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On the general linear recursive sequences

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Abstract

In this paper we investigate the properties of the general linear recursive sequences started from the Lucas sequence and give an application to matrices.

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1 Introduction

For $a_1, a_2 \in \mathbb{Z}$, the corresponding Lucas sequence $\{u_n\}$ is given by $u_0 = 0$, $u_1 = 1$, and $u_{n+1} + a_1u_n + a_2u_{n-1} = 0$ $(n \geq 1)$. The comparable series have been studied by many mathematicians [1]; [2]; [3]. The general linear recursive sequences $\{u_n\}$ is given by $u_n + a_1u_{n-1} + \cdots + a_mu_{n-m} = 0$ $(n \geq 0)$. Here we comply [4] the Lucas series extended to general linear recursive sequences by defining $\{u_n(a_1, ..., a_m)\}$ as follows:

$$u_{1-m} = \cdots = u_{-1} = 0, \quad u_0 = 1,$$
$$u_n + a_1u_{n-1} + \cdots + a_mu_{n-m} = 0 \quad (n = 0, \pm1, \pm2, ...),$$

where $m \geq 2$ and $a_m \neq 0$.

Throughout the Section 2 we assume that a_1,...,a_m are complex numbers with $a_m \neq 0$, $x^m + a_1x^{m-1} + \cdots + a_m = (x - \lambda_1) \cdots (x - \lambda_m)$, $s_n = \lambda_1^n + \lambda_2^n \cdots + \lambda_m^n$ and $u_n = u_n(a_1, ..., a_m)$. There we obtain convolution sums between u_n and s_n also state u_n by using s_n. After newly defining $Coef(u_n)$ which is the summation of the coefficients of s_i $(1 \leq i \leq n)$ and their multiplication terms in u_n, we prove $Coef(u_n) = 1$ for $n \in \mathbb{N}$. In that process, we especially find that

$$\sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1n_2 \cdots n_kk!} = n + 1.$$

In the Section 3 we treat the application of u_n in the powers of matrices and simplifies it by a modular p according to the Legendre symbol.

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2 Relations between u_n and s_n

Theorem 2.1. *For $n \in \mathbb{N}$ we have*

(a)

$$\sum_{k=0}^{n} u_k u_{n-k} = \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!},$$

(b)

$$\sum_{k=0}^{n} k u_k u_{n-k} = n \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^{k-1} s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!}.$$

Proof. (a) First in ([4], p. 345) we can see that

$$\ln \sum_{n=0}^{\infty} u_n x^n = \sum_{n=1}^{\infty} \frac{s_n}{n} x^n.$$

This leads that

$$\sum_{n=1}^{\infty} \frac{2s_n}{n} x^n = \ln \sum_{n_1=0}^{\infty} u_{n_1} x^{n_1} + \ln \sum_{n_2=0}^{\infty} u_{n_2} x^{n_2}$$
$$= \ln \sum_{n_1,n_2=0}^{\infty} u_{n_1} u_{n_2} x^{n_1+n_2}$$

and

$$\sum_{n_1,n_2=0}^{\infty} u_{n_1} u_{n_2} x^{n_1+n_2} = \exp \sum_{n=1}^{\infty} \frac{2s_n}{n} x^n. \tag{2.1}$$

Then by (2.1) and Maclaurin series of an exponential function we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{n_1=0}^{n} u_{n_1}u_{n-n_1}\right)x^n \\
&= \exp\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n \\
&= \sum_{N=0}^{\infty}\frac{1}{N!}\left(\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n\right)^N \\
&= 1+\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n+\frac{1}{2!}\left(\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n\right)^2+\frac{1}{3!}\left(\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n\right)^3+\cdots \\
&= 1+\sum_{n=1}^{\infty}\frac{2s_n}{n}x^n+\sum_{n=2}^{\infty}\left(\sum_{n_1+n_2=n}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\right)\frac{x^n}{2!} \\
&\quad +\sum_{n=3}^{\infty}\left(\sum_{n_1+n_2+n_3=n}\frac{2^3 s_{n_1}s_{n_2}s_{n_3}}{n_1n_2n_3}\right)\frac{x^n}{3!}+\cdots \\
&= 1+2s_1x+\left(s_2x^2+\sum_{n_1+n_2=2}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\cdot\frac{x^2}{2!}\right) \\
&\quad +\left(\frac{2s_3}{3}x^3+\sum_{n_1+n_2=3}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\cdot\frac{x^3}{2!}+\sum_{n_1+n_2+n_3=3}\frac{2^3 s_{n_1}s_{n_2}s_{n_3}}{n_1n_2n_3}\cdot\frac{x^3}{3!}\right) \\
&\quad +\cdots+\left(\frac{2s_n}{n}x^n+\sum_{n_1+n_2=n}\frac{2^2 s_{n_1}s_{n_2}}{n_1n_2}\cdot\frac{x^n}{2!}+\cdots\right. \\
&\qquad\qquad \left.+\sum_{n_1+n_2+\cdots+n_n=n}\frac{2^n s_{n_1}s_{n_2}\cdots s_{n_n}}{n_1n_2\cdots n_n}\cdot\frac{x^n}{n!}\right)+\cdots \\
&= 1+\sum_{n=1}^{\infty}\left(\sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n}}^{n}\frac{2^k s_{n_1}s_{n_2}\cdots s_{n_k}}{n_1n_2\cdots n_k}\cdot\frac{1}{k!}\right)x^n
\end{aligned}$$

and so

$$\sum_{k=0}^{n}u_ku_{n-k}=\sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n}}^{n}\frac{2^k s_{n_1}s_{n_2}\cdots s_{n_k}}{n_1n_2\cdots n_k k!}\quad \text{for } n\geq 1.$$

(b) Effortlessly we can know that

$$\begin{aligned}
\sum_{k=0}^{n}ku_ku_{n-k} &= \sum_{K=0}^{n}(n-K)u_{n-K}u_K \\
&= n\sum_{K=0}^{n}u_{n-K}u_K-\sum_{K=0}^{n}Ku_{n-K}u_K
\end{aligned}$$

and

$$\sum_{k=0}^{n}ku_ku_{n-k}=\frac{n}{2}\sum_{k=0}^{n}u_ku_{n-k}$$

so we refer to part (a).

□

Lemma 2.2. *We have*

(a)

$$u_1 = s_1,$$

(b)

$$u_2 = \frac{1}{2}s_1^2 + \frac{1}{2}s_2,$$

(c)

$$u_3 = \frac{1}{6}s_1^3 + \frac{1}{2}s_1 s_2 + \frac{1}{3}s_3.$$

Proof. (a) Let us put $n = 1$ in Theorem 2.1 (a):

$$u_0 u_1 + u_1 u_0 = \sum_{k=0}^{1} u_k u_{1-k} = \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=1}}^{1} \frac{2^k s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!} = 2s_1.$$

Since $u_0 = 1$, we obtain $u_1 = s_1$.

(b) Placing $n = 2$ in Theorem 2.1 (a), we note that

$$u_0 u_2 + u_1 u_1 + +u_2 u_0 = \sum_{k=0}^{2} u_k u_{2-k} = \sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=2}}^{2} \frac{2^k s_{n_1} s_{n_2} \cdots s_{n_k}}{n_1 n_2 \cdots n_k k!}$$
$$= s_2 + 2s_1^2$$

and so

$$2u_2 + u_1^2 = s_2 + 2s_1^2.$$

Using part (a) in the above identity, we conclude that

$$u_2 = \frac{1}{2}s_1^2 + \frac{1}{2}s_2.$$

(c) In a similar manner we set $n = 3$ in Theorem 2.1 (a) and use part (a) and (b).

□

Now Lemma 2.2 suggests that u_1, u_2, and u_3 are represented by s_1, s_2, s_3, and their multiplication terms, furthermore the summation of the coefficients of s_i $(1 \leq i \leq 3)$ and their multiplication terms is 1. For example, Lemma 2.2 (c) shows that

$Coef(u_3)$
$:=$ *The summation of the coefficients of s_i and their multiplication terms in u_3*
$= \frac{1}{6} + \frac{1}{2} + \frac{1}{3}$
$= 1.$

Thus we define $Coef(u_n)$ and generalize the above fact as follows:

Definition 2.1. $Coef(u_n)$ implies that the summation of the coefficients of s_i $(1 \leq i \leq n)$ and their multiplication terms in u_n for $n \in \mathbb{N}$.

Under this condition we can see that $Coef(u_n)$ is a linear transformation. To prove it let us put

$$u_n = a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n},$$
$$u_{n'} = a'_1 s_1^{p'_1} s_2^{p'_2} \cdots s_{n'}^{p'_{n'}} + a'_2 s_1^{q'_1} s_2^{q'_2} \cdots s_{n'}^{q'_{n'}} + \cdots + a'_{n'} s_1^{r'_1} s_2^{r'_2} \cdots s_{n'}^{r'_{n'}},$$

where $p_i, q_i, r_i, p'_i, q'_i, r'_i \in \mathbb{N} \cup \{0\}$ and $a_i, a'_j \in \mathbb{R}$ for $(1 \leq i \leq n, \quad 1 \leq j \leq n')$. Then there exists a constant α and it satisfies

$$\begin{aligned}
&Coef(\alpha u_n)\\
&= Coef\Big(\alpha(a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n})\Big)\\
&= Coef\Big(\alpha a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + \alpha a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + \alpha a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n}\Big)\\
&= \alpha a_1 + \alpha a_2 + \cdots + \alpha a_n\\
&= \alpha(a_1 + a_2 + \cdots + a_n)\\
&= \alpha Coef(u_n).
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
&Coef(u_n + u_{n'})\\
&= Coef\Big((a_1 s_1^{p_1} s_2^{p_2} \cdots s_n^{p_n} + a_2 s_1^{q_1} s_2^{q_2} \cdots s_n^{q_n} + \cdots + a_n s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n})\\
&\quad + (a'_1 s_1^{p'_1} s_2^{p'_2} \cdots s_{n'}^{p'_{n'}} + a'_2 s_1^{q'_1} s_2^{q'_2} \cdots s_{n'}^{q'_{n'}} + \cdots + a'_{n'} s_1^{r'_1} s_2^{r'_2} \cdots s_{n'}^{r'_{n'}})\Big)\\
&= (a_1 + a_2 + \cdots + a_n) + (a'_1 + a'_2 + \cdots + a'_{n'})\\
&= Coef(u_n) + Coef(u_{n'}).
\end{aligned} \tag{2.2}$$

In addition we can find

$$Coef(u_n u_{n'}) = Coef(u_n) Coef(u_{n'}). \tag{2.3}$$

Theorem 2.3. *We indicate u_n by s_i $(1 \leq i \leq n)$ and their multiplication terms, moreover $Coef(u_n) = 1$ for $n \in \mathbb{N}$.*

Proof. Obviously we can represent u_n as s_i $(1 \leq i \leq n)$ and their multiplication terms by Theorem 2.1 and Lemma 2.2. Next we use the induction to deduce that $Coef(u_n) = 1$. Let us put

$$s_1 = s_2 = \cdots = s_i = 1 \tag{2.4}$$

to exclude the effect of s_i $(1 \leq i \leq n)$. Then first since $u_1 = s_1$ in Lemma 2.2 (a), we have $Coef(u_1) = 1$. Second we suppose that $Coef(u_n) = 1$, which leads that

$$\sum_{k=0}^{n} u_k u_{n-k} = \sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1 n_2 \cdots n_k k!} \qquad \text{for } n \in \mathbb{N} \tag{2.5}$$

by Theorem 2.1 (a) and Eq. (2.4). And by (2.2) and (2.3) the above identity signifies

$$
\begin{aligned}
&Coef\left(\sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1 n_2 \cdots n_k k!}\right)\\
&= Coef\left(\sum_{k=0}^{n} u_k u_{n-k}\right)\\
&= Coef(u_0u_n + u_1u_{n-1} + u_2u_{n-2} + \cdots + u_{n-1}u_1 + u_nu_0)\\
&= Coef(u_0)Coef(u_n) + Coef(u_1)Coef(u_{n-1}) + Coef(u_2)Coef(u_{n-2})\\
&\quad + \cdots + Coef(u_{n-1})Coef(u_1) + Coef(u_n)Coef(u_0)\\
&= 2Coef(u_n) + n - 1\\
&= 2 \cdot 1 + n - 1\\
&= n + 1
\end{aligned}
$$

and

$$\sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1 n_2 \cdots n_k k!} = n + 1. \tag{2.6}$$

Similarly, by (2.5) and (2.6) we obtain

$$
\begin{aligned}
&n + 2\\
&= \sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n+1}}^{n+1} \frac{2^k}{n_1 n_2 \cdots n_k k!}\\
&= Coef\left(\sum_{\substack{k=1\\ n_1+n_2+\cdots+n_k=n+1}}^{n+1} \frac{2^k}{n_1 n_2 \cdots n_k k!}\right)\\
&= Coef\left(\sum_{k=0}^{n+1} u_k u_{n+1-k}\right)\\
&= Coef(u_0u_{n+1} + u_1u_n + u_2u_{n-1} + \cdots + u_nu_1 + u_{n+1}u_0)\\
&= Coef(u_0)Coef(u_{n+1}) + Coef(u_1)Coef(u_n) + Coef(u_2)Coef(u_{n-1})\\
&\quad + \cdots + Coef(u_n)Coef(u_1) + Coef(u_{n+1})Coef(u_0)\\
&= 2Coef(u_{n+1}) + n
\end{aligned}
$$

and so $Coef(u_{n+1}) = 1$. □

3 Application of u_n to matrices

Proposition 3.1. *Let p be an odd prime, $a, b, c, d \in \mathbb{Z}$, $p \nmid ad - bc$, $\Delta = (a-d)^2 + 4bc$. Then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p - \left(\frac{\Delta}{p}\right)} \equiv \begin{cases} I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \frac{a+d}{2} I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad - bc) I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1, \end{cases}$$

where I is the 2×2 identity matrix and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Proof. See Corollary 3.3 in [4]. □

Theorem 3.1. *Let p be an odd prime, $a, b, c, d \in \mathbb{Z}$, $p \nmid ad - bc$, $\Delta = (a-d)^2 + 4bc$. Then for $m, l \in \mathbb{N} \cup \{0\}$ satisfying $m \geq l$, we have*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm - l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad - bc)^l \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

In particular, if $m = l$ or $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \equiv I \pmod p$ with $m > l$, then we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm - l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad - bc)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

Proof. Let $u_{-1} = 0$, $u_0 = 1$, and

$$u_{n+1} = (a+d)u_n - (ad - bc)u_{n-1} \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

Then $u_n = u_n(-a - d, ad - bc)$. Moreover in ([4], p. 348) we can see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^n = \begin{pmatrix} u_n - du_{n-1} & bu_{n-1} \\ cu_{n-1} & u_n - au_{n-1} \end{pmatrix} \tag{3.2}$$

and

$$u_{p-1-\left(\frac{\Delta}{p}\right)} \equiv 0 \pmod p, \qquad u_{p-1} \equiv \left(\frac{\Delta}{p}\right) \pmod p. \tag{3.3}$$

Now, by Proposition 3.1, (3.2), and (3.3) we note that

$$\begin{aligned}
&\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p} \right\}^{m-l} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p-\left(\frac{\Delta}{p}\right)} \right\}^{l} \\
&= \begin{pmatrix} u_p - du_{p-1} & bu_{p-1} \\ cu_{p-1} & u_p - au_{p-1} \end{pmatrix}^{m-l} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p-\left(\frac{\Delta}{p}\right)} \right\}^{l} \\
&\equiv \begin{pmatrix} u_p - d\left(\frac{\Delta}{p}\right) & b\left(\frac{\Delta}{p}\right) \\ c\left(\frac{\Delta}{p}\right) & u_p - a\left(\frac{\Delta}{p}\right) \end{pmatrix}^{m-l} \\
&\quad \times \begin{cases} I^l \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2} I\right)^l \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ \left((ad-bc)I\right)^l \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1 \end{cases} \\
&\equiv \begin{cases} \begin{pmatrix} u_p - d & b \\ c & u_p - a \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^l \begin{pmatrix} u_p & 0 \\ 0 & u_p \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^l \begin{pmatrix} u_p + d & -b \\ -c & u_p + a \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}
\end{aligned} \tag{3.4}$$

Here when $\left(\frac{\Delta}{p}\right) = 1$, using (3.1) and (3.3) we deduce that

$$\begin{aligned}
u_p &= (a+d)u_{p-1} - (ad-bc)u_{p-2} \\
&\equiv (a+d)\left(\frac{\Delta}{p}\right) - (ad-bc)u_{p-1-\left(\frac{\Delta}{p}\right)} \pmod p \\
&\equiv (a+d)\cdot 1 - (ad-bc)\cdot 0 \pmod p \\
&\equiv a+d \pmod p
\end{aligned}$$

thus

$$\begin{pmatrix} u_p - d & b \\ c & u_p - a \end{pmatrix}^{m-l} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \pmod p.$$

And when $\left(\frac{\Delta}{p}\right) = 0$, referring to $u_{p-\left(\frac{\Delta}{p}\right)} = u_p \equiv \dfrac{a+d}{2} \pmod p$ in ([4], p. 349) we obtain

$$\begin{aligned}\left(\frac{a+d}{2}\right)^l \begin{pmatrix} u_p & 0 \\ 0 & u_p \end{pmatrix}^{m-l} &= \left(\frac{a+d}{2}\right)^l (u_p I)^{m-l} \\ &\equiv \left(\frac{a+d}{2}\right)^l \left(\frac{a+d}{2}\right)^{m-l} I \\ &\equiv \left(\frac{a+d}{2}\right)^m I \pmod p.\end{aligned}$$

Similarly when $\left(\frac{\Delta}{p}\right) = -1$, by (3.3) we have $u_p = u_{p-1-\left(\frac{\Delta}{p}\right)} \equiv 0 \pmod p$ and so

$$(ad-bc)^l \begin{pmatrix} u_p + d & -b \\ -c & u_p + a \end{pmatrix}^{m-l} \equiv (ad-bc)^l \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{m-l} \pmod p.$$

In consequence the above facts lead Eq. (3.4) to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\dfrac{a+d}{2}\right)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^l \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{m-l} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases} \tag{3.5}$$

Especially, if $m = l$ then Eq. (3.5) becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{0} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\dfrac{a+d}{2}\right)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^m \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{0} \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1 \end{cases}$$

$$\equiv \begin{cases} I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\dfrac{a+d}{2}\right)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^m I \pmod p, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

From the matrix theory we easily know when a matrix A satisfies $A^m = I$ for an identity matrix I and $m \in \mathbb{N}$, then the inverse matrix $A^{-1} = A^{m-1}$ since $A \cdot A^{m-1} = I$. Thus using this property we deduce as follows : If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l} \equiv I \pmod p$ with $m > l$ then the inverse matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \equiv$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l-1} \pmod{p}$ so

$$\begin{aligned} \left\{ \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\}^{m-l} &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right\}^{m-l} \\ &\equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{m-l-1} \right\}^{m-l} \pmod{p} \\ &\equiv (I^{-1})^{m-l} \pmod{p} \\ &\equiv I \pmod{p} \end{aligned}$$

and

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{m-l} \equiv (ad-bc)^{m-l} I \pmod{p}.$$

Therefore Eq. (3.5) shows that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pm-l\left(\frac{\Delta}{p}\right)} \equiv \begin{cases} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^l \cdot (ad-bc)^{m-l} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1 \end{cases}$$

$$\equiv \begin{cases} I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 1, \\ \left(\frac{a+d}{2}\right)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = 0, \\ (ad-bc)^m I \pmod{p}, & \text{if } \left(\frac{\Delta}{p}\right) = -1. \end{cases}$$

□

4 Conclusion

The essential point of this article is that we define a new concept $Coef(u_n)$ and obtain

$$\sum_{\substack{k=1 \\ n_1+n_2+\cdots+n_k=n}}^{n} \frac{2^k}{n_1 n_2 \cdots n_k k!} = n+1.$$

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