# Multiple Exact Travelling Solitary Wave Solutions of Nonlinear Evolution Equations

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**ABSTRACT:**

*An extended Tanh-function method with Riccati equation is presented for constructing multiple exact travelling wave solutions of some nonlinear evolution equations which are particular cases of a generalized equation. The results of solitary waves are general compact forms with non-zero constants of integration. Taking the full advantage of the Riccati equation improves the applicability and reliability of the Tanh method with its extension form.*

*Keywords: Extended Tanh method, Riccati equation, Solitary waves, Evolution equations.*

## 1. Introduction

Nonlinear partial differential equations (NLPDEs) play a major role in the study of nonlinear science. In recent decades, constructing the exact solitary travelling wave solutions and solitons of NLPDEs have become an important research subject due to the constant proposing of analytical methods, say, [1]–[9]. Among these methods, the powerful Hyperbolic Tangent (Tanh) method [2], [10] has been tremendously developed in the literature – for instance [3], [4], [11]. More precisely, the Extended Tanh method (later known as Tanh-coth method) and its modified form was introduced by [3]–[5], which has been successively utilized to obtain solution of NLPDEs. The Modified Extended Tanh method with Riccati equation [5], [11], [12] is widely recognized as one of the most powerful tools used in a favor of obtaining the explicit solitary travelling wave solutions of NLPDEs.

The following NLPDE are proposed as a generalization of the mentioned equations, involving nonlinear dispersion and dissipation effects [13]:

$$u_t + \alpha u u_x + \beta u^2 u_x + \nu u_{xx} + \mu u_{px} = 0 \qquad (1)$$

where $\alpha\beta \neq 0, \nu\mu \neq 0$ and $p$ are all arbitrary constants. Considering the setting of these parameters to be equal to special values with $\beta = 0$, equation (1) is reduced to KdV-Burgers equation ( $p=3, \quad \alpha\nu\mu \neq 0$ ), and to Kuramoto-Sivanshinsky $p=4, \alpha\nu\mu \neq 0$. Which they have the following well-known forms (respectively):

$$u_t + \alpha u u_x + \nu u_{xx} + \mu u_{3x} = 0 \tag{2}$$

$$u_t + \alpha u u_x + \nu u_{xx} + \mu u_{4x} = 0 \tag{3}$$

However, the class of this NLPDE when $\beta \neq 0$ is considered in [14]. This paper is organized to fully present the algorithm of the considered method in Section.2, Analytical solution in the form of solitary travelling wave solutions of equation (1) with its special parameters' values are obtained in Section.3. Finally in Section.4 concluding remarks are presented.

## 2. Methodology of the method

The solitary wave solution of a NLPDE in two variables $x, t$ :

$$\Psi_1\left(u, u_t, u_x, u_{xt}, u_{xx}, \ldots\right) = 0 \tag{4}$$

are the solution of the nonlinear ordinary differential equation NLODE :

$$\Psi_2\left(U, U', U'', U''', \ldots\right) = 0 \tag{5}$$

which is obtained by the travelling wave transformation $u(x,t) = U(\zeta) = U(x - \omega t)$, and the prime denotes the ordinary derivative with respect to $\zeta$. Introducing a new variable $\psi = \psi(\zeta)$, that satisfies the Riccati equation of the form:

$$\frac{d}{d\zeta}\psi(\zeta) = k + \psi(\zeta)^2 \tag{6}$$

where $k$ is a real constant. The modified Extended Tanh method with Riccati equation admits that the solution of (5) can be expressed by a polynomial in $\psi^j$ :

$$\begin{aligned} u(x,t) = U(\zeta) = {} & a_N \psi^N + a_{N-1}\psi^N + \ldots + a_1\psi + a_0 \\ & + b_1\psi^{-1} + \ldots + b_{N-1}\psi^{-N-1} + b_N\psi^{-N} \end{aligned} \tag{7}$$

where $N$ is the balancing integer. Substituting (6) along with (7) into (5) then setting the coefficients of all powers of $\psi(\zeta)^j$ to zero, a nonlinear algebraic system is generated with respect to parameters $a_0, a_j, b_j, k, \omega$. By the test sign of $k$ , the Riccati equation (6) has the well-known general solutions:

$$\psi(\zeta) = -\frac{1}{\zeta} \qquad , k = 0 \tag{8}$$

$$\psi(\zeta) = \begin{cases} -\sqrt{-k}\tanh\left(\sqrt{-k}\left(x - \omega t\right)\right) \\ -\sqrt{-k}\coth\left(\sqrt{-k}\left(x - \omega t\right)\right) \end{cases} \qquad k < 0 \tag{9}$$

$$\psi(\zeta) = \begin{cases} \sqrt{k}\tan\left(\sqrt{k}\left(x - \omega t\right)\right) \\ -\sqrt{k}\cot\left(\sqrt{k}\left(x - \omega t\right)\right) \end{cases} \qquad k > 0 \tag{10}$$

## 3. The solitary travelling wave solutions

### 3.1 Explicit solution of KdV-Burgers equation

Using the wave transformation prescribed in the previous section, gives rise to the NLODE:

$$-\omega U' + \alpha U U' + \nu U'' + \mu U''' = 0 \tag{11}$$

Integrating (11) with respect to $\zeta$, to get:

$$-\omega U + \frac{\alpha}{2} U^2 + \nu U' + \mu U'' + \eta_0 = 0 \tag{12}$$

where $\eta_0$ is an arbitrary constant. With $N = 2$ (balancing $U^2$ and $U''$) therefore (7) admits the ansätz:

$$U(\zeta) = a_0 + a_1\psi(\zeta) + a_2\psi^2(\zeta) + b_1\psi^{-1}(\zeta) + b_2\psi^{-2}(\zeta) \tag{13}$$

Substituting (13) into (12) and with the use of (6), we obtain the following algebraic system by setting all the coefficients of $\psi^j, j = 0, \pm 1, \pm 2$ to zero:

$$\begin{aligned}
&6k^2\mu b_2 + \frac{\alpha b_2^2}{2} = 0,\\
&2k^2\mu b_1 - 2k\nu b_2 + \alpha b_1 b_2 = 0,\\
&-k\nu b_1 + \frac{\alpha b_1^2}{2} + 8k\mu b_2 - \omega b_2 + \alpha a_0 b_2 = 0,\\
&2k\mu b_1 - \omega b_1 + \alpha a_0 b_1 - 2\nu b_2 + \alpha a_1 b_2 = 0,\\
&\eta - \omega a_0 + \frac{\alpha a_0^2}{2} + k\nu a_1 + 2k^2\mu a_2 - \nu b_1 + \alpha a_1 b_1 + 2\mu b_2 + \alpha a_2 b_2 = 0,\\
&2k\mu a_1 - \omega a_1 + \alpha a_0 a_1 + 2k\nu a_2 + \alpha a_2 b_1 = 0,\\
&\nu a_1 + \frac{\alpha a_1^2}{2} + 8k\mu a_2 - \omega a_2 + \alpha a_0 a_2 = 0,\\
&2\mu a_1 + 2\nu a_2 + \alpha a_1 a_2 = 0,\\
&6\mu a_2 + \frac{\alpha a_2^2}{2} = 0
\end{aligned} \tag{14}$$

The system in (14) is solved by the aid of Mathematica, and taking in consideration the solution of Riccati equation (8) - (10), we obtain the following families of solution:

**Family1.**

$$k = -\frac{\nu^2}{100\mu^2},\ \alpha = \frac{144k\nu^2 + 25\omega^2}{50\eta},\ a_0 = \frac{-12k\mu + \omega}{\alpha},\ a_1 = a_2 = 0, b_1 = \frac{12k\nu}{5\alpha}, b_2 = -\frac{12k^2\mu}{\alpha} \tag{15}$$

**$\eta$ and $\omega$ are an arbitrary**

As it is noted the value of $k < 0$ whenever $(\nu\mu)^2 > 0$, thus the corresponding travelling wave solution is:

$$u_1(x,t)=\frac{1}{\alpha}(\frac{6\nu^2}{25\mu}+\omega)-\frac{3\nu^2}{25\alpha\mu}(\coth(\frac{\nu}{10\mu}(x-\omega t))-1)^2 \tag{16}$$

**Family2.**

$$k=-\frac{\nu^2}{100\mu^2},\alpha=\frac{144k\nu^2+25\omega^2}{50\eta},a_0=\frac{-12k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha},b_1=b_2=0 \tag{17}$$

$\omega$ **is an arbitrary.**

Since $k<0$ whenever $(\nu\mu)^2>0$ thus the corresponding travelling wave solution is:

$$u_2(x,t)=\frac{1}{\alpha}\left(\frac{6\nu^2}{25\mu}+\omega\right)-\frac{3\nu^2}{25\alpha\mu}\left(\tanh(\frac{\nu}{10\mu}(x-\omega t))-1\right)^2 \tag{18}$$

**Family3.**

$$k=-\frac{\nu^2}{400\mu^2},\alpha=\frac{576k\nu^2+25\omega^2}{50\eta},a_0=\frac{-24k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha}, \tag{19}$$

$b_1=-ka_1,b_2=k^2a_2$, $\omega$ **is an arbitrary**)

Since $k<0$ whenever $(\nu\mu)^2>0$ thus the corresponding travelling wave solution is:

$$u_3(x,t)=\frac{1}{\alpha}\left(\frac{3\mu q^2}{50}+\omega\right)+\frac{3q}{25\alpha}\tanh(z)\left(\nu-\frac{q\mu}{4}\tanh(z)\right)+\frac{3q}{25\alpha}\coth(z)\left(\nu-\frac{\mu q}{4}\coth(z)\right)$$

$$q=\frac{\nu}{\mu},z=\frac{1}{20}\frac{\nu}{\mu}(x-\omega t) \tag{20}$$

This can be reduced to obtain solitary wave solution(16).

**Family4.**

$$\mu=\mp\frac{6\nu^2}{25\omega},\eta=0,k=-\frac{\nu^2}{100\mu^2},a_0=\frac{-12k\mu+\omega}{\alpha},a_1=a_2=0,$$

$$b_1=\frac{12k\nu}{5\alpha},b_2=-\frac{12k^2\mu}{\alpha},\ \omega\ \textit{is an arbitrary}) \tag{21}$$

Since $k<0$ whenever $(\nu\mu)^2>0$ thus the corresponding travelling wave solution is:

$$u_{4,5}(x,t)=\frac{1}{\alpha}(\frac{6\nu^2}{25\mu}+\omega)-\frac{3\nu^2}{25\alpha\mu}\left(\coth(\frac{\nu}{10\mu}(x-t\omega))-1\right)^2 \tag{22}$$

**Family5.**

$$\mu=\mp\frac{6\nu^2}{25\omega},\eta=0,k=-\frac{\nu^2}{100\mu^2},a_0=\frac{-12k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha}, \tag{23}$$

$b_1=0,b_2=0,$ $\omega$ **is an arbitrary**

Since $k<0$ whenever $(\nu\mu)^2>0$ thus the corresponding travelling wave solution is:

$$u_{6,7}(x,t)=\frac{1}{\alpha}\left(\frac{6\nu^2}{25\mu}+\omega\right)-\frac{3\nu^2}{25\alpha\mu}\left(\tanh(\frac{\nu}{10\mu}(x-\omega t))-1\right)^2 \tag{24}$$

**Family6.**

$$\mu=\mp\frac{6\nu^2}{25\omega},\eta=0,k=-\frac{\nu^2}{400\mu^2},a_0=\frac{-24k\mu+\omega}{\alpha},a_1=-\frac{12\nu}{5\alpha},a_2=-\frac{12\mu}{\alpha}, \tag{25}$$
$b_1=-ka_1,b_2=k^2a_2$ $\alpha$ **and** $\omega$ **is an arbitrary**)

Since $k<0$ whenever $(\nu\mu)^2>0$ thus the corresponding travelling wave solution is:

$$u_{8,9}(x,t)=\frac{1}{\alpha}(\frac{3\nu^2}{10\mu}+\omega)-\frac{3\nu^2}{100\alpha\mu}(\tanh(t)-2)^2-\frac{3\nu^2}{100\alpha\mu}(\coth(z)-2)^2$$
$$q=\frac{\nu}{\mu},\ \ z=\frac{1}{20}q(x-\omega t) \tag{26}$$

which is reduced to obtain solitary wave solution (15) .

The graphical representation of some solitary wave solutions of (2) is illustrated as follows:

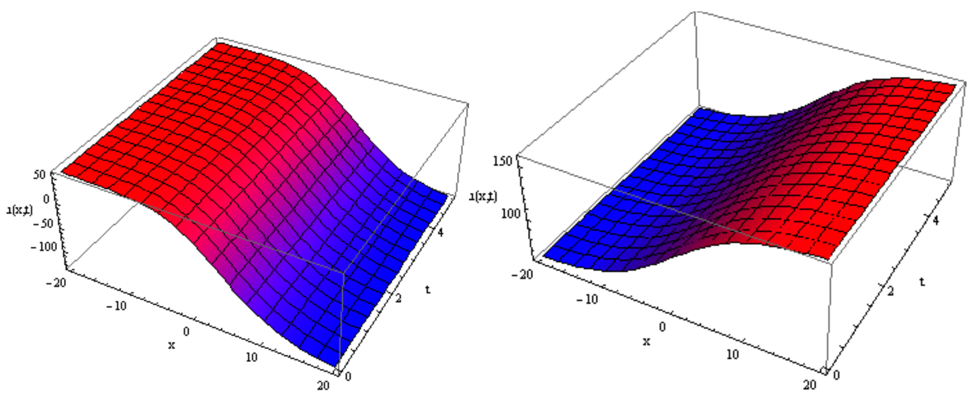


**Figure 1 The plots of solitary wave solutions (18)** $(\eta=10)$ **and (24) when** $\nu=1,\mu=-1,\omega=0.1$ **.**

### 3.2 Explicit solution of Kuramoto-Sivashinsky equation

Making the wave transformation prescribed in the previous section, KS equation(3) is reduced to the following NLODE:

$$-\omega U'+\alpha UU'+\nu U''+\mu U^{(4)}=0 \tag{27}$$

Integrating (27) with respect to $\zeta$ once yields:

$$-\omega U+\frac{\alpha}{2}U^2+\nu U'+\mu U'''+\varepsilon_0=0 \tag{28}$$

where $\varepsilon_0$ is an arbitrary constant. With $N = 3$ (balancing $U'''$ and $U^2$) therefore (7) admits the ansätz:

$$U(\zeta) = a_0 + a_1\psi(\zeta) + a_2\psi^2(\zeta) + a_3\psi^3(\zeta) + b_1\psi^{-1}(\zeta) + b_2\psi^{-2}(\zeta) + b_3\psi^{-3}(\zeta) \tag{29}$$

Substituting (13) into (12) and with the use of (6), we obtain the following algebraic system by setting all the coefficients of $\psi^j, j = 0, \pm 1, \pm 2, \pm 3$ to zero:

$$\begin{aligned}
&-60k^3\mu b_3 + \frac{\alpha b_3^2}{2} = 0,\\
&-24k^3\mu b_2 + \alpha b_2 b_3 = 0,\\
&-6k^3\mu b_1 + \frac{\alpha b_2^2}{2} - 114k^2\mu b_3 - 3k\nu b_3 + \alpha b_1 b_3 = 0,\\
&-40k^2\mu b_2 - 2k\nu b_2 + \alpha b_1 b_2 - \omega b_3 + \alpha a_0 b_3 = 0,\\
&-8k^2\mu b_1 - k\nu b_1 + \frac{\alpha b_1^2}{2} - \omega b_2 + \alpha a_0 b_2 - 60k\mu b_3 - 3\nu b_3 + \alpha a_1 b_3 = 0,\\
&-\omega b_1 + \alpha a_0 b_1 - 16k\mu b_2 - 2\nu b_2 + \alpha a_1 b_2 + \alpha a_2 b_3 = 0,\\
&-\omega a_0 + \frac{\alpha a_0^2}{2} + 2k^2\mu a_1 + k\nu a_1 + 6k^3\mu a_3 - 2k\mu b_1 - \nu b_1 + \alpha a_1 b_1\\
&\qquad + \alpha a_2 b_2 - 6\mu b_3 + \alpha a_3 b_3 + \grave{o}_0 = 0,\\
&-\omega a_1 + \alpha a_0 a_1 + 16k^2\mu a_2 + 2k\nu a_2 + \alpha a_2 b_1 + \alpha a_3 b_2 = 0,\\
&8k\mu a_1 + \nu a_1 + \frac{\alpha a_1^2}{2} - \omega a_2 + \alpha a_0 a_2 + 60k^2\mu a_3 + 3k\nu a_3 + \alpha a_3 b_1 = 0,\\
&40k\mu a_2 + 2\nu a_2 + \alpha a_1 a_2 - \omega a_3 + \alpha a_0 a_3 = 0,\\
&6\mu a_1 + \frac{\alpha a_2^2}{2} + 114k\mu a_3 + 3\nu a_3 + \alpha a_1 a_3 = 0,\\
&24\mu a_2 + \alpha a_2 a_3 = 0,\\
&60\mu a_3 + \frac{\alpha a_3^2}{2} = 0
\end{aligned} \tag{30}$$

The system in (30) is solved by the aid of Mathematica, and taking in consideration the solution of Riccati equation (8) - (10), we obtain the following families of solution:

**Family 1.**

$$\begin{aligned}
&k = -\frac{11\nu}{76\mu}, \alpha = \frac{3600k\nu^2 + 361\omega^2}{722\grave{o}_0}, a_0 = \frac{\omega}{\alpha}, a_1 = 0, a_2 = 0, a_3 = 0,\\
&b_1 = \frac{60(38k^2\mu + k\nu)}{19\alpha}, b_2 = 0, b_3 = \frac{120k^3\mu}{\alpha}, \omega \textbf{ and } \grave{o}_0 \textbf{ are arbitraries}
\end{aligned} \tag{31}$$

As $\nu\mu > 0$, we see that $k < 0$. Consequently, we obtain:

$$u_1(x,t)=\frac{\omega}{\alpha}-\frac{15\nu}{19\alpha}\sqrt{\frac{11\nu}{19\mu}}\coth(z)\left[9-11\coth^2(z)\right],\quad z=\frac{1}{2}\sqrt{\frac{11\nu}{19\mu}}(x-\omega t) \tag{32}$$

As $\nu\mu<0$, we see that $k>0$, the corresponding solution is:

$$u_2(x,t)=\frac{\omega}{\alpha}-\frac{15\nu}{19\alpha}\sqrt{-\frac{11\nu}{19\mu}}\cot(z)\left[9+11\cot^2(z)\right],\quad z=\frac{1}{2}\sqrt{-\frac{11\nu}{19\mu}}(x-\omega t) \tag{33}$$

**Family 2.**

$$\begin{gathered} k=\frac{\nu}{76\mu},\alpha=\frac{3600k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=a_2=a_3=0, \\ b_1=\frac{60(38k^2\mu+k\nu)}{19\alpha},b_2=0,b_3=\frac{120k^3\mu}{\alpha},\ \omega \text{ and } \grave{o}_0 \text{ are arbitararies} \end{gathered} \tag{34}$$

If $\nu\mu>0$, then $k<0$. Consequently, we obtain:

$$u_3(x,t)=\frac{\omega}{\alpha}+\frac{15}{19\alpha}\nu\sqrt{\frac{-\nu}{19\mu}}\coth(z)(\left[3-\coth^2(z)\right]),\quad z=\frac{1}{2}\sqrt{\frac{-\nu}{19\mu}}(x-\omega t) \tag{35}$$

If $\frac{\nu}{\mu}<0$, then $k>0$. Consequently, we obtain:

$$u_4(x,t)=\frac{\omega}{\alpha}+\frac{15\nu}{19\alpha}\sqrt{\frac{\nu}{19\mu}}\cot(z)\left[3+\cot^2(z)\right],\quad z=\frac{1}{2}\sqrt{\frac{\nu}{19\mu}}(x-\omega t) \tag{36}$$

**Family 3.**

$$\begin{gathered} k=-\frac{11\nu}{76\mu},\alpha=\frac{3600k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha}, \\ b_1=b_2=b_3=0,\ \omega \textbf{ and } \grave{o}_0 \textbf{ are arbitrarairs} \end{gathered} \tag{37}$$

If $\nu\mu>0$, then $k<0$. Consequently, we obtain:

$$u_5(x,t)=\frac{\omega}{\alpha}-\frac{15}{19\alpha}\nu\sqrt{\frac{11\nu}{19\mu}}\tanh(z)\left[9-11\tanh^2(z)\right],\quad z=\frac{1}{2}\sqrt{\frac{11\nu}{19\mu}}(x-\omega t) \tag{38}$$

If $\nu\mu<0$, then $k>0$. Consequently, we obtain:

$$u_6(x,t)=\frac{\omega}{\alpha}+\frac{15\nu}{19\alpha}\sqrt{\frac{-11\nu}{19\mu}}\tan(z)\left[9+11\tan(z)\right],\quad z=\frac{1}{2}\sqrt{\frac{-11\nu}{19\mu}}(x-\omega t) \tag{39}$$

**Family 4.**

$$k=\frac{\nu}{76\mu},\alpha=\frac{3600k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha},$$

$b_1=b_2=b_3=0$, $\omega$ **and** $\grave{o}_0$ **are arbitraries**

(40)

If $\nu\mu<0$, then $k<0$ and vice versa . Respectively, we obtain:

$$u_7(x,t)=\frac{\omega}{\alpha}+\frac{15\nu}{19\alpha}\sqrt{\frac{-\nu}{19\mu}}\tanh(z)\left[3-\tanh(z)^2\right],\ z=\frac{1}{2}\sqrt{\frac{-\nu}{19\mu}}(x-\omega t) \quad (41)$$

$$u_8(x,t)=\frac{\omega}{\alpha}-\frac{15\nu}{19\alpha}\sqrt{\frac{\nu}{19\mu}}\tan(z)[3+\tan^2(z)],\quad z=\frac{1}{2}\sqrt{\frac{\nu}{19\mu}}(x-\omega t) \quad (42)$$

**Family 5.**

$$k=-\frac{11\nu}{304\mu},\alpha=\frac{14400k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha},$$

$b_1=-ka_1, b_2=0, b_3=-k^3a_3$, $\omega$ **and**$\grave{o}_0$ **are arbitraries**

(43)

If $\nu\mu<0$, then $k<0$ and vice versa . Respectively, we obtain:

$$\begin{aligned}u_9(x,t)=&\frac{\omega}{\alpha}+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\tanh(z)+\frac{15\mu q^3}{8\alpha}\tanh^3(z)\\&+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\coth(z)+\frac{15\mu q^3}{8\alpha}\coth^3(z)\\q=&\sqrt{\frac{11\nu}{19\mu}},\quad z=\frac{1}{4}\sqrt{\frac{11\nu}{19\mu}}(x-\omega t)\end{aligned} \quad (44)$$

$$\begin{aligned}u_{10}(x,t)=&\frac{\omega}{\alpha}-\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\tan(z)-\frac{15\mu q^3}{8\alpha}\tan^3(z)\\&+\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\cot(z)+\frac{15\mu q^3}{8\alpha}\cot^3(z)\\q=&\sqrt{\frac{-11\nu}{19\mu}},\ z=\frac{1}{4}\sqrt{\frac{-11\nu}{19\mu}}(x-\omega t)\end{aligned} \quad (45)$$

The solitary wave solutions (44) and (45) can be simplified so that $u_1(x,t)$ and $u_2(x,t)$ are obtained respectively.

**Family 6.**

$$k=\frac{\nu}{304\mu},\alpha=\frac{14400k\nu^2+361\omega^2}{722\grave{o}_0},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,a_3=-\frac{120\mu}{\alpha}, \tag{46}$$
$$b_1=-ka_1,b_2=0,b_3=-k^3a_3,\quad \omega\ \textbf{and}\ \grave{o}_0\ \textbf{are arbitraries}$$

If $\nu\mu<0$, then $k<0$ and vice versa . Respectively, we obtain:

$$u_{11}(x,t)=\frac{\omega}{\alpha}+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\tanh(\frac{1}{4}q(x-\omega t))+\frac{15\mu q^3}{8\alpha}\tanh^3(z)^3$$
$$+\frac{15q}{19\alpha}(-\frac{19}{8}\mu q^2+\nu)\coth(z)+\frac{15\mu q^3}{8\alpha}\coth^3(z) \tag{47}$$
$$q=\sqrt{\frac{-\nu}{19\mu}},\ z=\frac{1}{4}q(x-\omega t)$$

$$u_{12}(x,t)=\frac{\omega}{\alpha}-\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\tan(z)-\frac{15\mu q^3}{8\alpha}\tan^3(z)$$
$$+\frac{15q}{19\alpha}(\frac{19}{8}\mu q^2+\nu)\cot(z)+\frac{15\mu q^3}{8\alpha}\cot^3(z) \tag{48}$$
$$q=\sqrt{\frac{\nu}{19\mu}},\ z=\frac{1}{4}q(x-\omega t)$$

The solitary wave solutions (47) and (48) can be simplified so that $u_3(x,t)$ and $u_4(x,t)$ are obtained respectively.

**Family 7.**

$$\grave{o}_0=0,\mu=-\frac{900\nu^3}{6859\omega^2},k=-\frac{361\omega^2}{3600\nu^2},a_0=\frac{\omega}{\alpha},a_1=a_2=a_3=0,$$
$$b_1=\frac{60(38k^2\mu+k\nu)}{19\alpha},b_2=0,b_3=\frac{120k^3\mu}{\alpha},\quad \omega \text{ and } \alpha \text{ are arbitaraies} \tag{49}$$

Since $k<0$, it follows that:

$$u_{13}(x,t)=\frac{\omega}{\alpha}+\frac{\omega}{2\alpha}\coth(\frac{19\omega}{60\nu}(x-\omega t))\left(3-\coth^2(\frac{19\omega}{60\nu}(x-\omega t)\right) \tag{50}$$

**Family 8.**

$$\grave{o}_0=0,\mu=\frac{9900\nu^3}{6859\omega^2},k=-\frac{361\omega^2}{3600\nu^2},a_0=\frac{\omega}{\alpha},a_1=a_2=a_3=0,$$
$$b_1=\frac{60(38k^2\mu+k\nu)}{19\alpha},b_2=0,b_3=\frac{120k^3\mu}{\alpha},\quad \omega \text{ and } \alpha \text{ are arbitaraies} \tag{51}$$

Since $k<0$, it follows that:

$$u_{14}(x,t)=\frac{\omega}{\alpha}-\frac{\omega}{2\alpha}\coth(\frac{19\omega}{60\nu}(x-\omega t))\left(9-11\coth^{2}[\frac{19\omega}{60\nu}(x-\omega t)]\right) \quad (52)$$

**Family 9.**

$$\grave{o}_0=0,\mu=-\frac{900\nu^3}{6859\omega^2},k=-\frac{361\omega^2}{3600\nu^2},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},$$
$$a_2=0,a_3=-\frac{120\mu}{\alpha},b_1=b_2=b_3=0 \quad (53)$$

Since $k<0$, it follows that:

$$u_{15}(x,t)=\frac{\omega}{\alpha}+\frac{\omega}{2\alpha}\tanh(\frac{19\omega}{60\nu}(x-\omega t))\left(3-\tanh^{2}(\frac{19\omega}{60\nu}(x-t\omega))\right) \quad (54)$$

**Family 10.**

$$\grave{o}_0=0,\mu=\frac{9900\nu^3}{6859\omega^2},k=-\frac{361\omega^2}{3600\nu^2},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,$$
$$a_3=-\frac{120\mu}{\alpha},b_1=b_2=b_3=0 \quad (55)$$

Since $k<0$, it follows that:

$$u_{16}(x,t)=\frac{\omega}{\alpha}-\frac{\omega}{2\alpha}\tanh(\frac{19\omega}{60\nu}(x-\omega t))\left(9-11\tanh^{2}(\frac{19\omega}{60\nu}(x-\omega t))\right) \quad (56)$$

**Family 11.**

$$\grave{o}_0=0,\mu=-\frac{900\nu^3}{6859\omega^2},k=-\frac{361\omega^2}{14400\nu^2},a_0=\frac{\omega}{\alpha},a_1=-\frac{60(38k\mu+\nu)}{19\alpha},a_2=0,$$
$$a_3=-\frac{120\mu}{\alpha},b_1=-ka_1,b_2=0,b_3=-k^3a_3 \quad (57)$$

Since $k<0$, it follows that:

$$u_{17}(x,t)=\frac{\omega}{\alpha}+\frac{q}{2\alpha}\left(\frac{-19(361)}{7200}\mu q^2+\nu\right)\left(\tanh(z)+\coth(z)\right)$$
$$+\frac{6859\mu q^3}{14400\alpha}\left(\tanh^3(z)+\coth^3(z)\right), \quad (58)$$
$$q=\frac{\omega}{\nu},\ z=\frac{19}{120}q(x-\omega t)$$

Simplifying (58) the solitary wave solution (50) is obtained.

**Family 12.**

$$\grave{o}_0 = 0, \mu = \frac{9900\nu^3}{6859\omega^2}, k = -\frac{361\omega^2}{14400\nu^2}, a_0 = \frac{\omega}{\alpha}, a_1 = -\frac{60(38k\mu + \nu)}{19\alpha}, a_2 = 0,$$
$$a_3 = -\frac{120\mu}{\alpha}, b_1 = -ka_1, b_2 = 0, b_3 = -k^3 a_3 \qquad (59)$$

Since $k < 0$, it follows that:

$$u_{18}(x,t) = \frac{\omega}{\alpha} + \frac{q}{2\alpha}\left(\frac{-19(361)}{7200}\mu q^2 + \nu\right)\left(\tanh(z) + \coth(z)\right)$$
$$+\frac{6859\mu q^3}{14400\alpha}\left(\tanh^3(z) + \coth^3(z)\right) \qquad (60)$$
$$q =, \; z = \frac{19}{120} q(x - \omega t)$$

Simplifying (60) the solitary wave solution (52) is obtained.

The graphical representation of some solitary wave solutions of (3) is illustrated as follows:

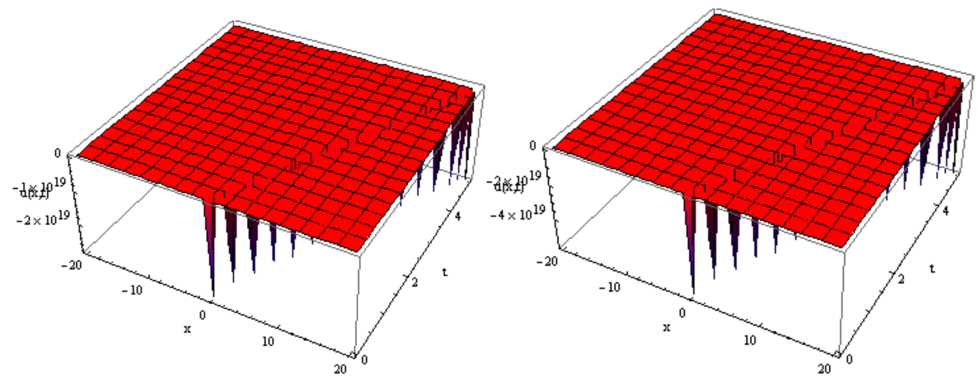


**Figure 2 The plots of solitary wave solutions (32) and (33) when** $\nu = 1, \mu = 1, \omega = 4; (\grave{o}_0 = -10)$ **.**

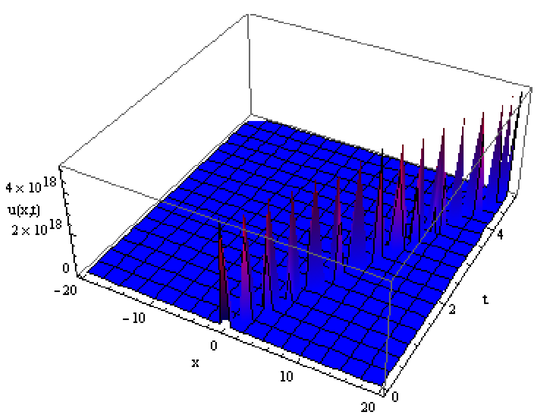


**Figure 3 The plot of solitary wave solutions (52) when** $\nu = 1, \mu = 1, \omega = 4$ **.**

**Remark:** All solutions are tested to satisfy their related PDEs and more generalized compact forms with nonzero constant of integration as mentioned in [15].

## 4. Conclusion

In this presented work, we have established and successfully employed the modified Extended Tanh method with Riccati equation for obtaining the solitary travelling wave solutions for a given class of NLPDEs. The method has an advantage of being direct and concise. In addition Enormous variety of solutions was obtained with the aid of Mathematica software.

## Competing Interests

Authors have declared that no competing interests exist.

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