Original Research Article

A sharp estimate of entropy solution to Euler-Poisson system for semiconductors in the whole domain

Abstract: In this paper, we are concerned with the global existence, large time behavior, and time-increasing-rate of entropy solutions to the one-dimensional unipolar hydrodynamic model for semiconductors in the form of Euler-Poisson equations. When the adiabatic index $\gamma > 2$, the L^∞ estimates of artificial viscosity approximate solutions are obtained by using entropy inequality and maximum principle. Then the L^∞ compensated compactness framework demonstrates the convergence of approximate solutions. Finally, the global entropy solutions are proved to decay exponentially fast to the stationary solution, without any assumption on the smallness of initial data and doping profile.

1 Introduction

In this paper, we consider the following one-dimensional Euler-Poisson system for semiconductor devices:

$$\begin{cases} n_t + J_x = 0, \\ J_t + (\dfrac{J^2}{n} + p(n))_x = nE - J, \qquad x \in \mathbf{R},\ t > 0, \\ E_x = n - b(x). \end{cases} \tag{1.1}$$

Here $n \geq 0,\ J,$ and E denote the electron density, (average) electron current density and the (negative) electric field, respectively. We assume the pressure p satisfies the γ-law: $p(n) = p_0 n^\gamma$ $(\gamma > 1)$, where $p_0 = \frac{\theta^2}{\gamma}$, $\theta = \frac{\gamma-1}{2}$. Several physical constants (such as the relaxation time) have been set to be unity for the simplicity of presentation. The function $b(x) > 0$, which is called doping profile, stands for the density of fixed, positively charged background ions. We refer to [26] for background on modeling and analysis.

System (1.1) is supplemented with the initial conditions

$$n(x,0) = n_0(x) \geq 0, \qquad J(x,0) = J_0(x), \tag{1.2}$$

which satisfy

$$\lim_{x\to\pm\infty} n_0(x) = n_\pm, \quad \lim_{x\to\pm\infty} J_0(x) = \bar{J}, \quad \lim_{x\to\pm\infty} n_0'(x) = 0.$$

$E(x,t)$ is added on the "boundary" condition

$$\lim_{x\to-\infty} E(x,t) = E_-. \tag{1.3}$$

Here the letters $n_\pm$, $\bar{J}$ and E_- are given constants. In this paper, we assume $b(x)$ satisfies

$$\begin{aligned} b(x) \in C^2(\mathbf{R}),\ b'(x) \in L^1(\mathbf{R}) \cap H^1(\mathbf{R}), \\ \lim_{x\to\pm\infty} b(x) = \quad b_\pm > 0, \quad B^* = \sup_x b(x) \geq \inf_x b(x) = B_* > 0. \end{aligned} \tag{1.4}$$

The corresponding steady-state model of equation (1.1) is

$$\begin{cases} \widetilde{J}_x = 0, \\ (\dfrac{\widetilde{J}^2}{\widetilde{N}} + p(\widetilde{N}))_x = \widetilde{N}\widetilde{E} - \widetilde{J}, \\ \widetilde{E}_x = \widetilde{N} - b(x). \end{cases} \tag{1.5}$$

A lot of efforts have been made for system (1.1) and (1.5) on the whole space or bounded domain. The mathematical study was initiated by Degond and Markowich, who obtained the existence of a unique subsonic smooth solution for the steady-state model (1.5) in [3]. Then, some other kinds of subsonic, transonic, and supersonic solutions are obtained, cf. [1,4,8,23,24,28] *etc.* As for the time-dependent problem (1.1), the existence of a local smooth solution was proved in [31], and the existence and asymptotic behavior of global smooth solution can be found in [9,10,13,14,17,18,22,27]. However, when the C^1 norm of the initial data is large, the corresponding solution shall develop singularities in finite time, see [29]. So weak solutions that include singularities should be considered. As far as weak solutions are concerned, Zhang [32], Marcati-Natalini [25], Li [20], and Li-Huang-Yu [11] investigated the global existence of entropy solutions by using numerical schemes and the compensated compactness method. As to the large time behavior of weak solutions, [16] first built the large time behavior framework for **any** uniformly bounded weak entropy solution, including vacuum case. It is worthy to be pointed out that in [16] the uniform bound of density is assumed to be independent of time. Later Yu in [30] considered the large time behavior of weak solutions with the density's bound increasing with time. However, the result in [30] need the density's bound increase **slowly**, i.e.

$$\|n(x,t)\|_{L^\infty} \leq Ct^\alpha, \quad \alpha < 2. \tag{1.6}$$

For the approximate solutions obtained by Lax-Friedrichs scheme, H. Yu [30] get the estimate

$$\|n^l(x,t)\|_{L^\infty} \leq C(1+t)^{\frac{2}{\theta}}, \quad \theta = \frac{\gamma-1}{2}. \tag{1.7}$$

Therefore, to ensure the assumption (1.6), the author required the adiabatic exponent $\gamma > 3$. A natural question is what about the case $1 < \gamma \leq 3$? One way to consider this problem is to improve the estimate of density. In this direction, X. Fang and H. Yu gave a sufficient condition which ensure the boundedness of the global weak solution (i.e. $\alpha = 0$ in (1.6)) in [7]. For problem (1.1) with insulating boundary condition, Huang *et. al.* [12] proved the vanishing viscosity weak solution are uniformly bounded by using a maximum principle, and then they proved the weak solution converge to the steady state with an exponential decay rate. However, since the boundedness of the space variable played an essential role in the proof, the method used in [12] can not be used in the whole line.

In this paper, we construct approximate solution $(n^\varepsilon, J^\varepsilon, E^\varepsilon)(x,t)$ of system $(1.1)-(1.3)$ by adding artificial viscosity and prove the approximate solution satisfy

$$\|n^\varepsilon(x,t)\|_{L^\infty} \leq M(1+t)^{\frac{1}{\theta}}, \ \|J^\varepsilon(x,t)\|_{L^\infty} \leq M(1+t)^{1+\frac{1}{\theta}}, \ \|E^\varepsilon(x,t)\|_{L^\infty} \leq M, \tag{1.8}$$

for $\theta = \frac{\gamma-1}{2}$ by using entropy estimate and maximum principle in the whole space. Then, the compensated compactness theory ensure the convergence and consistency. Finally, using the framework built in [30], we get the large time behavior of the obtained weak entropy solution.

Before stating our main results, we give the definition of entropy solution to $(1.1)-(1.3)$.

Definition 1. *For any fixed $T>0$, the bounded measurable function $(n,\ J,\ E)(x,\ t)$ is said to be a weak solution of problem $(1.1)-(1.3)$ in $(-\infty,\infty)\times[0,T)$, if it satisfies the system (1.1) in the sense of distribution and verifies the initial and limiting restrictions (1.2) and (1.3). Furthermore, a weak solution of system $(1.1)-(1.3)$ is called an weak entropy solution if it satisfies the entropy inequality*

$$\eta_t + q_x - \eta_J(nE-J) \leq 0 \tag{1.9}$$

in the sense of distribution, where (η, q) is mechanical entropy-entropy-flux pair satisfying

$$\eta(n,J) = \frac{J^2}{2n} + \frac{p_0 n^\gamma}{\gamma-1}, \ q(n,J) = \frac{J^3}{2n^2} + \frac{p_0\gamma}{\gamma-1} n^{\gamma-1} J. \tag{1.10}$$

Suppose $(n,J,E)(x,t)$ is the weak entropy solution of $(1.1)-(1.3)$, $(\widetilde{N}, \bar{J}, \widetilde{E})$ is the corresponding stationary solution. Due to the relaxation mechanism in the equation of (1.1), we expect that all its solutions converge to the corresponding steady solution. Therefore, we define the relative entropy

$$\begin{aligned} \eta_* &= \eta - \widetilde{\eta} - \nabla\widetilde{\eta}(U-\widetilde{U}) \\ &= \frac{J^2}{2n} + \frac{p_0 n^\gamma}{\gamma-1} - \frac{\bar{J}^2}{2\widetilde{N}} - \frac{p_0\widetilde{N}^\gamma}{\gamma-1} - [\frac{p_0\gamma}{\gamma-1}\widetilde{N}^{\gamma-1} - \frac{\bar{J}^2}{2\widetilde{N}^2}](n-\widetilde{N}) - \frac{\bar{J}}{\widetilde{N}}(J-\bar{J}). \end{aligned} \tag{1.11}$$

The main results of this paper are given as follows.

Theorem 1. *For any adiabatic index $\gamma>2$, suppose the initial data satisfy*

$$(E-\widetilde{E})(x,0) \in L^2(\mathbf{R}), \quad \eta_*(x,0) \in L^1(\mathbf{R}), \tag{1.12}$$

$$\int_{-\infty}^{\infty} \Big(n_0(s) - \widetilde{N}(s)\Big) ds = 0, \tag{1.13}$$

and

$$0 \leq n_0(x) \leq C_0, \quad |J_0(x)| \leq C_0 n_0(x), \tag{1.14}$$

for some constant $C_0>0$. Then
(a) *There exists a global L^∞ weak solution $(n,J,E)(x,t)$ to $(1.1)-(1.3)$ in the sense of Definition 1 such that*

$$0 \leq n(x,t) \leq M(1+t)^{\frac{1}{\theta}}, \ \ |J(x,t)| \leq M(1+t)^{1+\frac{1}{\theta}}, \ \ |E(x,t)| \leq M, \tag{1.15}$$

for some constant M depending solely on C_0 and γ, $\theta = \frac{\gamma-1}{2}$.
(b) *There exist positive constants $T^*(\gamma), C$, and $\tilde{C}$ such that*

$$\begin{aligned}\int_{-\infty}^{+\infty}\Big((E-\widetilde{E})^2(x,t) \;&+\; \eta^*(x,t)\Big)dx \\ &\le\; Ce^{-\tilde{C}t^{\frac{\gamma-2}{\gamma-1}}}\int_{-\infty}^{+\infty}\Big((E-\widetilde{E})^2(x,0)+\eta^*(x,0)\Big)dx\end{aligned} \tag{1.16}$$

for any $t > T^*(\gamma)$.

Remark: *Compared with the large time behavior result in [30], we generalize the adiabatic exponent to $\gamma > 2$.*

Throughout this paper, C_i $(i = 1, 2...)$ and M mean different constants, while $C(\cdot)$ denotes constant depending on the parameters in the bracket.

2 Preliminary and formulation

We first introduce some basic facts about the homogenous compressible Euler equation

$$\begin{cases} n_t + J_x = 0, \\ J_t + \left(\dfrac{J^2}{n} + p(n)\right)_x = 0. \end{cases} \tag{2.1}$$

The eigenvalues are

$$\lambda_1 = \frac{J}{n} - \theta n^\theta, \quad \lambda_2 = \frac{J}{n} + \theta n^\theta, \tag{2.2}$$

and the corresponding right eigenvectors are

$$r_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}. \tag{2.3}$$

The Riemann invariants are given by

$$W = \frac{J}{n} + n^\theta, \quad Z = \frac{J}{n} - n^\theta, \tag{2.4}$$

satisfying $\nabla W \cdot r_1 = 0$ and $\nabla Z \cdot r_2 = 0$, where $\nabla = (\partial_n, \partial_J)$ is the gradient with respect to $U = (n, J)$.

As for the steady system (1.5) under the conditions

$$\widetilde{N}(x) - b(x) \in H^1(R),\ \widetilde{J} = \bar{J},\ \widetilde{E}(-\infty) = E^-, \tag{2.5}$$

we consider the classical solution in the region where the subsonic condition

$$\inf_x (p'(\widetilde{N}) - \frac{\bar{J}^2}{\widetilde{N}^2}) > 0 \tag{2.6}$$

and the positivity of the density $\inf_x \widetilde{N}(x) > 0$ hold. Then we have the following theorem, see [16,19,22] etc for details.

Theorem A. *Suppose $b(x)$ satisfies (1.4) and $p'(B_*)B_*^2 > \bar{J}^2$, then problem (1.5) has a unique solution $(\widetilde{N}, \bar{J}, \widetilde{E})$ such that $b_\pm \widetilde{E}(\pm\infty) = \bar{J}$ and*

$$\begin{aligned} & B_* \leq \widetilde{N}(x) \leq B^*, \quad x \in \mathbf{R}, \\ & |\widetilde{N}(x) - b_\pm| = O(1)e^{-C_\pm |x|}, \quad \text{as } x \to \pm\infty, \\ & \|\widetilde{N}(x) - b(x)\|_{H^2} + \sup_{x\in\mathbf{R}}(|\widetilde{N}'(x)| + |\widetilde{N}''(x)| + |\widetilde{E}(x)|) \leq C_1, \end{aligned} \tag{2.7}$$

where $C_1 > 0$ depends only on $b(x)$ and $C_\pm = \dfrac{\widetilde{E}_\pm}{p'(b_\pm) - \widetilde{E}_\pm^2}$.

Theorem B comes from [15] Lemma 3.1, after a minor modification in the proof:

Theorem B. *Let $\gamma > 2$, $n \geq 0$ and $0 < a \leq N \leq b < +\infty$. Then there exists positive constant $\tilde{C}_1(a, b, \gamma)$ such that*

$$n^\gamma - N^\gamma - \gamma N^{\gamma-1}(n - N) \geq \tilde{C}_1(n - N)^2.$$

3 Boundedness of the approximate solution

For any fixed $T \in \mathbf{R}^+$, we consider the following approximate equation

$$\begin{cases} n_t^\varepsilon + J_x^\varepsilon = \varepsilon n_{xx}^\varepsilon, \\ J_t^\varepsilon + \left(\dfrac{(J^\varepsilon)^2}{n^\varepsilon} + p(n^\varepsilon)\right)_x = \varepsilon J_{xx}^\varepsilon + n^\varepsilon E^\varepsilon - J^\varepsilon, \\ E^\varepsilon = E_- + \displaystyle\int_{-\infty}^x (n^\varepsilon(s,t) - b(s))ds \end{cases} \tag{3.1}$$

in $(x,t) \in (-\infty, +\infty) \times [0, T)$ with the initial data

$$(n_0^\varepsilon(x), J_0^\varepsilon(x)) = (\max(n_0(x), \varepsilon), J_0(x)) * j^\varepsilon, \tag{3.2}$$

and the far field condition

$$\lim_{x\to-\infty} E^\varepsilon(x,t) = E_-. \tag{3.3}$$

Here j^ε is the standard mollifier and the positive parameter ε is small.

The local existence of the approximate solution can be proved by the same argument of [5]. To extend the local solution to the whole space, it is important to obtain a priori estimate of the upper bound of $(n^\varepsilon, J^\varepsilon)$ and the positive lower estimate of n^ε. For $\gamma > 1$, as stated in [2,12], the positive lower bound of n^ε can be given by the upper bound estimate of $\frac{J^\varepsilon}{n^\varepsilon}$. Therefore, we only need to estimate the bounds of Riemann invariants $(w^\varepsilon, z^\varepsilon)$ to gain the upper bound of $(n^\varepsilon, \frac{J^\varepsilon}{n^\varepsilon})$. However, it is worthy to be pointed out that the estimates of approximate electric field

$$E^\varepsilon = \int_{-\infty}^x (n^\varepsilon(s,t) - b(s))ds + E_-, \tag{3.4}$$

which plays a vital role in deriving the upper bound of density, are not trivial in the whole space. This is very different from the bounded domain case considered in [12]. Fortunately, with the help of relative entropy, we obtain our aim.

To prove Theorem 1, we first use the estimate on E^ε to derive the upper bound of $(n^\varepsilon, \frac{J^\varepsilon}{n^\varepsilon})$ by the following maximum principle:

Lemma 3.1 [2] *Let $(x,t) \in \mathbf{R} \times [0,T]$ and $(p,q)(x,t)$ be any bounded classical solution of the following quasilinear parabolic system*

$$\begin{cases} p_t + \mu_1 p_x = \varepsilon p_{xx} + a_{11}p + a_{12}q + R_1, \\ q_t + \mu_2 q_x = \varepsilon q_{xx} + a_{21}p + a_{22}q + R_2, \end{cases} \tag{3.5}$$

with initial data $p(x,0) \leq 0$, $q(x,0) \geq 0$, where the coefficients μ_i and a_{ij} are bounded with respect to (x,t) and may depend on p,q. The source terms R_i may also depend on p,q and x,t. Assume that $a_{12}, a_{21} \leq 0$, $R_1 \leq 0$, $R_2 \geq 0$. Then for any (x,t), $p(x,t) \leq 0$, $q(x,t) \geq 0$.

Then we have the following Lemma:
Lemma 3.2 *The smooth solution $(n^\varepsilon, J^\varepsilon, E^\varepsilon)$ of problem (3.1)-(3.3) satisfies the following a priori estimate:*

$$0 \leq n^\varepsilon(x,t) \leq M(1+t)^{\frac{1}{\theta}}, \quad |J^\varepsilon(x,t)| \leq M(1+t)^{1+\frac{1}{\theta}}, \quad |E^\varepsilon(x,t)| \leq M, \quad (x,t) \in \mathbf{R} \times [0,T], \tag{3.6}$$

where the positive constant M depends only on the initial data.

Proof: By the formula of Riemann invariants (2.4), system $(3.1)_{1,2}$ can be rewritten as

$$\begin{cases} w_t + \mu_2 w_x = \varepsilon w_{xx} + 2\varepsilon w_x \dfrac{n_x^\varepsilon}{n^\varepsilon} - \varepsilon\theta(\theta+1)(n^\varepsilon)^{\theta-2}(n_x^\varepsilon)^2 + \dfrac{1}{n^\varepsilon}\left(n^\varepsilon E^\varepsilon - J^\varepsilon\right), \\ z_t + \mu_1 z_x = \varepsilon z_{xx} + 2\varepsilon z_x \dfrac{n_x^\varepsilon}{n^\varepsilon} + \varepsilon\theta(\theta+1)(n^\varepsilon)^{\theta-2}(n_x^\varepsilon)^2 + \dfrac{1}{n^\varepsilon}\left(n^\varepsilon E^\varepsilon - J^\varepsilon\right), \end{cases} \tag{3.7}$$

where

$$w = \frac{J^\varepsilon}{n^\varepsilon} + (n^\varepsilon)^\theta, \quad z = \frac{J^\varepsilon}{n^\varepsilon} - (n^\varepsilon)^\theta, \tag{3.8}$$

$$\mu_1 = \frac{J^\varepsilon}{n^\varepsilon} - (\theta n^\varepsilon)^\theta, \quad \mu_2 = \frac{J^\varepsilon}{n^\varepsilon} + (\theta n^\varepsilon)^\theta. \tag{3.9}$$

Set the control function $\phi(t) = 2M_0(1+t)$, where M_0 will be determined later. Denote the modified Riemann invariants $(\bar{w}, \bar{z})$ as

$$\bar{w} = w - \phi, \quad \bar{z} = z + \phi. \tag{3.10}$$

Then (3.7) yields the decoupled equations for $(\bar{w}, \bar{z})$:

$$\begin{cases} \bar{w}_t + \mu_2 \bar{w}_x = \varepsilon \bar{w}_{xx} + 2\varepsilon \bar{w}_x \dfrac{n_x^\varepsilon}{n^\varepsilon} - \varepsilon\theta(\theta+1)(n^\varepsilon)^{\theta-2}(n_x^\varepsilon)^2 + E^\varepsilon - \dfrac{J^\varepsilon}{n^\varepsilon} - 2M_0, \\ \bar{z}_t + \mu_1 \bar{z}_x = \varepsilon \bar{z}_{xx} + 2\varepsilon \bar{z}_x \dfrac{n_x^\varepsilon}{n^\varepsilon} + \varepsilon\theta(\theta+1)(n^\varepsilon)^{\theta-2}(n_x^\varepsilon)^2 + E^\varepsilon - \dfrac{J^\varepsilon}{n^\varepsilon} + 2M_0. \end{cases} \tag{3.11}$$

Since
$$\frac{J^\varepsilon}{n^\varepsilon}=\frac{w+z}{2}=\frac{\bar{w}+\bar{z}}{2},$$
then system (3.11) becomes
$$\begin{cases}\bar{w}_t+(\mu_2-2\varepsilon\frac{n_x^\varepsilon}{n^\varepsilon})\bar{w}_x=\varepsilon\bar{w}_{xx}+a_{11}\bar{w}+a_{12}\bar{z}+R_1,\\ \bar{z}_t+(\mu_1-2\varepsilon\frac{n_x^\varepsilon}{n^\varepsilon})\bar{z}_x=\varepsilon\bar{z}_{xx}+a_{21}\bar{w}+a_{22}\bar{z}+R_2,\end{cases}$$
with
$$\begin{aligned}&a_{11}=-\frac{1}{2},\quad a_{12}=-\frac{1}{2},\quad a_{21}=-\frac{1}{2},\quad a_{22}=-\frac{1}{2},\\ &R_1=-\varepsilon\theta(\theta+1)(n^\varepsilon)^{\theta-2}(n_x^\varepsilon)^2+E^\varepsilon-2M_0,\\ &R_2=\varepsilon\theta(\theta+1)(n^\varepsilon)^{\theta-2}(n_x^\varepsilon)^2+E^\varepsilon+2M_0.\end{aligned}$$
In order to use Lemma 3.1, we need to get the estimate of approximate electric fields by the relative entropy. Therefore, we introduce a new variable
$$y^\varepsilon=-(E^\varepsilon-\widetilde{E})=-\int_{-\infty}^{x}\Big(n^\varepsilon(s,t)-\widetilde{N}(s)\Big)ds. \tag{3.12}$$
Then system (3.1) infer that
$$y_x^\varepsilon=-(n^\varepsilon-\widetilde{N}),\qquad y_t^\varepsilon=J^\varepsilon-\varepsilon n_x^\varepsilon-\bar{J}, \tag{3.13}$$
where we have used the facts that $n_x^\varepsilon(-\infty)=0$, $J^\varepsilon(-\infty)=\bar{J}$, which can be seen by inverting the heat operator, differentiating by space and then letting $x\to-\infty$. For simplicity, we only consider the case $E_-=\bar{J}=0$, the other case can be treated similarly as the way used in [16].

Define convex entropy-entropy flux pairs
$$\tilde{\eta}(n^\varepsilon,J^\varepsilon)=\frac{(J^\varepsilon)^2}{2n^\varepsilon}+\frac{p_0(n^\varepsilon)^\gamma}{\gamma-1},\quad \tilde{q}(n^\varepsilon,J^\varepsilon)=\frac{(J^\varepsilon)^3}{2(n^\varepsilon)^2}+\frac{\gamma p_0(n^\varepsilon)^{\gamma-1}}{\gamma-1}J^\varepsilon. \tag{3.14}$$
Multiplying $(3.1)_{1,2}$ with $\nabla\tilde{\eta}=(\tilde{\eta}_{n^\varepsilon},\tilde{\eta}_{J^\varepsilon})$, we have
$$\begin{aligned}\tilde{\eta}_t+\tilde{q}_x&=\varepsilon(\eta_{n^\varepsilon}n_{xx}^\varepsilon+\eta_{J^\varepsilon}J_{xx}^\varepsilon)+J^\varepsilon E^\varepsilon-\frac{(J^\varepsilon)^2}{n^\varepsilon}\\ &\le\varepsilon(\eta_{n^\varepsilon}n_x^\varepsilon+\eta_{J^\varepsilon}J_x^\varepsilon)_x+J^\varepsilon E^\varepsilon-\frac{(J^\varepsilon)^2}{n^\varepsilon},\end{aligned} \tag{3.15}$$
where the convexity of η is used. Introduce the relative entropy-entropy flux pair
$$\begin{aligned}&\tilde{\eta}_*=\tilde{\eta}-\frac{p_0\widetilde{N}^\gamma}{\gamma-1}-\frac{p_0\gamma}{\gamma-1}\widetilde{N}^{\gamma-1}(n^\varepsilon-\widetilde{N}),\\ &\tilde{q}_*=\tilde{q}-\frac{p_0\gamma}{\gamma-1}\widetilde{N}^{\gamma-1}J^\varepsilon.\end{aligned} \tag{3.16}$$

Since

$$J^\varepsilon E^\varepsilon = J^\varepsilon \widetilde{E} - y^\varepsilon J^\varepsilon = p_0\gamma \widetilde{N}^{\gamma-2} J^\varepsilon \widetilde{N}_x - y^\varepsilon J^\varepsilon, \tag{3.17}$$

then (3.15) turns into

$$\begin{aligned}
& \tilde{\eta}_{*t} + \tilde{q}_{*x} + \frac{(J^\varepsilon)^2}{n^\varepsilon} + J^\varepsilon y^\varepsilon \\
= \;& \tilde{\eta}_{*t} + \tilde{q}_{*x} + \frac{(J^\varepsilon)^2}{n^\varepsilon} + p_0\gamma \widetilde{N}^{\gamma-2} J^\varepsilon \widetilde{N}_x - J^\varepsilon E^\varepsilon \\
= \;& \tilde{\eta}_t + \tilde{q}_x - \tfrac{p_0\gamma}{\gamma-1} \widetilde{N}^{\gamma-1} n^\varepsilon_t - \tfrac{p_0\gamma}{\gamma-1} (\widetilde{N}^{\gamma-1} J^\varepsilon)_x + p_0\gamma \widetilde{N}^{\gamma-2} J^\varepsilon \widetilde{N}_x + \frac{(J^\varepsilon)^2}{n^\varepsilon} - J^\varepsilon E^\varepsilon \\
\leq \;& \varepsilon(\tilde{\eta}_{n^\varepsilon} n^\varepsilon_x + \tilde{\eta}_{J^\varepsilon} J^\varepsilon_x)_x - \frac{\varepsilon p_0 \gamma}{\gamma - 1} \widetilde{N}^{\gamma-1} n^\varepsilon_{xx}.
\end{aligned} \tag{3.18}$$

Integrating (3.18) over $(-\infty, +\infty)$, we have

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \tilde{\eta}_* dx + \int_{-\infty}^{+\infty} (J^\varepsilon y^\varepsilon + \frac{(J^\varepsilon)^2}{n^\varepsilon}) dx \leq -\frac{\varepsilon p_0 \gamma}{\gamma-1}\int_{-\infty}^{+\infty} \widetilde{N}^{\gamma-1} n^\varepsilon_{xx} dx. \tag{3.19}$$

Moreover, noticing (1.4), (2.7) and

$$\begin{aligned}
\int_{-\infty}^{+\infty} \widetilde{N}^{\gamma-1} n^\varepsilon_{xx} dx &= -\int_{-\infty}^{+\infty} (\gamma-1)\widetilde{N}^{\gamma-2}\widetilde{N}_x n^\varepsilon_x dx \\
&= (\gamma-1)\int_{-\infty}^{+\infty} [\widetilde{N}^{\gamma-2}\widetilde{N}_{xx} + (\gamma-2)\widetilde{N}^{\gamma-3}(\widetilde{N}_x)^2] n^\varepsilon dx \\
&\leq (\gamma-1)\|n^\varepsilon\|_{L^\infty}\int_{-\infty}^{+\infty} [\widetilde{N}^{\gamma-2}\widetilde{N}_{xx} + (\gamma-2)\widetilde{N}^{\gamma-3}(\widetilde{N}_x)^2] dx
\end{aligned}$$

we get

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \tilde{\eta}_* dx + \int_{-\infty}^{+\infty} (J^\varepsilon y^\varepsilon + \frac{(J^\varepsilon)^2}{n^\varepsilon}) dx \leq C_2 \varepsilon \|n^\varepsilon\|_{L^\infty} \tag{3.20}$$

from (3.19) and the Cauchy's inequality, where $C_2 = C(\gamma, \|\widetilde{N}\|_{H^2}, \|\widetilde{N}\|_{L^\infty})$.

On the other hand, form (3.13) we have

$$\int_{-\infty}^{+\infty} J^\varepsilon y^\varepsilon dx = \frac{d}{dt}\int_{-\infty}^{+\infty} \frac{(y^\varepsilon)^2}{2} dx + \varepsilon \int_{-\infty}^{+\infty} (y^\varepsilon_x)^2 dx - \varepsilon \int_{-\infty}^{+\infty} \widetilde{N} y^\varepsilon_x dx. \tag{3.21}$$

Again due to the Cauchy's inequality, (3.20) turns into

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty} (\tilde{\eta}_* + \frac{(y^\varepsilon)^2}{2}) dx + \frac{\varepsilon}{2}\int_{-\infty}^{+\infty} (y^\varepsilon_x)^2 dx + \int_{-\infty}^{+\infty} \frac{(J^\varepsilon)^2}{n^\varepsilon} dx \\
&\leq C_3 \varepsilon (\|n^\varepsilon\|_{L^\infty} + 1),
\end{aligned} \tag{3.22}$$

where C_3 depends only on γ, $\|\widetilde{N}\|_{H^2}$ and $\|\widetilde{N}\|_{L^\infty}$. Integrating this inequality from 0 to t gives

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (\tilde{\eta}_* + \frac{(y^\varepsilon)^2}{2}) dx \\
&\leq \int_{-\infty}^{+\infty} (\tilde{\eta}_* + \frac{(y^\varepsilon)^2}{2})(x,0) dx + \varepsilon C_3 \int_0^t (\|n^\varepsilon\|_{L^\infty} + 1) d\tau \\
&\leq \int_{-\infty}^{+\infty} (\tilde{\eta}_* + \frac{(y^\varepsilon)^2}{2})(x,0) dx + 1,
\end{aligned} \tag{3.23}$$

provided $0 < \varepsilon \le \varepsilon_1 = (C_3(C(T)+1)T)^{-1}$ and we priori assuming

$$\|n^\varepsilon\|_{L^\infty} \le C(T) = 2M_0(1+T)^{\frac{1}{\theta}}, \quad (x,t) \in R\times[0,T). \tag{3.24}$$

Define

$$\begin{aligned} \eta_*(x,0) &= \left[\frac{J^2}{2n} + \frac{p_0 n^\gamma}{\gamma-1} - \frac{p_0 \widetilde{N}^\gamma}{\gamma-1} - \frac{p_0\gamma}{\gamma-1}\widetilde{N}^{\gamma-1}(n-\widetilde{N})\right](x,0) \\ y(x,0) &= -(E-\widetilde{E})(x,0), \end{aligned} \tag{3.25}$$

and notice the definition in (3.2), we have

$$\int_{-\infty}^{+\infty} (\tilde{\eta}_* + \frac{(y^\varepsilon)^2}{2})(x,0)dx \le \int_{-\infty}^{+\infty} (\eta_* + \frac{y^2}{2})(x,0)dx + 1, \quad for \quad 0<\varepsilon\le\varepsilon_2 << 1. \tag{3.26}$$

On the other hand, form Theorem B, when $\gamma > 2$ there exists a positive constant $\tilde{C}_2(B_*, B^*, \gamma)$ such that

$$\tilde{\eta}_* \ge \tilde{C}_2\bigg(\frac{(J^\varepsilon)^2}{n^\varepsilon} + \frac{(y_x^\varepsilon)^2}{2}\bigg). \tag{3.27}$$

Combine (3.23), (3.26) and (3.27), we get

$$\int_{-\infty}^{+\infty}\bigg(\frac{(J^\varepsilon)^2}{n^\varepsilon} + \frac{(y_x^\varepsilon)^2}{2} + \frac{(y^\varepsilon)^2}{2}\bigg)dx \le \tilde{C}_3\bigg(\int_{-\infty}^{+\infty}(\eta_* + \frac{y^2}{2})(x,0)dx + 2\bigg) \tag{3.28}$$

for some $\tilde{C}_3 = \tilde{C}_3(B_*, B^*, \gamma)$ and $0<\varepsilon<\min\{\varepsilon_1,\varepsilon_2\}$. Then

$$\|y^\varepsilon\|_{L^\infty}^2 \le \int_{-\infty}^{+\infty}\Big((y^\varepsilon)^2 + (y_x^\varepsilon)^2\Big)dx \le 2\tilde{C}_3\bigg(\int_{-\infty}^{+\infty}(\eta_* + \frac{y^2}{2})(x,0)dx + 2\bigg).$$

Let

$$M_0 \ge 2\max\bigg\{\tilde{C}_3\bigg(\int_{-\infty}^{+\infty}(\eta_* + \frac{y^2}{2})(x,0)dx + 2\bigg), \quad \frac{J_0}{n_0} + n_0^\theta + 1, C_1\bigg\}$$

and notice the boundedness of $\tilde{E}$ in Theorem A, we have $|E^\varepsilon(x,t)| < \frac{3}{2}M_0$. Therefore,

$$R_1 \le \frac{3}{2}M_0 - 2M_0 \le 0, \tag{3.29}$$

$$R_2 \ge -\frac{3}{2}M_0 + 2M_0 \ge 0. \tag{3.30}$$

We also have the initial conditions

$$\bar{w}(x,0) = w(x,0) - \phi(0) = \frac{J_0}{n_0} + n_0^\theta - 2M_0 \le 0,$$

$$\bar{z}(x,0) = z(x,0) + \psi(0) = \frac{J_0}{n_0} - n_0^\theta + 2M_0 \ge 0.$$

Then Lemma 3.1 yields

$$\bar{w}(x,t) \le 0, \quad \bar{z}(x,t) \ge 0, \quad \forall (x,t) \in \mathbf{R}\times[0,T],$$

which implies that

$$\begin{aligned} w(x,t) &\le \phi(t) \le 2M_0(t+1), \\ z(x,t) &\ge -\phi(t) \ge -2M_0(t+1). \end{aligned}$$

Therefore, we finish the proof of Lemma 3.2.

4 Existence of global entropy solution and its large time behavior

The calculation in this section is standard, we just give the sketch. With the help of Lemma 3.2, and the compactness framework established in [5,6,21,25,32], we can prove that there exists a subsequence of $(n^\varepsilon, J^\varepsilon)$(still denoted by $(n^\varepsilon, J^\varepsilon)$) such that $(n^\varepsilon, J^\varepsilon) \to (n, J)$ in L^p_{loc}, $p \geq 1$ and (1.15) is proved. As to the large time behavior, use Theorem 3.1 in [30] we get (1.16) directly.

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