$, the rational solution corresponding to case 3 can be expressed as

$$u_{23} = a_0 - 2\lambda\left(-\frac{C}{\lambda(C\xi + D)}\right) \quad (4.2.19)$$

Where $\xi = x + y + z - \left(4\lambda\mu - \frac{3}{4}\right)t$

4.3 Example 3: The (3+1)-dimensional generalized Shallow water equation

We consider the following (3+1)-dimensional generalized Shallow water equation

$$\boldsymbol{u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} + u_{yt} - u_{xz} = 0.} \quad (4.3.1)$$

The traveling wave variable (2.2) permits us converting Eq.(4.3.1) into the following ODE:

$$\boldsymbol{c + u''' - 3u'^2 - (V + 1)u' = 0.} \quad (4.3.2)$$

Where $\boldsymbol{c}$ is a constant of integration. Consider the homogenus balance between $\boldsymbol{u'''}$ and $\boldsymbol{u'^2}$ in (4.3.2), we get $\boldsymbol{m = 1}$. Using the same idea in Sec 3.1, we may choose the solution of Eq.(4.3.2) in the form

$$\boldsymbol{U(\xi) = a_0 + a_1(G'/G) + b_1(F'/F)}$$
$$\boldsymbol{= a_0 - a_1F + b_1\left(F^{-1} - F\right),} \quad (4.3.3)$$

Substituting Eq. (4.3.3) into Eq. (4.3.2) , collecting the coefficients of $\boldsymbol{(F)^i(i = 0, \pm 2, \pm 4)}$,and letting it be zero, yields a set of simultaneous algebraic equations for $\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{b_1}$, $\boldsymbol{V}$ and $\boldsymbol{c}$

After solving these algebraic equations for $\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{b_1}, \boldsymbol{V}$ and $\boldsymbol{c}$ with the help of software Maple, yields the following results.

Case 1:

$$\boldsymbol{c = 0\,,\ \ V = 15\,,\ \ a_1 = 4\,,\ \ b_1 = -2\,,} \quad (4.3.4)$$

where $\boldsymbol{a_0}$ is arbitrary.

Case 2:

$$\boldsymbol{c = 0\,,\ V = 3\,,\ a_1 = 2\,,\ b_1 = -2}\ , \tag{4.3.5}$$

where $\boldsymbol{a_0}$ is arbitrary.

Substituting Eqs.(4.3.4),(4.3.5) into Eq.(4.3.3) we get two types of the exact solutions of Eq.(4.3.1) as follows:

According to case 1.

Type 1:

Class I: $\boldsymbol{u_{11}(x,t) = a_0 - 2\,tanh(x + y + z - 15t) - 2\ coth(x + y + z - 15t)}$. (4.3.6)

Class II: $\boldsymbol{u_{12}(x,t) = a_0 - 2\ coth(x + y + z - 15t) - 2\,tanh(x + y + z - 15t)}$.(4.3.7)

According to case 2.

Type 2:

Class I: $\boldsymbol{u_{21}(x,t) = a_0 - 2\,coth(x + y + z - 3t)}$. (4.3.8)

Class II: $\boldsymbol{u_{22}(x,t) = a_0 - 2\,tanh(x + y + z - 3t)}$. (4.3.9)

The solutions can be expressed as follows:

$$u(\xi) = \boldsymbol{a_0} + \boldsymbol{a_1}\left(\frac{\boldsymbol{G'}}{\boldsymbol{G^2}}\right) + \boldsymbol{b_1}\left(\frac{\boldsymbol{G'}}{\boldsymbol{G^2}}\right)^{-\boldsymbol{1}}, \tag{4.3.10}$$

Where a_0, a_1, b_1 are unknown constants. We substitute Eq.(4.3.10) into (4.3.2) along with Eq.(3.5) to collect all the coefficients with the same power of $\left(\frac{G'}{G^2}\right)^i$, $(i = 0, \pm 1, \pm 2, \dots)$. From Eq.(4.3.10) each coefficient of $\left(\frac{G'}{G^2}\right)^i$ is set to zero, and system of algebraic equations about a_0, a_1, b_1 is attained as follows:

The following results are obtained upon solving the above system of algebraic equations using Maple

Case 1:

$$\boldsymbol{c = 0\,, V = -4\lambda\mu - 1\ ,\ a_0 = a_0\ ,\ a_1 = 0\,, b_1 = -2\mu}$$

Case 2:

$$\boldsymbol{c = 0\,, V = -4\lambda\mu - 1\ ,\ a_0 = a_0\ , a_1 = 2\lambda\ , b_1 = 0}$$

Case 3:

$$\boldsymbol{c = 0\,, V = -16\lambda\mu - 1\,,\ a_0 = a_0\ , a_1 = 2\lambda\,, b_1 = -2\mu}$$

In Eq.(4.3.10) we substitute the above cases along with ratios (3.6)-(3.8), and three groups of solutions for Eq.(4.3.1) exist.

Solution 1: When $\boldsymbol{\mu\lambda} > 0$, the trigonometric solution corresponding to case 1 can be expressed as

$$u_{11} = a_0 - 2\mu\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right)^{-1} \tag{4.3.11}$$

When $\boldsymbol{\mu\lambda} < 0$ the hyperbolic solution corresponding to case 1 can be expressed as

$$u_{12} = a_0 - 2\mu\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) - D}\right)\right)^{-1} \tag{4.3.12}$$

When $\mu = 0\,, \lambda \neq 0$, the rational solution corresponding to case 1 can be expressed as

$$u_{13} = a_0 \tag{4.3.13}$$

Where $\xi = x + y + z - (-4\lambda\mu - 1)t$

Solution 2:

When $\boldsymbol{\mu\lambda} > 0$, the trigonometric solution corresponding to case 2 can be expressed as

$$u_{21} = a_0 + 2\lambda\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right) \tag{4.3.14}$$

When $\boldsymbol{\mu\lambda} < 0$ the hyperbolic solution corresponding to case 2 can be expressed as

$$u_{22} = a_0 + 2\lambda\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) - D}\right)\right) \tag{4.3.15}$$

When $\mu = 0\,, \lambda \neq 0$, the rational solution corresponding to case 2 can be expressed as

$$u_{23} = a_0 + 2\lambda\left(-\frac{C}{\lambda(C\xi + D)}\right) \tag{4.3.16}$$

Where $\xi = x + y + z - (-4\lambda\mu - 1)t$

Solution 3:

When $\boldsymbol{\mu\lambda} > 0$, the trigonometric solution corresponding to case 3 can be expressed as

$$u_{31} = a_0 + 2\lambda\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right) - 2\mu\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\ \cos\sqrt{\mu\lambda}\xi + D\sin\sqrt{\mu\lambda}\xi}{D\ \cos\sqrt{\mu\lambda}\xi - C\sin\sqrt{\mu\lambda}\xi}\right)\right)^{-1} \tag{4.3.17}$$

When $\boldsymbol{\mu\lambda} < 0$ the hyperbolic solution corresponding to case 3 can be expressed as

$$u_{22} = a_0 + 2\lambda\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) - D}\right)\right) -$$

$$2\mu\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + D}{C\ \sinh\left(2\sqrt{|\mu\lambda|}\,\xi\right) + C\cosh\left(2\sqrt{|\mu\lambda|}\,\xi\right) - D}\right)\right)^{-1} \tag{4.3.18}$$

When $\mu = 0\,, \lambda \neq 0$, the rational solution corresponding to case 3 can be expressed as

$$u_{23} = a_0 + 2\lambda\left(-\frac{C}{\lambda(C\xi + D)}\right) \tag{4.3.19}$$

Where $\xi = x + y + z - (-16\lambda\mu - 1)t$

5. Geometry of the exact solution:

The geometry of the exact solutions of various equations has been intensely studied by different authors in various ways[19-24]. In this section, we are going to investigate the exact solution and the numerical solutions in the 3-dimensional space-time known as Lorentz-Minkowski space $\mathbb{R}_1^3$. The main reason for choosing to work in this space is that the Lorentz-Minkowski space plays an important role in both special relativity and general relativity with space coordinates and time coordinates.

First, we need to recall some basic facts and notations in $\mathbb{R}_1^3$ [25-29].

Let $\boldsymbol{X} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3})$ and $\boldsymbol{Y} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3})$ be any two vector fields in $\mathbb{R}_1^3$. Then inner product of X and Y is defined by

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \boldsymbol{x_1 y_1} + \boldsymbol{x_2 y_2} - \boldsymbol{x_3 y_3}. \qquad (5.1)$$

Note that a vector field X is called

(i) a timelike vector if $\langle \boldsymbol{X}, \boldsymbol{X} \rangle < \boldsymbol{0}$,
(ii) a spacelike vector if $\langle \boldsymbol{X}, \boldsymbol{X} \rangle > \boldsymbol{0}$,
(iii) a lightlike (or degenerate) vector if $\langle \boldsymbol{X}, \boldsymbol{X} \rangle = \boldsymbol{0}$ and $\boldsymbol{X} \neq \boldsymbol{0}$.

Thus, the inner product in $\mathbb{R}_1^3$ splits each vector field into three categories, namely

(i) spacelike, (ii) timelike, and (iii) lightlike (degenerate) vectors. The category is known as causal character of a vector. The set of all lightlike vectors is called null cone. Furthermore, the norm of a vector X is defined by its causal character as follows:

(i) $\|X\| = \sqrt{\langle X, X \rangle}$ if X is a spacelike vector,
(ii) $\|X\| = -\sqrt{\langle X, X \rangle}$ if X is a timelike vector.

Let X be a unit timelike vector and $e = (0,0,1)$ in $\mathbb{R}_1^3$. Then X is called

(i) a timelike future pointing vector if $\langle X, e \rangle > 0$,
(ii) a timelike past pointing vector if $\langle X, e \rangle < 0$.

Now, let $r(x,t)$ be a surface in $\mathbb{R}_1^3$. Then the normal vector N at a point in $r(x,t)$ is given by

$$\boldsymbol{N} = \frac{\boldsymbol{r_x} \wedge \boldsymbol{r_t}}{\|\boldsymbol{r_x} \wedge \boldsymbol{r_t}\|}, \qquad (5.2)$$

Where $\wedge$ denotes the wedge product in $\mathbb{R}_1^3$. A surface is called

(i) a timelike surface if N is spacelike,
(ii) a spacelike surface if N is timelike,
(iii) a lightlike (or degenerate) surface if N is lightlike.

We note that a point is called regular if $N \neq 0$ and singular if $N = 0$.

Now, let us consider a surface given by

$$\boldsymbol{r}(\boldsymbol{x},\boldsymbol{t}) = \big(\boldsymbol{x},\boldsymbol{t},\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})\big), \tag{5.3}$$

Where $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ is the exact solution of the (3+1)-dimensional Zakhrov-Kuznetsov (ZK) equation, the (3+1)-dimensional Potential-YTSF Equation and the (3+1)-dimensional generalized Shallow water equation given by (4.1.7) , (4.2.8) , (4.3.8) respectively

In view of (5.2), the normal vector field of $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{t})$ becomes

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{t}) = -\frac{6}{\sqrt{1+\frac{36}{a^2\cosh(ata_0-x-y-z)^4}+\frac{36a_0^2}{\cosh(ata_0-x-y-z)^4}}\,a\cosh(ata_0-x-y-z)^2}\,\boldsymbol{e_x} + \frac{6a_0}{\sqrt{1+\frac{36}{a^2\cosh(ata_0-x-y-z)^4}+\frac{36a_0^2}{\cosh(ata_0-x-y-z)^4}}\,\cosh(ata_0-x-y-z)^2}\,\boldsymbol{e_y} + \frac{1}{\sqrt{1+\frac{36}{a^2\cosh(ata_0-x-y-z)^4}+\frac{36a_0^2}{\cosh(ata_0-x-y-z)^4}}}\,\boldsymbol{e_z}$$

(5.4)

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{t}) = \frac{4}{\sqrt{4+\frac{377}{\cosh\left(x+y+z+\frac{19}{4}t\right)^4\left(\cosh\left(x+y+z+\frac{19}{4}t\right)^2-1\right)^2}}\,\cosh\left(x+y+z+\frac{19}{4}t\right)^2\left(\cosh\left(x+y+z+\frac{19}{4}t\right)^2-1\right)}\,\boldsymbol{e_x} + \frac{19}{\sqrt{4+\frac{377}{\cosh\left(x+y+z+\frac{19}{4}t\right)^4\left(\cosh\left(x+y+z+\frac{19}{4}t\right)^2-1\right)^2}}\,\cosh\left(x+y+z+\frac{19}{4}t\right)^2\left(\cosh\left(x+y+z+\frac{19}{4}t\right)^2-1\right)}\,\boldsymbol{e_y} + \frac{2}{\sqrt{4+\frac{377}{\cosh\left(x+y+z+\frac{19}{4}t\right)^4\left(\cosh\left(x+y+z+\frac{19}{4}t\right)^2-1\right)^2}}}\,\boldsymbol{e_z} \tag{5.5}$$

$$\boldsymbol{N}(\boldsymbol{x},\boldsymbol{t}) = -\frac{2}{\sqrt{1+\frac{40}{(\cosh(-x-y-z+3t)^2-1)^2}}\,(\cosh(-x-y-z+3t)^2-1)}\,\boldsymbol{e_x} + \frac{6}{\sqrt{1+\frac{40}{(\cosh(-x-y-z+3t)^2-1)^2}}\,(\cosh(-x-y-z+3t)^2-1)}\,\boldsymbol{e_y} + \frac{1}{\sqrt{1+\frac{40}{(\cosh(-x-y-z+3t)^2-1)^2}}}\,\boldsymbol{e_z} \tag{5.6}$$

Form (5.4), (5.5), (5.6) it is clear that $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{t})$ is a regular surface, that is, every point of it is a regular point.

6. Gaussian curvature and Mean curvature of node points

Another important fact for a surface is to compute the Gaussian curvature and Mean curvature which are an intrinsic character of it. The Gaussian curvature is the determinant of the shape operator. For a surface $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{t})$, we shall apply the following useful way to compute the Gaussian curvature:

Consider $\langle \boldsymbol{N},\boldsymbol{N}\rangle = \boldsymbol{\varepsilon}\|\boldsymbol{N}\|$, where $\boldsymbol{\varepsilon} = \mp \boldsymbol{1}$. Let us define

$$\boldsymbol{E} = \langle \boldsymbol{r_x},\boldsymbol{r_x}\rangle, \qquad \boldsymbol{F} = \langle \boldsymbol{r_x},\boldsymbol{r_t}\rangle, \qquad \boldsymbol{G} = \langle \boldsymbol{r_t},\boldsymbol{r_t}\rangle$$

and (6.1)

$$\boldsymbol{e} = \langle \boldsymbol{u_{xx}}, \boldsymbol{N} \rangle, \qquad \boldsymbol{f} = \langle \boldsymbol{u_{xt}}, \boldsymbol{N} \rangle, \qquad \boldsymbol{g} = \langle \boldsymbol{u_{tt}}, \boldsymbol{N} \rangle.$$

Then the Gaussian curvature $K(p)$ at a point p of a surface satisfies

$$\boldsymbol{K(p) = \varepsilon \frac{eg-f^2}{EG-F^2}}. \tag{6.2}$$

We note that

(i) $\boldsymbol{K(p) > 0}$ means that the surface $\boldsymbol{r(x,t)}$ is shaped like an elliptic paraboloid near $\boldsymbol{p}$. In this case, $\boldsymbol{p}$ is called an elliptic point.

(ii) $\boldsymbol{K(p) < 0}$ means that the surface $\boldsymbol{r(x,t)}$ is shaped like a hyperbolic paraboloid near $\boldsymbol{p}$. In this case, $\boldsymbol{p}$ is called a hyperbolic point.

(iii) $\boldsymbol{K(p) = 0}$ means that the surface $\boldsymbol{r(x,t)}$ is shaped like a parabolic cylinder or a plane near $\boldsymbol{p}$. In this case, $\boldsymbol{p}$ is called a parabolic point.

Now, let us consider the surface given in (5.3). Form (6.1) and (6.2), by a straightforward computation, we get $\boldsymbol{K = 0}$ for equations (4.1.7) ,(4.2.8) ,(4.3.8) .

Another important kind of curvatures is mean curvature which measures the surface tension from the surrounding space at a point. The mean curvature is a trace of the second fundamental form. For a surface $\boldsymbol{r(x,t)}$, we shall apply the following useful way to compute the mean curvature $\boldsymbol{H(p)}$:

$$\boldsymbol{H(p) = \varepsilon \frac{1}{2} \frac{eG-2fF+gE}{EG-F^2}}. \tag{6.3}$$

If $\boldsymbol{H(p) = 0}$ for all points of $\boldsymbol{r(x,t)}$, then the surface is called minimal. Furthermore, if the value of the mean curvature at a point $\boldsymbol{p}$ receives at least a possible amount of tension from the surrounding space, then $\boldsymbol{p}$ is called ideal point. That is, if a point in a surface is affected as little as possible from the external influence, then it becomes ideal.

From (6.3), for equation (4.1.7) we obtain

$$\boldsymbol{H = \frac{6\sinh(ata_0-x-y-z)\, a\cosh(ata_0-x-y-z)(a^2a_0^2+1)}{\sqrt{\frac{\cosh(ata_0-x-y-z)^4+36a_0^2a^2+36}{a^2\cosh(ata_0-x-y-z)^4}}+(\mathrm{a}^2\cosh(ata_0-x-y-z)^4+36a_0^2a^2+36)}}$$

From (6.3), for equation (4.2.8) we obtain

$$\boldsymbol{H = \left(377\sinh\left(x+y+z+\tfrac{19}{4}t\right)\left(2\cosh\left(x+y+z+\tfrac{19}{4}t\right)^2-1\right)\cosh\left(x+y+z+\tfrac{19}{4}t\right)\right)/}$$

$$\left[\sqrt{4\cosh\left(x+y+z+\frac{19}{4}t\right)^8-8\cosh\left(x+y+z+\frac{19}{4}t\right)^6+4\cosh\left(x+y+z+\frac{19}{4}t\right)^4+377}\,.\right.$$
$$\left(4\cosh\left(x+y+z+\frac{19}{4}t\right)^8-8\cosh\left(x+y+z+\frac{19}{4}t\right)^6+4\cosh\left(x+y+z+\frac{19}{4}t\right)^4+377\right)$$
$$\left./\left(\sqrt{\cosh\left(x+y+z+\frac{19}{4}t\right)^4\left(\cosh\left(x+y+z+\frac{19}{4}t\right)^2-1\right)^2}\right)\right]$$

From (6.3), for equation (4.3.8) we obtain

$$H=\frac{20\cosh(-x-y-z+3t)\sinh(-x-y-z+3t)}{\sqrt{\frac{\cosh(-x-y-z+3t)^4-2\cosh(-x-y-z+3t)^2+41}{\left(\cosh(-x-y-z+3t)^2-1\right)^2}}\left(\cosh(-x-y-z+3t)^4-2\cosh(-x-y-z+3t)^2+41\right)}$$

7. Numerical solutions for the exact solutions for the above NPD equations:

We can study the behavior of the travelling wave solutions which obtained above by illustrating the following figures:

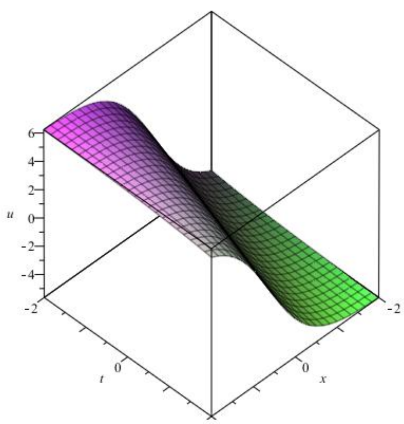

Figure 1.The plot of the solution (4.1.7) When $a_0=0.3, a=1, y=0, z=0$

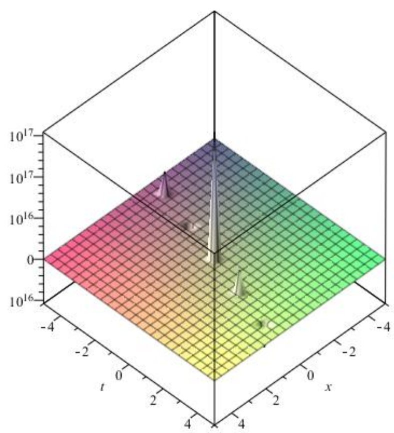

Figure 2.The plot of the solution (4.1.9) when $a_0=0.3, a=1, y=0, z=0$

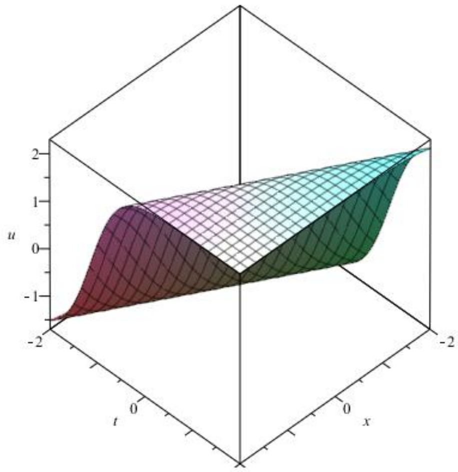

Figure 3. The plot of the solution (4.2.7) When $a_0 = 0.3$, $y = 0$, $z = 0$

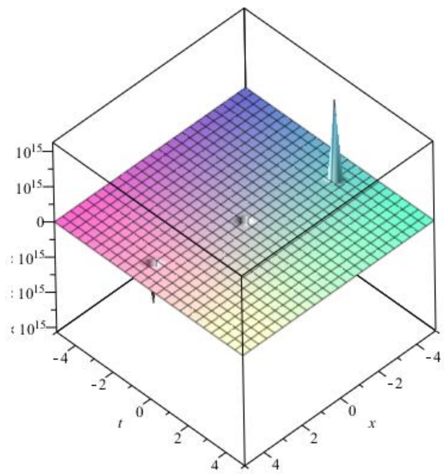

Figure 4. The plot of the solution (4.2.8) when $a_0 = -0.5$, $y = 0$, $z = 0$

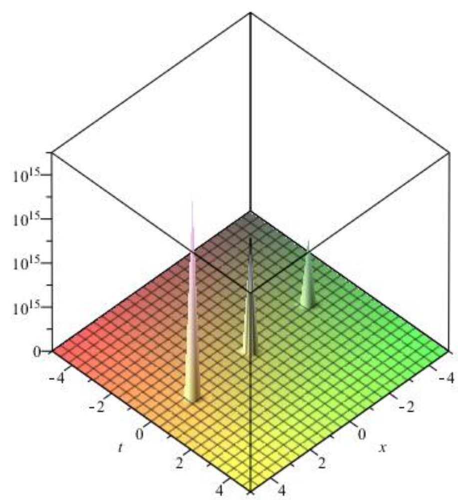

Figure 5. The plot of the solution (4.3.6) When $a_0 = -0.5$, $y = 0$, $z = 0$

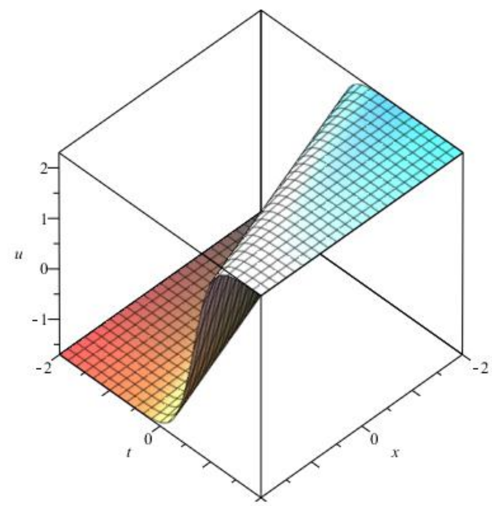

Figure 6. The plot of the solution (4.3.9) when $a_0 = 0.3$, $y = 0$, $z = 0$

7. Conclusions:

In this article, the variation of the (G'/G)-expansion method is developed , by knowing the advantage solution of the coupled Riccati equation and the $\left(\frac{G'}{G^2}\right)$-expansion method, are used to find new exact solutions of the (3+1)-dimensional Zakhrov-Kuznetsov equation , the (3+1)-dimensional Potential-YTSF Equation and the (3+1)-dimensional generalized Shallow water equation, then we can find its geometrical properties by calculating its Gaussian Curvature and Mean curvature . Our results show that the methods can be used for solving many nonlinear partial differential equations in mathematical physics.

- Ethics approval and consent to participate

Not applicable.

- Consent for publication

Not applicable.

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