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Notes on "Constructions of triangular norms on lattices by means of irreducible elements"

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Commentary

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Abstract

In this note, we show by an counterexample that a paper by Yılmaz and Kazancı (Ş. Yılmaz, O. Kazancı, Constructions of triangular norms on lattices by means of irreducible elements, Inform. Sci. 397–398 (2017) 110–117) suffers from certain mistakes.

Keywords: $\wedge$*-semilattice;* $\vee$*-semilattice; Irreducible element; Triangular norm*

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1 Introduction

Triangular norms (t-norms for short) on the unit interval were originally studied by Schweizer and Sklar in the framework of probabilistic metric spaces [10]. In fuzzy logic, the set of truth values is modelled by the real unit interval and the truth function for a conjunction connective is taken as minimum. Due to the close connection between order theory and fuzzy set theory, the real unit interval is replaced by some more general structure, for example, a bounded lattice [5]. Therefore, recently an increasing interest of t-norms on bounded lattices can be observed, see e.g. [2, 3, 4, 7, 8, 9] and many others.

A poset $(L, \leq)$ is a $\vee$-semilattice (dually, $\wedge$-semilattice) iff $\sup\{x, y\}$ (dually, $\inf\{x, y\}$) exists for any two elements x and y. Denote $x \vee y = \sup\{x, y\}$ and call it the join of x and y. Dually, denote $x \wedge y = \inf\{x, y\}$ and call it the meet of x and y. A lattice is an ordered set $(E, \leq)$ which is both an $\vee$-semilattice and an $\wedge$-semilattice with respect to its order [1].

A sublattice is a non-empty subset S of a lattice L, such that S is closed under meet and join. A lattice is complete if for every subset there exist the meet and the join. A lattice which possesses the smallest (the bottom) and the greatest (the top) elements, 0 and 1, respectively, is bounded. If L is a lattice with bottom element 0, then by an atom of L we mean an element $a \in L$ such that $0 < a$ and there is no $x \in L$ such that $0 < x < a$. A lattice L is a chain if either $x \leq y$ or $y \leq x$ for all $x, y \in L$.

Let $(L, \wedge, \vee)$ be a lattice. It is known that

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z) \text{ holds for all } x, y, z \in L \tag{1.1}$$

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and

$$(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z) \text{ holds for all } x, y, z \in L \tag{1.2}$$

are equivalent. A lattice satisfying identities (1.1) or (1.2) is called distributive. In a bounded lattice L, an element x is a complement of y if and only if $x \wedge y = 0$ and $x \vee y = 1$. A complemented lattice is a bounded lattice in which every element has a complement. By a boolean lattice we mean a complemented distributive lattice.

If L is a lattice then $a \in L$ (with $a \neq 0$ if L has a bottom element 0) is said to be $\vee$-irreducible if $x \vee y = a$ implies $x = a$ or $y = a$. Thus, $a \in L$ is $\vee$-irreducible if it cannot be expressed as the join of two elements that are strictly less than a. We denote the set of $\vee$-irreducible elements of a lattice L by $J(L)$. Dually, we can define the set $M(L)$ of $\wedge$-irreducible elements. For further information see [1, 6].

Definition 1.1. [11] Let $(L, \leq, 0, 1)$ be a bounded lattice. The set $J(L)^* = J(L) \cup \{0, 1\}$ and $M(L)^* = M(L) \cup \{0, 1\}$ are defined as the extended set of $\vee$-irreducible elements and $\wedge$-irreducible elements of L, respectively.

We point out an assertion in [11] is incorrect by counterexamples.

2 Mistakes and counterexamples

On page 112 in [11], the last line, the authors claimed that $J(L_1 \times L_2)^* = J(L_1 \times L_2) \cup \{(0, 0), (1, 1)\} = J(L_1)^* \times J(L_2)^*$. It follows from Definition 1.1 that $J(L_1 \times L_2)^* = J(L_1 \times L_2) \cup \{(0, 0), (1, 1)\}$. However, the following example shows that $J(L_1 \times L_2)^* \neq J(L_1)^* \times J(L_2)^*$ in general.

Example 2.1. *Let $L_1 = \{0, a, b, 1\}$ be a boolean lattice with two atoms, and let $L_2 = \{0, 1\}$ be a two-element chain. Then $L_1 \times L_2$ is a boolean lattice with three atoms. It is clear that $J(L_1) = \{a, b\}$, $J(L_1)^* = L_1$, $J(L_2) = \{1\}$ and $J(L_2)^* = L_2$. It follows that*

$$J(L_1 \times L_2)^* = \{(0, 0), (a, 0), (b, 0), (0, 1), (1, 1)\},$$

and $J(L_1)^* \times J(L_2)^* = L_1 \times L_2$. Thus $J(L_1 \times L_2)^* \neq J(L_1)^* \times J(L_2)^*$.

It is clear that $J(L_1 \times L_2)^*$ is a subposet of $L_1 \times L_2$. In Example 2.1, $(a, 0) \vee_{L_1 \times L_2} (b, 0) = (1, 0)$ and $(a, 0) \vee_{J(L_1 \times L_2)^*} (b, 0) = (1, 1)$. It follows that $J(L_1 \times L_2)^*$ is not a sublattice of $L_1 \times L_2$.

Competing Interests

Author has declared that no competing interests exist.

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