Integrability of very weak Solutions for Boundary value problems of Nonhomogeneous p-Harmonic equations

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***Abstract*—The paper deals with very weak solutions u to boundary value problems of the nonhomogeneous p-harmonic equation. We show that, any very weak solution u to the boundary value problem is integrable provided that r is sufficiently close to p.**

Keywords—Integrability; Very weak solution; Boundary value problem; p-harmonic equation.

I. INTRODUCTION

Let $1<p<n$, $\theta:\overline{\Omega}\to R$, $\theta(x)\in W^{1,q}(\Omega)$, $q>r$, $f(x)\in L^{\frac{nq}{n(p-1)+r}}(\Omega)$. We shall examine the boundary value problem of the p-harmonic equation

$$\begin{cases}-\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))=f(x), & x\in\Omega,\\ u(x)=\theta(x), & x\in\partial\Omega,\end{cases} \tag{1.1}$$

Throughout this paper Ω will stand for a bounded regular domain in $\mathbf{R}^n(n\geq 2)$. By a regular domain we understand any domain of finite measure for which the estimates (3.3) and (3.4) below for the Hodge decomposition are satisfied, see [1], [2]. A Lipschitz domain, for example, is regular.

Definition 1.1. A function $u\in\theta+W_0^{1,r}(\Omega)$, $\max\{1,p-1\}<r<p$, is called a very weak solution to the boundary value problem (1.1) if for all $\Phi\in W_0^{1,r/(r-p+1)}(\Omega)$ with compact support sets in Ω, there is

$$\int_\Omega\left\langle|\nabla u|^{p-2}\nabla u,\nabla\Phi\right\rangle dx=\int_\Omega f(x)\Phi dx \tag{1.2}$$

where $f(x)\in L^{\frac{nq}{n(p-1)+r}}(\Omega)$.

Recall that a function $u\in\theta+W_0^{1,p}(\Omega)$ is called the weak solution of the boundary value problem (1.1) if (1.2) holds true for all $\Phi\in W_0^{1,p}(\Omega)$. The words very weak in Definition 1.1 mean that the Sobolev integrable exponent r of u can be smaller than the natural one p, see [1], Theorem 1, page 602.

In this paper we will need the definition of weak L^t-space (see [2]): for $t>0$, the weak L^t- space, $L^t_{weak}(\Omega)$, consists of all measurable functions f such that

$$\left|\left\{x\in\Omega:\left|f(x)\right|>s\right\}\right|\le\frac{k}{s^t}$$

for some positive constant $k=k(f)$ and every $s>0$, where $|E|$ is the n-dimensional Lebesgue measure of E.

Integrability property is important in the regularity theories of nonlinear elliptic PDEs and systems. In [3], Zhu et al. studied the global integrability of nonhomogeneous quasilinear elliptic equations $-\mathrm{div}A(x,u,\nabla u)=f(x)+\mathrm{div}(|\nabla u|^{p-2}\nabla u)$. In [4], Guo et al. studied the higher order integrability of the divergence elliptic equation $-\mathrm{div}A(x,\nabla u)=-\mathrm{div}f$. In [5], Zhang et al. studied the global integrability of A-harmonic equation $-\mathrm{div}A(x,\nabla u)=-\mathrm{div}f$. In this paper, we consider the global integrability of the very weak solutions of the boundary value problem (1.1) [10]. The main result is the following theorem.

Theoerm 1.1. Let $\theta\in W^{1,q}(\Omega)$, $q>r$, There exists $\varepsilon_0=\varepsilon_0(n,p)>0$, such that for each very weak solution $u\in\theta+W_0^{1,r}(\Omega)$, $\max\{1,p-1\}<r<p<n$, to the boundary value problem (1.1), we have

$$u\in\begin{cases}\theta+L^{q^*}_{weak}(\Omega) & for\ q<r,\\ \theta+L^{\tau}_{weak}(\Omega) & for\ q=r\ and\ \tau<\infty,\\ \theta+L^{\infty}(\Omega) & for\ q>n,\end{cases}\tag{1.3}$$

provided that $|p-r|<\varepsilon_0$.

Note that we have restricted ourselves to the case $r<n$ since otherwise any function in $W^{1,r}(\Omega)$ is in the spce $L^t(\Omega)$ for any $t<\infty$ by the Sobolev embedding theorem. At the same time, it is also noted that the very weak solution u to the boundary value problem (1.1) is taken from the Sobolev space $W^{1,r}(\Omega)$, and the embedding theorem ensures that the integrability of u reaches from r to r^*. And our result theorem 1.1 improves this integrability. Note that the key to proving the theorem 1.1 is to use Hodge decomposition[1][6] to construct the appropriate test function.

II. PRELIMINARY LEMMAS

Lemma 1.1[6] For $p\ge 2$ and any $X,Y\in\mathbf{R}^n$, one has

$$2^{2-p}|X-Y|^p\le\left\langle|X|^{p-2}X-|Y|^{p-2}Y,X-Y\right\rangle.$$

Here $|\cdot|$ is the Euclidian norm in $\mathbf{R}^n$, and $\langle\cdot,\cdot\rangle$ is the euclidian scalar product.

Lemma 1.2[7] For any $X,Y\in\mathbf{R}^n$, one has

$$\left\| X|^{\varepsilon}X-|Y|^{\varepsilon}Y\right| \le \begin{cases}(1+\varepsilon)(|Y|+|X-Y|)^{\varepsilon}|X-Y|, & \varepsilon>0,\\ \frac{1-\varepsilon}{2^{\varepsilon}(1+\varepsilon)}|X-Y|^{1+\varepsilon}, & -1<\varepsilon\le 0.\end{cases}$$

Lemma 1.3[2] For $1<p<2$ and any $X,Y\in\mathbf{R}^n$, one has

$$\left\langle |X|^{p-2}X-|Y|^{p-2}Y, X-Y\right\rangle \ge |X-Y|\left((|X-Y|+|Y|)^{p-1}-|Y|^{p-1}\right).$$

Lemma 1.4[2] Let $\varepsilon_0>0$, $\phi:(s_0,\infty)\to[0,\infty)$ is a decrement function such that for each r, s $(r>s>s_0)$, if

$$\phi(r)\le\frac{c}{(r-s)^{\alpha}}(\phi(s))^{\beta}$$

where c,α,β are constants, we have

(1) if $\beta>1$ we have that $\phi(s_0+d)=0$, where $d^{\alpha}=c2^{\alpha\beta/(\beta-1)}(\phi(s_0))^{\beta-1}$;

(2) If $\beta<1$ we have that $\phi(s)\le 2^{\mu/(1-\beta)}(c^{1/(1-\beta)}+(2s_0)^{\mu}\phi(s_0))s^{-\mu}$, where $\mu=\alpha/(1-\beta)$.

III. PROOF OF THEOREM 1.1

For any $L>0$, let

$$v=\begin{cases}u-\theta+L & for\ u-\theta<-L,\\ 0 & for\ -L\le u-\theta\le L,\\ u-\theta-L & for\ u-\theta>L.\end{cases} \tag{3.1}$$

Then according to the hypothesis, we have $v\in W_0^{1,r}(\Omega)$ and $\nabla v=(\nabla u-\nabla\theta)\cdot 1_{\{|u-\theta|>L\}}$, where 1_E is the characteristic function of the set E. We introduce the Hodge decomposition of vector field $|\nabla v|^{p-2}\nabla v\in L^{r/(r-p+1)}(\Omega)$. So that

$$|\nabla v|^{r-p}\nabla v=\nabla\Phi+h. \tag{3.2}$$

Here $\Phi\in W_0^{1,r/(r-p+1)}$, $h\in L^{r/(r-p+1)}(\Omega,\mathbf{R}^n)$ is a vector field with zero divergence, and satisfied

$$\|\nabla\Phi\|_{r/(r-p+1)}\le C(n,p)\|\nabla v\|_r^{r-p+1} \tag{3.3}$$

and

$$\|h\|_{r/(r-p+1)}\le C(n,p)|p-r|\|\nabla v\|_r^{r-p+1}. \tag{3.4}$$

From the counter-proof method, it is inevitable to exist φ such that $\Phi=\varphi-\varphi_{\Omega}$. Taken Φ as a test function of the integral identity (1.2), that is

$$\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u, |\nabla u-\nabla\theta|^{r-p} (\nabla u-\nabla\theta) \right\rangle dx = \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u, h \right\rangle dx + \int_{\{|u-\theta|>L\}} f(x)\Phi dx.$$

This implies

$$\begin{aligned}
&\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p} (\nabla u-\nabla\theta) \right\rangle dx \\
&= \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla\theta|^{p-2}\nabla\theta, h \right\rangle dx \\
&\quad + \int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, h \right\rangle dx \\
&\quad - \int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p} (\nabla u-\nabla\theta) \right\rangle dx \\
&\quad + \int_{\{|u-\theta|>L\}} f(x)\Phi dx \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.5}$$

Now we shall distinguish between two cases.

Case 1: $p \ge 2$. Using Lemma 2.1, (3.5) can be estimated as

$$\begin{aligned}
&\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p} (\nabla u-\nabla\theta) \right\rangle dx \\
&\ge 2^{2-p} \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r \, dx.
\end{aligned} \tag{3.6}$$

Using the Lemma 2.2, Hölder inequality and Young inequality, $|I_1|$ can be estimated as

$$\begin{aligned}
|I_1| &= \left| \int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2} \nabla u - |\nabla\theta|^{p-2}\nabla\theta, h \right\rangle dx \right| \\
&\le (p-1)\int_{\{|u-\theta|>L\}} (|\nabla\theta| + |\nabla u-\nabla\theta|)^{p-2} |\nabla u-\nabla\theta| \, |h| \, dx \\
&\le 2^{p-2}(p-1)\Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-2} |\nabla u-\nabla\theta| \, |h| \, dx + \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{p-1} |h| \, dx\Big) \\
&\le 2^{p-2}(p-1)\Big[\Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^r \, dx\Big)^{\frac{p-2}{r}} \Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r \, dx\Big)^{\frac{1}{r}} \\
&\quad \cdot \Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}} dx\Big)^{\frac{r-p+1}{r}} + \Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r \, dx\Big)^{\frac{p-1}{r}} \cdot \Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}} dx\Big)^{\frac{r-p+1}{r}}\Big] \\
&\le 2^{p-2}(p-1)C(n,p)|p-r| \Big[\Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^r \, dx\Big)^{\frac{p-2}{r}} \\
&\quad \cdot \Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r \, dx\Big)^{\frac{r-p+2}{r}} + \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r \, dx\Big].
\end{aligned} \tag{3.7}$$

Using the Hölder inequality, (3.4) and Young inequality, $|I_2|$ and $|I_3|$ can be estimated as

$$\begin{aligned}
|I_2| &= \left| \int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, h \right\rangle dx \right| \\
&\le \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1} |h| \, dx \\
&\le \Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^r \, dx\Big)^{\frac{p-1}{r}} \Big(\int_{\{|u-\theta|>L\}} |h|^{\frac{r}{r-p+1}} dx\Big)^{\frac{r-p+1}{r}} \\
&\le C(n,p)|p-r| \Big(\int_{\{|u-\theta|>L\}} |\nabla\theta|^r \, dx\Big)^{\frac{p-1}{r}} \cdot \Big(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r \, dx\Big)^{\frac{r-p+1}{r}} \\
&\le C(n,p)|p-r| \Big[C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^r \, dx + \varepsilon \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^r \, dx\Big],
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
|I_3| &= \left| -\int_{\{|u-\theta|>L\}} \left\langle |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx \right| \\
&\le \int_{\{|u-\theta|>L\}} |\nabla\theta|^{p-1}|\nabla u-\nabla\theta|^{r-p+1}\, dx \\
&\le \left(\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx\right)^{\frac{p-1}{r}} \left(\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx + \varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx.
\end{aligned}
\tag{3.9}
$$

Using the Hölder inequality, Sobolev-Poincáre inequality[8],

$$
\left(\int_\Omega |u-u_\Omega|^{pn/(n-p)}dx\right)^{(n-p)/pn} \le C\left(\int_\Omega |\nabla u|^{p} dx\right)^{1/p}, (1\le p<n),
$$

and using (3.3) and Young inequality, $|I_4|$ can be estimated as

$$
\begin{aligned}
|I_4| &= \left|\int_{\{|u-\theta|>L\}} f(x)\Phi dx\right| \\
&\le \left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\varphi-\varphi_\Omega|^{\frac{nr}{n(r-p+1)-r}}\, dx\right)^{\frac{n(r-p+1)-r}{nr}} \\
&\le C(n,p)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\nabla\Phi|^{\frac{r}{r-p+1}}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{nr}} \cdot \left(\int_{\{|u-\theta|>L\}} |\nabla v|^{r}\, dx\right)^{\frac{r-p+1}{r}} \\
&\le C(n,p)\Big[C(\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{n(p-1)}} \\
&\qquad + \varepsilon\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx\Big].
\end{aligned}
\tag{3.10}
$$

Combining (3.5)-(3.10), we arrive at

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx \\
&\le C(n,p,\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla\theta|^{r}\, dx \\
&\quad + (C(n,p)|p-r|+\varepsilon)\int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r}\, dx \\
&\quad + C(n,p,\varepsilon)\left(\int_{\{|u-\theta|>L\}} |f(x)|^{\frac{nr}{n(p-1)+r}}\, dx\right)^{\frac{n(p-1)+r}{n(p-1)}} ,
\end{aligned}
\tag{3.11}
$$

Case 2: $1<p<2$. Lemma 2.3 yields

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}} \left\langle |\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta, |\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx \\
&\ge \int_{\{|u-\theta|>L\}} |\nabla u-\nabla\theta|^{r-p+1} \\
&\qquad \cdot\left((|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1} - |\nabla\theta|^{p-1}\right)dx.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r\,dx \\
&\le \int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r-p+1}(|\nabla u-\nabla\theta|+|\nabla\theta|)^{p-1}dx \\
&\le \int_{\{|u-\theta|>L\}}\left\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx \\
&\quad+\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{r-p+1}|\nabla\theta|^{p-1}\,dx \\
&\le \int_{\{|u-\theta|>L\}}\left\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,|\nabla u-\nabla\theta|^{r-p}(\nabla u-\nabla\theta)\right\rangle dx \\
&\quad+\varepsilon\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r\,dx+C(\varepsilon)\int_{\{|u-\theta|>L\}}|\nabla\theta|^r\,dx.
\end{aligned} \tag{3.12}
$$

Using Lemma 2.2 and (3.4), $|I_1|$ can be estimated as

$$
\begin{aligned}
|I_1|&=\left|\int_{\{|u-\theta|>L\}}\left\langle|\nabla u|^{p-2}\nabla u-|\nabla\theta|^{p-2}\nabla\theta,h\right\rangle dx\right| \\
&\le\frac{3-p}{2^{p-2}(p-1)}\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^{p-1}|h|\,dx \\
&\le\frac{3-p}{2^{p-2}(p-1)}\left(\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r\,dx\right)^{\frac{p-1}{r}}\cdot\left(\int_{\{|u-\theta|>L\}}|h|^{\frac{r}{r-p+1}}\,dx\right)^{\frac{r-p+1}{r}} \\
&\le\frac{3-p}{2^{p-2}(p-1)}C(n,p)|p-r|\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r\,dx.
\end{aligned} \tag{3.13}
$$

For the case $1<p<2$, $|I_2|$ and $|I_3|$ can also be estimated by (3.8)-(3.9). Combining (3.5), (3.12) and (3.13), we arrive at (3.11).

Let $\varepsilon_0=1/C(n,p)$. Then for $|p-r|<\varepsilon_0$ we have $C(n,p)|p-r|<1$. Taking ε small enough, such that $C(n,p)|p-r|+\varepsilon<1$, then the second term on the right-hand side of (3.11) can be absorbed by the left-hand side; thus we obtain

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r\,dx \\
&\le C(n,p)\int_{\{|u-\theta|>L\}}|\nabla\theta|^r\,dx+C(n,p)\left(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nr}{n(p-1)+r}}\,dx\right)^{\frac{n(p-1)+r}{n(p-1)}}.
\end{aligned} \tag{3.14}
$$

Since $\theta\in W^{1,q}(\Omega)$, $q>r$, using the Hölder inequality, we have

$$
\begin{aligned}
&\int_{\{|u-\theta|>L\}}|\nabla\theta|^r\,dx \\
&\le\left(\int_{\{|u-\theta|>L\}}|\nabla\theta|^q\,dx\right)^{r/q}|\{|u-\theta|>L\}|^{(q-r)/q} \\
&=\|\nabla\theta\|_q^r|\{|u-\theta|>L\}|^{(q-r)/q}.
\end{aligned} \tag{3.15}
$$

By the proof idea of reference [9](Page 442), and the Hölder inequality, we get

$$
\begin{aligned}
&\left(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nr}{n(p-1)+r}}\,dx\right)^{\frac{n(p-1)+r}{n(p-1)}} \\
&\le\left(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nq}{n(p-1)+r}}\,dx\right)^{\frac{nr(p-1)+r^2}{qn(p-1)}}|\{|u-\theta|>L\}|^{(q-r)/q} \\
&\le M\,|\{|u-\theta|>L\}|^{(q-r)/q},
\end{aligned} \tag{3.16}
$$

where $M=(\int_{\{|u-\theta|>L\}}|f(x)|^{\frac{nq}{n(p-1)+r}}dx)^{\frac{nr(p-1)+r^2}{qn(p-1)}}$, M is bounded and is a constant dependent only on n, p. Then (3.14) can be collated into the following results

$$\begin{aligned}&\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r dx\\&\le C(n,p)(\int_{\{|u-\theta|>L\}}|\nabla\theta|^q dx)^{r/q}|\{|u-\theta|>L\}|^{(q-r)/q}\\&\quad+C(n,p)M|\{|u-\theta|>L\}|^{(q-r)/q}\\&=C|\{|u-\theta|>L\}|^{(q-r)/q}(1+\|\nabla\theta\|_q^r)\ ,\end{aligned}\tag{3.17}$$

where $C=C(n,p,M)$.

We now turn our attention back to the function $v\in W_0^{1,r}(\Omega)$. By the Sobolev embedding theorem, we have

$$\begin{aligned}(\int_\Omega|v|^{r^*}dx)^{1/r^*}&\le C(n,r)(\int_\Omega|\nabla v|^r dx)^{1/r}\\&=C(n,r)(\int_{\{|u-\theta|>L\}}|\nabla u-\nabla\theta|^r dx)^{1/r}\ ,\end{aligned}\tag{3.18}$$

since $|v|=(|u-\theta|-L)\cdot 1_{\{|u-\theta|>L\}}$, we have

$$(\int_{\{|u-\theta|>L\}}(|\nabla u-\nabla\theta|-L)^{r^*}dx)^{1/r^*}=(\int_\Omega|v|^{r^*}dx)^{1/r^*}\ ,\tag{3.19}$$

and for $L'>L$,

$$\begin{aligned}&(L'-L)^{r^*}|\{|u-\theta|>L'\}|\\&=\int_{\{|u-\theta|>L\}}(L'-L)^{r^*}dx\\&\le\int_{\{|u-\theta|>L'\}}(|u-\theta|-L)^{r^*}dx\\&\le\int_{\{|u-\theta|>L\}}(|u-\theta|-L)^{r^*}dx\ .\end{aligned}\tag{3.20}$$

By collecting (3.17)-(3.20), we deduce that

$$\begin{aligned}&((L'-L)^{r^*}|\{|u-\theta|>L'\}|)^{1/r^*}\\&\le C(n,r)(\|\nabla\theta\|_q+1)|\{|u-\theta|>L\}|^{1/r-1/q}\ .\end{aligned}\tag{3.21}$$

Thus

$$\begin{aligned}&|\{|u-\theta|>L'\}|\\&\le\frac{1}{(L'-L)^{r^*}}(C(n,r)(\|\nabla\theta\|_q+1))^{r^*}|\{|u-\theta|>L\}|^{r^*(1/r-1/q)}\ .\end{aligned}\tag{3.22}$$

Let $\phi(s)=|\{|u-\theta|>s\}|$, $\alpha=r^*$, $c=(C(n,r)(\|\nabla\theta\|_q+1))^{r^*}$, $\beta=r^*(1/r-1/q)$, $s_0>0$, Then (3.22) become

$$\phi(L')\le\frac{c}{(L'-L)^\alpha}\phi(L)^\beta\tag{3.23}$$

for $L'>L>0$.

(1) For the case $q<n$, one has $\beta<1$. In this case, if $s\ge 1$, we get from Lemma 2.3 that

$$|\{|u-\theta|>s\}|\le c(\alpha,\beta,s_0)s^{-t},$$

where $t=\alpha/(1-\beta)=q*$. For $0<s<1$, one has

$$|\{|u-\theta|>s\}|\le|\Omega|=|\Omega|s^{q^*}s^{-q^*}\le|\Omega|s^{-q^*}.$$

Thus

$$u\in\theta+L^{q^*}_{weak}(\Omega).$$

(2) For the case $q=n$, one has $\beta=1$. For any $\tau<\infty$, (3.23) implies

$$\phi(L')\le\frac{c}{(L'-L)^{\alpha}}\phi(L)=\frac{c}{(L'-L)^{\alpha}}\phi(L)^{1-\alpha/\tau}\phi(L)^{\alpha/\tau}$$
$$\le\frac{c|\Omega|^{\alpha/\tau}}{(L'-L)^{\alpha}}\phi(L)^{1-\alpha/\tau}.$$

As about, we derive

$$u\in\theta+L^{\tau}_{weak}(\Omega).$$

(3) For the case $q>n$, one has $\beta>1$. Lemma 2.3 implies $\phi(d)=0$ for some $d=d(\alpha,\beta,s_0,r,(\|\nabla\theta\|_q+1))$. Thus $|\{|u-\theta|>d\}|=0$, which means $u-\theta\le d$ a.e. in Ω. Therefore

$$u\in\theta+L^{\infty}(\Omega),$$

completing the proof of Theorem 1.1.

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