ON THE GENERAL LINEAR RECURSIVE SEQUENCES

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ON THE GENERAL LINEAR RECURSIVE SEQUENCES

ABSTRACT. In this paper we investigate the properties of the general linear recursive sequences started from the Lucas sequence and give an application to matrices.

1. INTRODUCTION

For $a_1, a_2 \in \mathbb{Z}$, the corresponding Lucas sequence $\{u_n\}$ is given by $u_0 = 0$, $u_1 = 1$, and $u_{n+1} + a_1 u_n + a_2 u_{n-1} = 0$ $(n \geq 1)$. The comparable series have been studied by many mathematicians [1, 2, 3]. The general linear recursive sequence $\{u_n\}$ is given by $u_n + a_1 u_{n-1} + \cdots + a_m u_{n-m} = 0$ $(n \geq 0)$. Here we comply [4] the Lucas series extended to general linear recursive sequences by defining $\{u_n(a_1, ..., a_m)\}$ as follows:

$$u_{1-m} = \cdots = u_{-1} = 0, \quad u_0 = 1,$$
$$u_n + a_1 u_{n-1} + \cdots + a_m u_{n-m} = 0 \quad (n = 0, \pm 1, \pm 2, ...),$$

where $m \geq 2$ and $a_m \neq 0$.

Throughout the Section 2 we assume that $a_1,...,a_m$ are complex numbers with $a_m \neq 0$, $x^m + a_1 x^{m-1} + \cdots + a_m = (x - \lambda_1) \cdots (x - \lambda_m)$, $s_n = \lambda_1^n + \lambda_2^n \cdots + \lambda_m^n$ and $u_n = u_n(a_1, ..., a_m)$. There we obtain convolution sums between u_n and s_n also state u_n by using s_n. After newly defining $Coef(u_n)$ which is the summation of the coefficients of s_i $(1 \leq i \leq n)$ and their multiplication terms in u_n, we prove $Coef(u_n) = 1$ for $n \in \mathbb{N}$. In that process, we especially find that

$$\sum_{k=1}^{n} \sum_{n_1+n_2+\cdots+n_k=n} \frac{2^k}{n_1 n_2 \cdots n_k k!} = n + 1.$$

In the Section 3 we treat the application of u_n in the powers of matrices and simplifies it by a modular p according to the Legendre symbol.

2. RELATIONS BETWEEN u_n AND s_n

Theorem 2.1. *For $n \in \mathbb{N}$ we have*

(a)

$$\sum_{k=0}^{n} u_k u_{n-k} = \sum_{k=1}^{n} \sum_{n_1+n_2+\cdots+n_k=n} \frac{2^k s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!},$$

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(b)

$$\sum_{k=0}^{n} k u_k u_{n-k} = n \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^{k-1} s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!}.$$

Proof. (a) First in [4, p. 345] we can see that

$$\ln \sum_{n=0}^{\infty} u_n x^n = \sum_{n=1}^{\infty} \frac{s_n}{n} x^n.$$

This leads that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2s_n}{n} x^n &= \ln \sum_{n_1=0}^{\infty} u_{n_1} x^{n_1} + \ln \sum_{n_2=0}^{\infty} u_{n_2} x^{n_2} \\ &= \ln \sum_{n_1,n_2=0}^{\infty} u_{n_1} u_{n_2} x^{n_1+n_2}\end{aligned}$$

and

$$\sum_{n_1,n_2=0}^{\infty} u_{n_1} u_{n_2} x^{n_1+n_2} = \exp \sum_{n=1}^{\infty} \frac{2s_n}{n} x^n. \qquad (1)$$

Then by (1) and Maclaurin series of an exponential function we have

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$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{n_1=0}^{n} u_{n_1}u_{n-n_1}\right)x^n \\
&= \exp \sum_{n=1}^{\infty}\frac{2s_n}{n}x^n \\
&= \sum_{N=0}^{\infty}\frac{1}{N!}\left(\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n\right)^N \\
&= 1+\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n+\frac{1}{2!}\left(\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n\right)^2+\frac{1}{3!}\left(\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n\right)^3+\cdots \\
&= 1+\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n+\sum_{n=2}^{\infty}\left(\sum_{n_1+n_2=n}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\right)\frac{x^n}{2!} \\
&\quad+\sum_{n=3}^{\infty}\left(\sum_{n_1+n_2+n_3=n}\frac{2^3 s_{n_1}s_{n_2}s_{n_3}}{n_1n_2n_3}\right)\frac{x^n}{3!}+\cdots \\
&= 1+2s_1x+\left(s_2x^2+\sum_{n_1+n_2=2}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\cdot\frac{x^2}{2!}\right) \\
&\quad+\left(\frac{2s_3}{3}x^3+\sum_{n_1+n_2=3}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\cdot\frac{x^3}{2!}+\sum_{n_1+n_2+n_3=3}\frac{2^3 s_{n_1}s_{n_2}s_{n_3}}{n_1n_2n_3}\cdot\frac{x^3}{3!}\right) \\
&\quad+\cdots+\left(\frac{2s_n}{n}x^n+\sum_{n_1+n_2=n}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\cdot\frac{x^n}{2!}+\cdots\right. \\
&\qquad\left.+\sum_{n_1+n_2+\cdots+n_n=n}\frac{2^n s_{n_1}s_{n_2}\cdots s_{n_n}}{n_1n_2\cdots n_n}\cdot\frac{x^n}{n!}\right)+\cdots \\
&= 1+\sum_{n=1}^{\infty}\left(\sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n}}^{n}\frac{2^k s_{n_1}s_{n_2}\cdots s_{n_k}}{n_1n_2\cdots n_k}\cdot\frac{1}{k!}\right)x^n
\end{aligned}
$$

and so

$$
\sum_{k=0}^{n}u_ku_{n-k}=\sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n}}^{n}\frac{2^k s_{n_1}s_{n_2}\cdots s_{n_k}}{n_1n_2\cdots n_k k!}\quad\text{for } n\geq 1.
$$

(b) Effortlessly we can know that

$$
\begin{aligned}
\sum_{k=0}^{n}ku_ku_{n-k} &= \sum_{K=0}^{n}(n-K)u_{n-K}u_K \\
&= n\sum_{K=0}^{n}u_{n-K}u_K-\sum_{K=0}^{n}Ku_{n-K}u_K
\end{aligned}
$$

and

$$\sum_{k=0}^{n} k u_k u_{n-k} = \frac{n}{2} \sum_{k=0}^{n} u_k u_{n-k}$$

so we refer to part (a).

□

Lemma 2.2. *We have*

(a)

$$u_1 = s_1,$$

(b)

$$u_2 = \frac{1}{2}s_1^2 + \frac{1}{2}s_2,$$

(c)

$$u_3 = \frac{1}{6}s_1^3 + \frac{1}{2}s_1 s_2 + \frac{1}{3}s_3.$$

Proof. (a) Let us put $n = 1$ in Theorem 2.1 (a):

$$u_0 u_1 + u_1 u_0 = \sum_{k=0}^{1} u_k u_{1-k} = \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=1}}^{1} \frac{2^k s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!} = 2s_1.$$

Since $u_0 = 1$, we obtain $u_1 = s_1$.

(b) Placing $n = 2$ in Theorem 2.1 (a), we note that

$$u_0 u_2 + u_1 u_1 + +u_2 u_0 = \sum_{k=0}^{2} u_k u_{2-k} = \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=2}}^{2} \frac{2^k s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!}$$
$$= s_2 + 2s_1^2$$

and so

$$2u_2 + u_1^2 = s_2 + 2s_1^2.$$

Using part (a) in the above identity, we conclude that

$$u_2 = \frac{1}{2}s_1^2 + \frac{1}{2}s_2.$$

(c) In a similar manner we set $n = 3$ in Theorem 2.1 (a) and use part (a) and (b).

□

Now Lemma 2.2 suggests that u_1, u_2, and u_3 are represented by s_1, s_2, s_3, and their multiplication terms, furthermore the summation of the coefficients of s_i

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$(1 \le i \le 3)$ and their multiplication terms is 1. For example, Lemma 2.2 (c) shows that

$$\begin{aligned} &Coef(u_3) \\ &:= \textit{The summation of the coefficients of } s_i \textit{ and their multiplication terms in } u_3 \\ &= \frac{1}{6} + \frac{1}{2} + \frac{1}{3} \\ &= 1. \end{aligned}$$

Thus we define $Coef(u_n)$ and generalize the above fact as follows:

Definition 2.3. $Coef(u_n)$ implies that the summation of the coefficients of s_i $(1 \le i \le n)$ and their multiplication terms in u_n for $n \in \mathbb{N}$.

Under this condition we can see that $Coef(u_n)$ is a linear transformation. To prove it let us put

$$u_n = a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n},$$
$$u_{n'} = a'_1 s_1^{p'_1} s_2^{p'_2} \cdots s_{n'}^{p'_{n'}} + a'_2 s_1^{q'_1} s_2^{q'_2} \cdots s_{n'}^{q'_{n'}} + \cdots + a'_{n'} s_1^{r'_1} s_2^{r'_2} \cdots s_{n'}^{r'_{n'}},$$

where $p_i, q_i, r_i, p'_i, q'_i, r'_i \in \mathbb{N} \cup \{0\}$ and $a_i, a'_j \in \mathbb{R}$ for $(1 \le i \le n, \quad 1 \le j \le n')$. Then there exists a constant α and it satisfies

$$\begin{aligned} &Coef(\alpha u_n) \\ &= Coef\Big(\alpha(a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n})\Big) \\ &= Coef\Big(\alpha a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + \alpha a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + \alpha a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n}\Big) \\ &= \alpha a_1 + \alpha a_2 + \cdots + \alpha a_n \\ &= \alpha(a_1 + a_2 + \cdots + a_n) \\ &= \alpha Coef(u_n). \end{aligned}$$

In a similar manner,

$$\begin{aligned} &Coef(u_n + u_{n'}) \\ &= Coef\Big((a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n}) \\ &\qquad + (a'_1 s_1^{p'_1} s_2^{p'_2} \cdots s_{n'}^{p'_{n'}} + a'_2 s_1^{q'_1} s_2^{q'_2} \cdots s_{n'}^{q'_{n'}} + \cdots + a'_{n'} s_1^{r'_1} s_2^{r'_2} \cdots s_{n'}^{r'_{n'}})\Big) \\ &= (a_1 + a_2 + \cdots + a_n) + (a'_1 + a'_2 + \cdots + a'_{n'}) \\ &= Coef(u_n) + Coef(u_{n'}). \end{aligned} \tag{2}$$

In addition we can find

$$Coef(u_n u_{n'}) = Coef(u_n) Coef(u_{n'}). \tag{3}$$

Theorem 2.4. *We indicate u_n by s_i $(1 \le i \le n)$ and their multiplication terms, moreover $Coef(u_n) = 1$ for $n \in \mathbb{N}$.*

Proof. Obviously we can represent u_n as s_i $(1 \le i \le n)$ and their multiplication terms by Theorem 2.1 and Lemma 2.2. Next we use the induction to deduce that $Coef(u_n) = 1$. Let us put

$$s_1 = s_2 = \cdots = s_i = 1 \tag{4}$$

to exclude the effect of s_i $(1 \le i \le n)$. Then first since $u_1 = s_1$ in Lemma 2.2 (a), we have $Coef(u_1) = 1$. Second we suppose that $Coef(u_n) = 1$, which leads that

$$\sum_{k=0}^{n} u_k u_{n-k} = \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1 n_2 \cdots n_k k!} \quad \text{for } n \in \mathbb{N} \tag{5}$$

by Theorem 2.1 (a) and Eq. (4). And by (2) and (3) the above identity signifies

$$\begin{aligned}
&Coef\left(\sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1 n_2 \cdots n_k k!}\right) \\
&= Coef\left(\sum_{k=0}^{n} u_k u_{n-k}\right) \\
&= Coef(u_0 u_n + u_1 u_{n-1} + u_2 u_{n-2} + \cdots + u_{n-1} u_1 + u_n u_0) \\
&= Coef(u_0)Coef(u_n) + Coef(u_1)Coef(u_{n-1}) + Coef(u_2)Coef(u_{n-2}) \\
&\quad + \cdots + Coef(u_{n-1})Coef(u_1) + Coef(u_n)Coef(u_0) \\
&= 2Coef(u_n) + n - 1 \\
&= 2 \cdot 1 + n - 1 \\
&= n + 1
\end{aligned}$$

and

$$\sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1 n_2 \cdots n_k k!} = n + 1. \tag{6}$$

Similarly, by (5) and (6) we obtain

$$\begin{aligned}
&n + 2 \\
&= \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n+1}}^{n+1} \frac{2^k}{n_1 n_2 \cdots n_k k!} \\
&= Coef\left(\sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n+1}}^{n+1} \frac{2^k}{n_1 n_2 \cdots n_k k!}\right) \\
&= Coef\left(\sum_{k=0}^{n+1} u_k u_{n+1-k}\right) \\
&= Coef(u_0 u_{n+1} + u_1 u_n + u_2 u_{n-1} + \cdots + u_n u_1 + u_{n+1} u_0) \\
&= Coef(u_0)Coef(u_{n+1}) + Coef(u_1)Coef(u_n) + Coef(u_2)Coef(u_{n-1}) \\
&\quad + \cdots + Coef(u_n)Coef(u_1) + Coef(u_{n+1})Coef(u_0) \\
&= 2Coef(u_{n+1}) + n
\end{aligned}$$

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and so $Coef(u_{n+1}) = 1$.

□

3. Application of u_n to matrices

Proposition 3.1. *Let p be an odd prime, $a, b, c, d \in \mathbb{Z}$, $p \nmid ad - bc$, $\Delta = (a-d)^2 + 4bc$. Then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p-\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \frac{a+d}{2} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1, \end{cases}$$

where I is the 2×2 identity matrix and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Proof. See Corollary 3.3 in [4]. □

Theorem 3.2. *Let p be an odd prime, $a, b, c, d \in \mathbb{Z}$, $p \nmid ad - bc$, $\Delta = (a-d)^2 + 4bc$. Then for $m, l \in \mathbb{N} \cup \{0\}$ satisfying $m \geq l$, we have*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^l \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{m-l} \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

In particular, if $m = l$ or $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \equiv I \pmod{p}$ with $m > l$, then we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

Proof. Let $u_{-1} = 0$, $u_0 = 1$, and

(7) $$u_{n+1} = (a+d)u_n - (ad-bc)u_{n-1} \quad \text{for } n \in \mathbb{N} \cup \{0\}.$$

Then $u_n = u_n(-a-d, ad-bc)$. Moreover in [4, p. 348] we can see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} u_n - du_{n-1} & bu_{n-1} \\ cu_{n-1} & u_n - au_{n-1} \end{pmatrix} \tag{8}$$

and

$$u_{p-1-\left(\frac{\Delta}{p}\right)} \equiv 0 \pmod p, \qquad u_{p-1} \equiv \left(\frac{\Delta}{p}\right) \pmod p. \tag{9}$$

Now, by Proposition 3.1, (8), and (9) we note that

$$\begin{aligned}
&\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \\
&= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^p\right\}^{m-l} \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p-\left(\frac{\Delta}{p}\right)}\right\}^l \\
&= \begin{pmatrix} u_p - du_{p-1} & bu_{p-1} \\ cu_{p-1} & u_p - au_{p-1} \end{pmatrix}^{m-l} \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p-\left(\frac{\Delta}{p}\right)}\right\}^l \\
&\equiv \begin{pmatrix} u_p - d\left(\frac{\Delta}{p}\right) & b\left(\frac{\Delta}{p}\right) \\ c\left(\frac{\Delta}{p}\right) & u_p - a\left(\frac{\Delta}{p}\right) \end{pmatrix}^{m-l} \\
&\quad \times \begin{cases} I^l \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2} I\right)^l \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ \left((ad-bc)I\right)^l \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1 \end{cases}
\end{aligned} \tag{10}$$

$$\equiv \begin{cases} \begin{pmatrix} u_p - d & b \\ c & u_p - a \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^l \begin{pmatrix} u_p & 0 \\ 0 & u_p \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^l \begin{pmatrix} u_p + d & -b \\ -c & u_p + a \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

Here when $\left(\frac{\Delta}{p}\right) = 1$, using (7) and (9) we deduce that

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$$\begin{aligned}u_p &= (a+d)u_{p-1} - (ad-bc)u_{p-2}\\ &\equiv (a+d)\left(\frac{\Delta}{p}\right) - (ad-bc)u_{p-1-(\frac{\Delta}{p})} \pmod p\\ &\equiv (a+d)\cdot 1 - (ad-bc)\cdot 0 \pmod p\\ &\equiv a+d \pmod p\end{aligned}$$

thus

$$\begin{pmatrix} u_p - d & b \\ c & u_p - a\end{pmatrix}^{m-l} \equiv \begin{pmatrix} a & b \\ c & d\end{pmatrix}^{m-l} \pmod p.$$

And when $\left(\frac{\Delta}{p}\right) = 0$, referring to $u_{p-(\frac{\Delta}{p})} = u_p \equiv \frac{a+d}{2} \pmod p$ in [4, p. 349] we obtain

$$\begin{aligned}\left(\frac{a+d}{2}\right)^l \begin{pmatrix} u_p & 0 \\ 0 & u_p\end{pmatrix}^{m-l} &= \left(\frac{a+d}{2}\right)^l (u_p I)^{m-l}\\ &\equiv \left(\frac{a+d}{2}\right)^l \left(\frac{a+d}{2}\right)^{m-l} I\\ &\equiv \left(\frac{a+d}{2}\right)^m I \pmod p.\end{aligned}$$

Similarly when $\left(\frac{\Delta}{p}\right) = -1$, by (9) we have $u_p = u_{p-1-(\frac{\Delta}{p})} \equiv 0 \pmod p$ and so

$$(ad-bc)^l \begin{pmatrix} u_p + d & -b \\ -c & u_p + a\end{pmatrix}^{m-l} \equiv (ad-bc)^l \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}^{m-l} \pmod p.$$

In consequence the above facts lead Eq. (10) to

$$(11)\quad \begin{pmatrix} a & b \\ c & d\end{pmatrix}^{pm-l(\frac{\Delta}{p})} \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d\end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1,\\ \left(\frac{a+d}{2}\right)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0,\\ (ad-bc)^l \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1.\end{cases}$$

Especially, if $m = l$ then Eq. (11) becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{0} \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^{m} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^{m} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{0} \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1 \end{cases}$$

$$\equiv \begin{cases} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^{m} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^{m} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

From the matrix theory we easily know when a matrix A satisfies $A^m = I$ for an identity matrix I and $m \in \mathbb{N}$, then the inverse matrix $A^{-1} = A^{m-1}$ since $A \cdot A^{m-1} = I$. Thus using this property we deduce as follows :

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \equiv I \pmod{p}$ with $m > l$ then the inverse matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l-1} \pmod{p}$ so

$$\begin{aligned} \left\{ \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\}^{m-l} &\equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right\}^{m-l} \\ &\equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l-1} \right\}^{m-l} \pmod{p} \\ &\equiv (I^{-1})^{m-l} \pmod{p} \\ &\equiv I \pmod{p} \end{aligned}$$

and

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{m-l} \equiv (ad-bc)^{m-l} I \pmod{p}.$$

Therefore Eq. (11) shows that

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^l \cdot (ad-bc)^{m-l} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1 \end{cases}$$

$$\equiv \begin{cases} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

□

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