Original research papers

Pullback Absorbing set for the stochastic lattice Selkov equations

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Abstract: In this paper, pullback absorbing property for the stochastic reversible Selkov system in an infinite lattice with additive noises is proved. In order to obtain the proof of absorption, a transformation of addition involved with an Ornstein-Uhlenbeck process is used.
Keywords: pullback absorbing set; additive noise; Selkov system

1 Introduction

In this paper, we consider the stochastic lattice Selkov system with the cubic nonlinearity and additive white noises on an infinite lattice as follows:

$$\begin{cases} du_i = [d_1(u_{i+1} - 2u_i + u_{i-1}) - a_1 u_i + b_1 u_i^2 v_i - b_2 u_i^3 + f_{1i}]dt + \alpha_i dw_i, \quad i \in \mathbb{Z}, \ t > 0, \\ dv_i = [d_2(v_{i+1} - 2v_i + v_{i-1}) - a_2 v_i - b_1 u_i^2 v_i + b_2 u_i^3 + f_{2i}]dt + \alpha_i dw_i, \quad i \in \mathbb{Z}, \ t > 0, \end{cases} \tag{1.1}$$

with initial conditions

$$u_i(0) = u_{i,0}, \qquad v_i(0) = v_{i,0}, \qquad i \in \mathbb{Z}, \tag{1.2}$$

where $\mathbb{Z}$ denotes the integer set, $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, d_1, d_2, a_1, a_2, b_1, b_2 are positive constants, $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \ell^2$, $\{w_i | i \in \mathbb{Z}\}$ is independent Brownian motions.

Equation (1.1) can be regarded as a Selkov system (see [8]) on $\mathbb{R}$:

$$\begin{cases} u_t = d_1 \triangle u - a_1 u + b_1 u^2 v - b_2 u^3 + f_1 + \alpha w_t, \\ v_t = d_2 \triangle v - a_2 v - b_1 u^2 v + b_2 u^3 + f_2 + \alpha w_t. \end{cases} \tag{1.3}$$

We have obtained the random dynamical system, see [5]. Pullback absorbing property is very important to describe the long-time behavior of the equations for the mathematics and physics, especially, to prove the existence of random attractor. Therefore, in this paper, we prove the pullback absorbing property for the Selkov equations (1.1).

2 Preliminaries

In this section, we introduce the relevant definitions of absorbing property, which are taken from [2], [3], [4], [7].

Let (H, d) be a complete separable metric space, $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. If $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ measurable, and

$$\theta_0 = I, \theta_{s+t} = \theta_s \circ \theta_t, \quad \forall s, t \in \mathbb{R},$$

$$\theta_t P = P, \quad \forall t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system.

Definition 2.2. If a mapping

$$\psi : \mathbb{R}^+ \times \Omega \times H \to H, \quad (t, \omega, x) \mapsto \psi(t, \omega, x),$$

is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable and satisfies, for every $\omega \in \Omega$,

(i) $\psi(0, \omega, \cdot)$ is the identity on H;

(ii) Cocycle property: $\psi(t + s, \omega, \cdot) = \psi(t, \theta_s\omega, \psi(s, \omega, \cdot))$ for all $t, s \in \mathbb{R}^+$;

(iii) $\psi(\cdot, \omega, \cdot) : \mathbb{R}^+ \times H \to H$ is strongly continuous.

Then the map ψ is a continuous random dynamical system on H over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

Definition 2.3. If for every $\omega \in \Omega$ and a random bounded set $D(\omega) \subset H$,

$$\lim_{t\to\infty} e^{\gamma t} d(D(\theta_{-t}\omega)) = 0 \text{ for all } \gamma > 0,$$

where $d(D) = \sup_{x\in D} \|x\|_H$. Then $D(\omega) \subset H$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$.

Definition 2.4. A random set $J(\omega)$ is called a pullback absorbing set in $\mathcal{D}$, if for all $D \in \mathcal{D}$ and every $\omega \in \Omega$, there exists a $t_D(\omega) > 0$ such that

$$\psi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset J(\omega), \forall t \geq t_D(\omega).$$

Where $\mathcal{D}$ is a collection of random sets of H.

3 Ornstein-Uhlenbeck process

Let $\iota^2 = \{u = (u_i)_{i\in\mathbb{Z}}, \quad u_i \in \mathbb{R} : \sum_{i\in\mathbb{Z}} |u_i|^2 < +\infty\}$, with the inner product and norm as follows:

$$\langle u, v\rangle = \sum_{i\in\mathbb{Z}} u_i v_i, \quad \|u\|^2 = \langle u, u\rangle, \quad u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}} \in \iota^2.$$

Then $\iota^2 = (\iota^2, \langle\cdot,\cdot\rangle, \|\cdot\|)$ is a Hilbert space. Set $E = \iota^2 \times \iota^2$ be the product Hilbert space. In view of the cubic term $\pm u^2 v, \pm u^3$, we need $u \in \iota^6, v \in \iota^6$ to make (1.1) hold in ι^2.

To convert the stochastic equation to a deterministic one with random parameters, we introduce an Ornstein-Uhlenbeck process (O-U process) (see [6]) in ι^2 on $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ given by the Wiener process:

$$y(\theta_t\omega) = -(a_1 + a_2)\int_{-\infty}^{0} e^{(a_1+a_2)s}(\theta_t\omega)(s)ds, \qquad t \in \mathbb{R}, \qquad \omega \in \Omega,$$

and y solve the following Itô equations respectively:

$$dy + (a_1 + a_2)ydt = dw(t), \qquad t > 0.$$

There exists a θ_t-invariant set $\Omega' \subset \Omega$ of full P measure such that

(1) the mappings $s \to y(\theta_s\omega)$, is continuous for each $\omega \in \Omega$;

(2) the random variables $\|y(\theta_t\omega)\|$ is tempered.

Let

$$\tilde{u}(t) = u(t) - y(\theta_t\omega), \qquad \tilde{v}(t) = v(t) - y(\theta_t\omega).$$

From (1.3), we have

$$\begin{cases} \tilde{u}_t = -d_1 A(\tilde{u} + y(\theta_t\omega)) - a_1\tilde{u} + a_2 y(\theta_t\omega) + b_1(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)) \\ \qquad -b_2(\tilde{u} + y(\theta_t\omega))^3 + f_1 \\ \\ \tilde{v}_t = -d_2 A(\tilde{v} + y(\theta_t\omega)) - a_2\tilde{v} + a_1 y(\theta_t\omega) - b_1(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)) \\ \qquad +b_2(\tilde{u} + y(\theta_t\omega))^3 + f_2 \end{cases} \tag{3.1}$$

with the initial value condition

$$\tilde{u}(0, \omega, \tilde{u}_0) = \tilde{u}_0(\omega) = u_0 - y(\omega), \qquad \tilde{v}(0, \omega, \tilde{v}_0) = \tilde{v}_0(\omega) = v_0 - y(\omega).$$

4 pullback absorbing property

Lemma 4.1. *There exists a θ_t-invariant set $\Omega' \subset \Omega$ of full P measure and an absorbing random set $J(\omega), \omega \in \Omega'$, $\forall D \in \mathcal{D}$ and $\forall \omega \in \Omega'$, there exists $T_D(\omega) > 0$ such that*

$$\psi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset J(\omega) \qquad \forall t \geq T_D(\omega).$$

Moreover, $J \in \mathcal{D}$.

Proof. Taking the inner product to (3.1) with $(\tilde{u}, \tilde{v})^T$ in E, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2 &= -d_1\langle A\tilde{u}, \tilde{u}\rangle - d_1\langle Ay(\theta_t\omega), \tilde{u}\rangle - a_1\|\tilde{u}\|^2 + b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{u}\rangle \\
&\quad -b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{u}\rangle + \langle f_1, \tilde{u}\rangle + a_2\langle y(\theta_t\omega), \tilde{u}\rangle, \\
\frac{1}{2}\frac{d}{dt}\|\tilde{v}\|^2 &= -d_2\langle A\tilde{v}, \tilde{v}\rangle - d_2\langle Ay(\theta_t\omega), \tilde{v}\rangle - a_2\|\tilde{v}\|^2 - b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{v}\rangle \\
&\quad +b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{v}\rangle + \langle f_2, \tilde{v}\rangle + a_1\langle y(\theta_t\omega), \tilde{v}\rangle. \qquad (4.1)
\end{aligned}$$

Summing the two equations up, we have

$$\begin{aligned}
&\frac{d}{dt}[\|\tilde{u}\|^2 + \|\tilde{v}\|^2] + 2d_1\langle A\tilde{u}, \tilde{u}\rangle + 2d_2\langle A\tilde{v}, \tilde{v}\rangle + 2a_1\|\tilde{u}\|^2 + 2a_2\|\tilde{v}\|^2 \\
= \;&-2d_1\langle Ay(\theta_t\omega), \tilde{u}\rangle - 2d_2\langle Ay(\theta_t\omega), \tilde{v}\rangle + 2\langle f_1, \tilde{u}\rangle + 2\langle f_2, \tilde{v}\rangle + 2a_2\langle y(\theta_t\omega), \tilde{u}\rangle + 2a_1\langle y(\theta_t\omega), \tilde{v}\rangle \\
&+2b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{u}\rangle - 2b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{v}\rangle \\
&-2b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{u}\rangle + 2b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{v}\rangle. \qquad (4.2)
\end{aligned}$$

Then we have

$$\begin{aligned}
&2b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{u}\rangle - 2b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{v}\rangle \\
&-2b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{u}\rangle + 2b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{v}\rangle \\
= \;&2b_1\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega)), \tilde{u} - \tilde{v}\rangle - 2b_2\langle(\tilde{u} + y(\theta_t\omega))^3, \tilde{u} - \tilde{v}\rangle \\
\leq \;&2\max\{b_1, b_2\}\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{v} + y(\theta_t\omega) - \tilde{u} - y(\theta_t\omega)), \tilde{u} - \tilde{v}\rangle \\
= \;&-2\max\{b_1, b_2\}\langle(\tilde{u} + y(\theta_t\omega))^2(\tilde{u} - \tilde{v}), \tilde{u} - \tilde{v}\rangle \\
= \;&-2\sum_{i\in\mathbb{Z}}(\tilde{u}_i + y_i(\theta_t\omega))^2(\tilde{u}_i - \tilde{v}_i)^2 \leq 0. \qquad (4.3)
\end{aligned}$$

By Young's inequality in [9], we have the following estimate

$$-2d_1\langle Ay(\theta_t\omega), \tilde{u}\rangle \leq \frac{a_1}{3}\|\tilde{u}\|^2 + \frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2, \qquad (4.4)$$

$$-2d_2\langle Ay(\theta_t\omega), \tilde{v}\rangle \leq \frac{a_2}{3}\|\tilde{v}\|^2 + \frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2, \qquad (4.5)$$

$$2a_2\langle y(\theta_t\omega), \tilde{u}\rangle \leq \frac{a_1}{3}\|\tilde{u}\|^2 + \frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2, \qquad (4.6)$$

$$2a_1\langle y(\theta_t\omega), \tilde{v}\rangle \leq \frac{a_2}{3}\|\tilde{v}\|^2 + \frac{3a_1^2)^2}{a_2}\|y(\theta_t\omega)\|^2, \qquad (4.7)$$

$$2\langle f_1, \tilde{u}\rangle \leq \frac{a_1}{3}\|\tilde{u}\|^2 + \frac{3}{a_1}\|f_1\|^2, \qquad (4.8)$$

$$2\langle f_2, \tilde{v}\rangle \leq \frac{a_2}{3}\|\tilde{v}\|^2 + \frac{3}{a_2}\|f_2\|^2. \qquad (4.9)$$

By (4.2)-(4.9), we obtain that

$$\begin{aligned}
&\frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2]+2d_1\langle A\tilde{u},\tilde{u}\rangle+2d_2\langle A\tilde{v},\tilde{v}\rangle+2a_1\|\tilde{u}\|^2+2a_2\|\tilde{v}\|^2\\
\leq\ & \frac{a_1}{3}\|\tilde{u}\|^2+\frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2+\frac{a_2}{3}\|\tilde{v}\|^2+\frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2+\frac{3}{a_1}\|f_1\|^2+\frac{a_1}{3}\|\tilde{u}\|^2\\
&+\frac{3}{a_2}\|f_2\|^2+\frac{a_2}{3}\|\tilde{v}\|^2+\frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2+\frac{a_1}{3}\|\tilde{u}\|^2+\frac{3a_1^2}{a_2}\|y(\theta_t\omega)\|^2+\frac{a_2}{3}\|\tilde{v}\|^2\\
=\ & a_1\|\tilde{u}\|^2+\frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2+a_2\|\tilde{v}\|^2+\frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2+\frac{3}{a_1}\|f_1\|^2\\
&+\frac{3}{a_2}\|f_2\|^2+\frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2+\frac{3a_1^2}{a_2}\|y(\theta_t\omega)\|^2,
\end{aligned}$$

hence we have,

$$\begin{aligned}
&\frac{d}{dt}[\|\tilde{u}\|^2+\|\tilde{v}\|^2]+a_1\|\tilde{u}\|^2+a_2\|\tilde{v}\|^2\\
\leq\ & \frac{3d_1^2}{a_1}\|Ay(\theta_t\omega)\|^2+\frac{3d_2^2}{a_2}\|Ay(\theta_t\omega)\|^2+\frac{3}{a_1}\|f_1\|^2+\frac{3}{a_2}\|f_2\|^2+\frac{3a_2^2}{a_1}\|y(\theta_t\omega)\|^2+\frac{3a_1^2}{a_2}\|y(\theta_t\omega)\|^2\\
\leq\ & C_1\|Ay(\theta_t\omega)\|^2+C_2\|y(\theta_t\omega)\|^2+C_3(\|f_1\|^2+\|f_2\|^2)\\
\leq\ & C_4(\|y(\theta_t\omega)\|^2+\|Ay(\theta_t\omega)\|^2+\|f_1\|^2+\|f_2\|^2+\|f_3\|^2), \qquad (4.10)
\end{aligned}$$

where

$$\begin{aligned}
C_1&=\max\{\frac{3d_1^2}{a_1},\frac{3d_2^2}{a_2}\}, \qquad C_2=\max\{\frac{3a_2^2}{a_1},\frac{3a_1^2}{a_2}\},\\
C_3&=\max\{\frac{3}{a_1},\frac{3}{a_2}\}, \qquad C_4=\max\{C_1,C_2,C_3\}.
\end{aligned}$$

By Gronwall's inequality in [9], it follows that

$$\begin{aligned}
&\|\tilde{u}(t,\omega,\tilde{u}_0(\omega))\|^2+\|\tilde{v}(t,\omega,\tilde{v}_0(\omega))\|^2\\
\leq\ & e^{-\min\{a_1,a_2\}t}[\|\tilde{u}_0(\omega)\|^2+\|\tilde{v}_0(\omega)\|^2]+\frac{C_4}{\min\{a_1,a_2\}}(\|f_1\|^2+\|f_2\|^2)\\
&+C_4\int_0^t e^{-\min\{a_1,a_2\}(t-s)}(\|y(\theta_s\omega)\|^2+\|Ay(\theta_s\omega)\|^2)ds. \qquad (4.11)
\end{aligned}$$

Let $c_1=\min\{a_1,a_2\}$. Now that the random variable $y(\theta_t\omega)$ is tempered and continuous in t. It follows from Proposition 4.3.3 in [1], there is a tempered function $\jmath(\omega)>0$ that satisfies

$$\|y(\theta_t\omega)\|^2+\|Ay(\theta_t\omega)\|^2\leq \jmath(\theta_t\omega)\leq \jmath(\omega)e^{\frac{c_1}{2}|t|}. \qquad (4.12)$$

Replacing ω by $\theta_{-t}\omega$ in (4.11), by (4.12), we get

$$\begin{aligned}
&\|\tilde{u}(t,\theta_{-t}\omega,\tilde{u}_0(\theta_{-t}\omega))\|^2+\|\tilde{v}(t,\theta_{-t}\omega,\tilde{v}_0(\theta_{-t}\omega))\|^2\\
\leq\ & e^{-c_1t}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2]+\frac{C_4}{c_1}(\|f_1\|^2+\|f_2\|^2)\\
&+C_4\int_0^t e^{-c_1(t-s)}(\|y(\theta_{s-t}\omega)\|^2+\|Ay(\theta_{s-t}\omega)\|^2)ds\\
\leq\ & e^{-c_1t}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2]+\frac{C_4}{c_1}(\|f_1\|^2+\|f_2\|^2)\\
&+C_4\int_{-t}^0 e^{c_1\tau}(\|y(\theta_\tau\omega)\|^2+\|Ay(\theta_\tau\omega)\|^2)d\tau\\
\leq\ & e^{-c_1t}[\|\tilde{u}_0(\theta_{-t}\omega)\|^2+\|\tilde{v}_0(\theta_{-t}\omega)\|^2]+\frac{C_4}{c_1}(\|f_1\|^2+\|f_2\|^2)+\frac{2C_4l(\omega)}{c_1}. \qquad (4.13)
\end{aligned}$$

Define $R^2(\omega) = 2[C_4(\|f_1\|^2 + \|f_2\|^2) + 2C_4 l(\omega)]/c_1$; $\jmath(\omega)$ is a tempered function, so $R(\omega)$ is also tempered.

Define

$$\tilde{J}(\omega) = \{(\tilde{u}, \tilde{v}) \in \imath^2 \times \imath^2, \|\tilde{u}\|^2 + \|\tilde{v}\|^2 \leq R^2(\omega)\}.$$

From Theorem 4.2 in [5], $\tilde{J}(\omega)$ is an absorbing set for the random dynamical system $(\tilde{u}(t,\omega,\tilde{u}_0), \tilde{v}(t,\omega,\tilde{v}_0))$, i.e., $\forall D \in \mathcal{D}$ and $\forall \omega \in \Omega'$, there exists $T_D(\omega)$ such that

$$\Phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset \tilde{J}(\omega) \quad \text{for} \quad t \geq T_D(\omega).$$

Let

$$J(\omega) = \{(u, v) \in \imath^2 \times \imath^2, \|u\|^2 + \|v\|^2 \leq R_1^2(\omega)\},$$

where

$$R_1^2(\omega) = 2R^2(\omega) + 4\|y(\theta_t\omega)\|^2.$$

since

$$\begin{aligned} & \psi(t, \omega, (u_0, v_0, z_0)) \\ = \; & \Phi(t, \omega, (u_0 - y(\omega), v_0 - y(\omega))) + (y(\theta_t\omega), y(\theta_t\omega)) \\ = \; & (\tilde{u}(t, \omega, u_0 - y(\omega)) + y(\theta_t\omega), \tilde{v}(t, \omega, v_0 - y(\omega)) + y(\theta_t\omega)), \end{aligned}$$

so $J(\omega)$ is an absorbing random set for $\psi(t, \omega)$ and $J \in \mathcal{D}$. The proof of Lemma 4.1 is completed. $\square$

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