Integrability of very weak Solutions for Boundary value problems of Nonhomogeneous A-Harmonic equations

***Abstract*—The paper deals with very weak solutions u to boundary value problems of the nonhomogeneous A-harmonic equation. We show that, any very weak solution u to the boundary value problem is integrable provided that r is sufficiently close to p.**

Keywords—Integrability; Very weak solution; Boundary value problem; A-harmonic equation.

I. INTRODUCTION

Let $1<p<n$, $\theta(x)\in W^{1,q}(\Omega)$, $q>r$, $f(x)\in L^{\frac{nq}{n(p-1)+r}}(\Omega,\mathbb{R}^n)$. We shall examine the boundary value problem of the A-harmonic equation

$$\begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = f(x), & x\in\Omega, \\ u(x)=\theta(x), & x\in\partial\Omega, \end{cases} \tag{1.1}$$

Throughout this paper Ω will stand for a bounded regular domain in $\mathbb{R}^n (n\geq 2)$. By a regular domain we understand any domain of finite measure for which the estimates (2.4) and (2.5) below for the Hodge decomposition are satisfied, see [1], [2]. A Lipschitz domain, for example, is regular.

Definition 1.1. A function $u\in\theta+W_0^{1,r}(\Omega)$, $\max\{1,p-1\}<r<p$, is called a very weak solution to the boundary value problem (1.1), for all $\Phi\in W_0^{1,r/(r-p+1)}(\Omega)$ with compact support sets in Ω, there is

$$\int_\Omega \left\langle |\nabla u|^{p-2}\nabla u, \nabla\Phi \right\rangle dx = \int_\Omega f(x)\Phi dx \tag{1.2}$$

where $f(x)\in L^{\frac{nq}{n(p-1)+r}}(\Omega,\mathbb{R}^n)$.

Recall that a function $u\in\theta+W_0^{1,p}(\Omega)$ is called the weak solution of the boundary value problem (1.1) if (1.2) holds true for all $\Phi\in W_0^{1,p}(\Omega)$. The words very weak in Definition 1.1 mean that the Sobolev integrable exponent r of u can be small than the natural one p. see [1], Theorem 1, page 602.

In this paper we will need the definition of weak L^t-space (see [2]): for $t>0$, the weak L^t- space, $L^t_{weak}(\Omega)$, consists of all measurable functions f such that

$$\left|\left\{x\in\Omega : |f(x)|>s\right\}\right| \leq \frac{k}{s^t}$$

for some positive constant $k = k(f)$ and every $s > 0$, where $|E|$ is the n-dimensional Lebesgue measure of E.

Integrability property is important in the regularity theories of nonlinear elliptic PDEs and systems. In[3], Zhu et al. studied the global integrability of nonhomogeneous quasilinear elliptic equations

$$-\mathrm{div}A(x,u,\nabla u) = f(x) + \mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

In [4], Guo et al. studied the higher order integrability of the divergence elliptic equation $-\mathrm{div}A(x,\nabla u) = -\mathrm{div}f$. In [5], Zhang et al. studied the global integrability of A- harmonic equation $-\mathrm{div}A(x,\nabla u) = -\mathrm{div}f$. In this paper, we consider the global integrability of the very weak solutions of the boundary value problem (1.1). The main result is the following theorem.

Theoerm 1.1. Let $\theta \in W^{1,q}(\Omega)$, $q > r$, There exists $\varepsilon_0 = \varepsilon_0(n,p) > 0$, such that for each very weak solution $u \in \theta + W_0^{1,r}(\Omega)$, $\max\{1, p-1\} < r < p < n$, to the boundary value problem (1.1), we have

$$u \in \begin{cases} \theta + L^{q^*}_{weak}(\Omega) & for\ q < r, \\ \theta + L^{\tau}_{weak}(\Omega) & for\ q = r\ and\ \tau < \infty, \\ \theta + L^{\infty}(\Omega) & for\ q > n, \end{cases} \tag{1.3}$$

provided that $|p-r| < \varepsilon_0$.

Note that we have restricted ourselves to the case $r < n$ since otherwise any function in $W^{1,r}(\Omega)$ is in the spce $L^t(\Omega)$ for any $t < \infty$ by the Sobolev embedding theorem. At the same time, it is also noted that the very weak solution u of the boundary value problem (1.1) is taken from the Sobolev space $W^{1,r}(\Omega)$, and the embedding theorem ensures that the integrability of u reaches from r to r^*. And our result theorem 1.1 improves this integrability. Note that the key to proving the theorem 1.1 is to use Hodge decomposition[1][6] to construct the appropriate test function.

II. PRELIMINARY LEMMAS

Lemma 1.1. For $p \geq 2$ and any $X, Y \in \mathbb{R}^n$, one has

$$2^{2-p}|X-Y|^p \leq \left\langle |X|^{p-2}X - |Y|^{p-2}Y, X-Y \right\rangle.$$

Lemma 1.2. For any $X, Y \in \mathbb{R}^n$ and $\varepsilon > 0$, one has

$$\begin{aligned} & \left| |X|^{\varepsilon} X - |Y|^{\varepsilon} Y \right| \\ \leq & \begin{cases} (1+\varepsilon)(|Y| + |X-Y|)^{\varepsilon}|X-Y|, & \varepsilon > 0, \\ \frac{1-\varepsilon}{2^{\varepsilon}(1+\varepsilon)}|X-Y|^{1+\varepsilon}, & -1 < \varepsilon \leq 0. \end{cases} \end{aligned}$$

Lemma 1.3. For $1 < p < 2$ and any $X, Y \in \mathbb{R}^n$, one has

$$\begin{aligned} & \left\langle |X|^{p-2}X - |Y|^{p-2}Y, X-Y \right\rangle \\ \geq & |X-Y|\left((|X-Y| + |Y|)^{\mathrm{p}p-1} - |Y|^{p-1} \right). \end{aligned}$$

Lemma 1.4. Let $\varepsilon_0 > 0$, $\phi:(s_0,\infty)\to[0,\infty)$ is a decrement function such that for each r, s $(r > s > s_0)$, if

$$\phi(r) \le \frac{c}{(r-s)^{\alpha}}(\phi(s))^{\beta}$$

where c, α, β are constants, we have

(1) if $\beta > 1$ we have that $\phi(s_0 + d) = 0$, where $d^{\alpha} = c2^{\alpha\beta/(\beta-1)}(\phi(s_0))^{\beta-1}$;

(2) If $\beta < 1$ we have that $\phi(s) \le 2^{\mu/(1-\beta)}(c^{1/(1-\beta)} + (2s_0)^{\mu}\phi(s_0))s^{-\mu}$, where $\mu = \alpha/(1-\beta)$.

III. PROOF OF THEOREM 1.1

For any $L > 0$, let

$$v = \begin{cases} u-\theta+L & for\ u-\theta < -L, \\ 0 & for\ -L \le u-\theta \le L, \\ u-\theta-L & for\ u-\theta > L, \end{cases} \tag{3.1}$$

Then according to the hypothesis, we have $v \in W_0^{1,r}(\Omega)$ and $\nabla v = (\nabla u - \nabla\theta)\cdot 1_{\{|u-\theta|>L\}}$, Where 1_E is the characteristic function of the set E. We introduce the Hodge decomposition of vector field $|\nabla v|^{p-2}\nabla v \in L^{r/(r-p+1)}(\Omega)$. So that

$$|\nabla v|^{r-p}\nabla v = \nabla\Phi + h, \tag{3.2}$$

Here $\Phi \in W_0^{1,r/(r-p+1)}, h \in L^{r/(r-p+1)}(\Omega, \mathbb{R}^n)$ is a vector field with zero divergence, and satisfied

$$\|\nabla\Phi\|_{r/(r-p+1)} \le C(n,p)\|\nabla v\|_r^{r-p+1} \tag{3.3}$$

and

$$\|h\|_{r/(r-p+1)} \le C(n,p)|p-r|\|\nabla v\|_r^{r-p+1}. \tag{3.4}$$

From the counter-proof method, it is inevitable to exist φ such that $\Phi = \varphi - \varphi_\Omega$. Taken Φ as a test function of the integral identity (1.2), that is

$$\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u, |\nabla u - \nabla\theta|^{r-p}(\nabla u - \nabla\theta)\right\rangle dx = \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u, h\right\rangle dx + \int_{\{|u-\theta|>L\}} f(x)\Phi dx.$$

This implies

$$\begin{aligned} &\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u - \nabla\theta|^{r-p}(\nabla u - \nabla\theta)\right\rangle dx \\ =& \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, h\right\rangle dx \\ &+ \int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, h\right\rangle dx \\ &- \int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, |\nabla u - \nabla\theta|^{r-p}(\nabla u - \nabla\theta)\right\rangle dx \\ &+ \int_{\{|u-\theta|>L\}} f(x)\Phi dx \\ =& I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.5}$$

Now we shall distinguish between two cases.

Case 1: $p \geq 2$. using Lemma 2.1, (3.5) can be estimated as

$$\begin{aligned}&\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx\\ &\geq 2^{2--p--}\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx.\end{aligned} \tag{3.6}$$

Using the Lemma 2.2, Hölder inequality and Young inequality, $|I_1|$ can be estimated as

$$\begin{aligned}|I_1| &= \left|\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, h\right\rangle dx\right|\\ &\leq (p-1)\int_{\{|u-\theta|>L\}} (|\nabla\theta| + |\nabla u-\nabla\theta|)^{p-2}|\nabla u-\nabla\theta||h|\,dx\\ &\leq 2^{p-2}(p-1)\left(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-2}|\nabla u-\nabla\theta||h|\,dx + \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{p-1}|h|\,dx\right)\\ &\leq 2^{p-2}(p-1)\Big[\Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\,dx\Big)^{\frac{p-2}{r}}\Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\Big)^{\frac{1}{r}}\\ &\quad\cdot\Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}}\,dx\Big)^{\frac{r-p+1}{r}} + \Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\Big)^{\frac{p-1}{r}}\cdot\Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}}\,dx\Big)^{\frac{r-p+1}{r}}\Big]\\ &\leq 2^{p-2}(p-1)C(n,p)|p-r|\Big[\Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\,dx\Big)^{\frac{p-2}{r}}\\ &\quad\cdot\Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\Big)^{\frac{r-p+2}{r}} + \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\Big]\end{aligned} \tag{3.7}$$

Using the Hölder inequality, (3.4) and Young inequality, $|I_2|$ and $|I_3|$ can be estimated as

$$\begin{aligned}|I_2| &= \left|\int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, h\right\rangle dx\right|\\ &\leq \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|h|\,dx\\ &\leq \Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\,dx\Big)^{\frac{1}{r}}\Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}}\,dx\Big)^{\frac{r-p+1}{r}}\\ &\leq C(n,p)|p-r|\Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\,dx\Big)^{\frac{p-1}{r}}\cdot\Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\Big)^{\frac{r-p+1}{r}}\\ &\leq C(n,p)|p-r|\Big[C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\,dx + \varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\Big]\ ,\end{aligned} \tag{3.8}$$

$$\begin{aligned}|I_3| &= \left|-\int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx\right|\\ &\leq \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|\nabla u-\nabla\theta|^{r-p+1}\,dx\\ &\leq \Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\,dx\Big)^{\frac{p-1}{r}}\Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\Big)^{\frac{r-p+1}{r}}\\ &\leq C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\,dx + \varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\,dx\,.\end{aligned} \tag{3.9}$$

Using the Hölder inequality, Sobolev-Poincáre inequality[7],

$$\Big(\int_{\Omega} |u-u_{\Omega}|^{pn/(n-p)}dx\Big)^{(n-p)/pn} \leq C\Big(\int_{\Omega} |\nabla u|^{p}\Big)^{1/p}, (1 \leq p < n),$$

and using (3.3) and Young inequality, $|I_4|$ can be estimated as

$$
\begin{aligned}
|I_4| &= \left|\int_{\{|u-\theta|>L\}} f(x)\Phi dx\right| \\
&\le \left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}} dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\varphi-\varphi_\Omega|^{\frac{nr}{n(r-p+1)-r}} dx\right)^{\frac{n(r-p+1)-r}{nr}} \\
&\le C(n,p)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}} dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\nabla\Phi|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}} dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\nabla v|^{r} dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)[C(\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}} dx\right)^{\frac{n(p-1)+r}{n(p-1)}} \\
&\qquad +\varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r} dx].
\end{aligned} \tag{3.10}
$$

Combining (3.5)-(3.10), we arrive at

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r} dx \\
&\le C(n,p,\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r} dx \\
&\quad +(C(n,p)\,|p-r|+\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r} dx \\
&\quad +C(n,p,\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}} dx\right)^{\frac{n(p-1)+r}{n(p-1)}} ,
\end{aligned} \tag{3.11}
$$

Case 2: $1 < p < 2$. Lemma 2.3 yields

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx \\
&\ge \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1} \\
&\qquad \cdot((|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1} - |\nabla\theta|^{p-1})dx.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r} dx \\
&\le \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}(|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1} dx \\
&\le \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx \\
&\quad +\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1}|\nabla\theta|^{p-1} dx \\
&\le \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx \\
&\quad +\varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r} dx + C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r} dx .
\end{aligned} \tag{3.12}
$$

Using Lemma 2.2 and (3.4), $|I_1|$ can be estimated as

$$
\begin{aligned}
|I_1| &= \left|\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, h\right\rangle dx\right| \\
&\le \frac{3-p}{2^{p-2}(p-1)}\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{p-1}|h| dx \\
&\le \frac{3-p}{2^{p-2}(p-1)}\left(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r} dx\right)^{\frac{p-1}{r}} \cdot \left(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}} \\
&\le \frac{3-p}{2^{p-2}(p-1)} C(n,p)\,|p-r| \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r} dx.
\end{aligned} \tag{3.13}
$$

For the case $1 < p < 2$, $|I_2|$-$|I_3|$ can also be estimated by (3.8)-(3.9). Combining (3.5), (3.12) and (3.13) , we arrive at (3.11).

Let $\varepsilon_0 = 1/C(n,p)$, Then for $|p-r| < \varepsilon_0$ we have $C(n,p)|p-r| < 1$, Taking ε small enough, such that $C(n,p)|p-r| + \varepsilon < 1$, then the second term on the right-hand side of (3.11) can be absorbed by the left-hand side; thus we obtain

$$\begin{aligned}&\int_{\{|u-\theta|>L\}} |\nabla u - \nabla\theta|^r \, dx \\ &\le C(n,p)\int_{\{|u-\theta|>L\}} |\nabla\theta|^r \, dx + C(n,p)\Big(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}} \, dx\Big)^{\frac{n(p-1)+r}{n(p-1)}}.\end{aligned} \tag{3.14}$$

Since $\theta \in W^{1,q}(\Omega)$, $q > r$, using the Hölder inequality, we have

$$\begin{aligned}&\int_{\{|u-\theta|>L\}} |\nabla\theta|^r \, dx \\ &\le \Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^q \, dx\Big)^{r/q} |\{|u-\theta|>L\}|^{(q-r)/q} \\ &= \|\nabla\theta\|_q^r |\{|u-\theta|>L\}|^{(q-r)/q} .\end{aligned} \tag{3.15}$$

By the proof idea of reference [9](Page 442), and the Hölder inequality, we get

$$\begin{aligned}&\Big(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}} \, dx\Big)^{\frac{n(p-1)+r}{n(p-1)}} \\ &\le \Big(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nq}{n(p-1)+r}} \, dx\Big)^{\frac{nr(p-1)+r^2}{qn(p-1)}} |\{|u-\theta|>L\}|^{(q-r)/q} \\ &\le M\,|\{|u-\theta|>L\}|^{(q-r)/q},\end{aligned} \tag{3.16}$$

where $M = \Big(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nq}{n(p-1)+r}} \, dx\Big)^{\frac{nr(p-1)+r^2}{qn(p-1)}}$, M is bounded and is a constant dependent only on n, p. Then (3.14) can be collated into the following results

$$\begin{aligned}&\int_{\{|u-\theta|>L\}} |\nabla u - \nabla\theta|^r \, dx \\ &\le C(n,p)\Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^q \, dx\Big)^{r/q} |\{|u-\theta|>L\}|^{(q-r)/q} \\ &\quad + C(n,p)M\,|\{|u-\theta|>L\}|^{(q-r)/q} \\ &= C\,|\{|u-\theta|>L\}|^{(q-r)/q}\,(1 + \|\nabla\theta\|_q^r) ,\end{aligned} \tag{3.17}$$

where $C = C(n, p, \varepsilon, \varsigma, M)$,

We now turn our attention back to the function $v \in W_0^{1,r}(\Omega)$. By the Sobolev embedding theorem, we have

$$\begin{aligned}\Big(\int_\Omega |v|^{r^*} dx\Big)^{1/r^*} &\le C(n,r)\Big(\int_\Omega |\nabla v|^r dx\Big)^{1/r} \\ &= C(n,r)\Big(\int_{\{|u-\theta|>L\}} |\nabla u - \nabla\theta|^r dx\Big)^{1/r} ,\end{aligned} \tag{3.18}$$

since $|v| = (|u-\theta| - L) \cdot 1_{\{|u-\theta|>L\}}$, we have

$$\Big(\int_{\{|u-\theta|>L\}} (|\nabla u - \nabla\theta| - L)^{r^*} dx\Big)^{1/r^*} = \Big(\int_\Omega |v|^{r^*} dx\Big)^{1/r^*} , \tag{3.19}$$

and for $\tilde{L} > L$,

$$
\begin{aligned}
&(\tilde{L}-L)^{r^*}|\{|u-\theta|>\tilde{L}\}| \\
&=\int_{\{|u-\theta|>\tilde{L}\}}(\tilde{L}-L)^{r^*}dx \\
&\le\int_{\{|u-\theta|>\tilde{L}\}}(|u-\theta|-L)^{r^*}dx \\
&\le\int_{\{|u-\theta|>\tilde{L}\}}(|u-\theta|-L)^{r^*}dx.
\end{aligned} \tag{3.20}
$$

By collecting (3.17)-(3.20), we deduce that

$$
\begin{aligned}
&((\tilde{L}-L)^{r^*}|\{|u-\theta|>\tilde{L}\}|)^{1/r^*} \\
&\le C(n,r)(\|\nabla\theta\|_q+1)|\{|u-\theta|>L\}|^{1/r-1/q}
\end{aligned} \tag{3.21}
$$

Thus

$$
\begin{aligned}
&|\{|u-\theta|>\tilde{L}\}| \\
&\le\frac{1}{(\tilde{L}-L)^{r^*}}(C(n,r)(\|\nabla\theta\|_q+1))^{r^*}|\{|u-\theta|>L\}|^{r^*(1/r-1/q)}
\end{aligned} \tag{3.22}
$$

Let $\phi(s)=|\{|u-\theta|>s\}|$, $\alpha=r*$, $c=(C(n,r)(\|\nabla\theta\|_q+1))^{r^*}$, $\beta=r*(1/r-1/q)$, $s_0>0$, Then (3.22) become

$$
\phi(\tilde{L})\le\frac{c}{(\tilde{L}-L)^{\alpha}}\phi(L)^{\beta} \tag{3.23}
$$

for $\tilde{L}>L>0$.

(1) For the case $q<n$, one has $\beta<1$. In this case, if $s\ge 1$, we get from Lemma 2.3 that

$$
|\{|u-\theta|>s\}|\le c(\alpha,\beta,s_0)s^{-t},
$$

where $t=\alpha/(1-\beta)=q*$. For $0<s<1$, one has

$$
|\{|u-\theta|>s\}|\le|\Omega|=|\Omega|s^{q^*}s^{-q^*}\le|\Omega|s^{-q^*}.
$$

Thus

$$
u\in\theta+L^{q^*}_{weak}(\Omega).
$$

(2) For the case $q=n$, one has $\beta=1$. For any $\tau<\infty$, (3.23) implies

$$
\begin{aligned}
\phi(\tilde{L})&\le\frac{c}{(\tilde{L}-L)^{\alpha}}\phi(L)=\frac{c}{(\tilde{L}-L)^{\alpha}}\phi(L)^{1-\alpha/\tau}\phi(L)^{\alpha/\tau} \\
&\le\frac{c|\Omega|^{\alpha/\tau}}{(\tilde{L}-L)^{\alpha}}\phi(L)^{1-\alpha/\tau}.
\end{aligned}
$$

As about, we derive

$$
u\in\theta+L^{\tau}_{weak}(\Omega).
$$

(3) For the case $q>n$, one has $\beta>1$. Lemma 2.3 implies $\phi(d)=0$ for some $d=d(\alpha,\beta,s_0,r,(\|\nabla\theta\|_q+1))$. Thus $|\{|u-\theta|>d\}|=0$, which means $u-\theta\le d$ a.e. in Ω, Therefore

$$
u\in\theta+L^{\infty}(\Omega),
$$

completing the proof of Theorem 1.1.

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