__Original Research Article__

BAYESIAN ESTIMATION OF THE SCALE PARAMETER OF THE WEIMAL DISTRIBUTION

Abstract

This article aims at estimating the scale parameter of the Weimal distribution using Bayesian method and comparing the estimators obtained to the estimator of the scale parameter obtained from the method of maximum likelihood. Under Bayesian approach, the estimators are obtained by using uniform prior and Jeffrey's prior with the adoption of the precautionary, quadratic and square error loss functions. A derivation and discussion under maximum likelihood estimation is also done. The above methods of estimation employed in this paper are compared based on their mean square errors (MSEs) through a simulation study carried out in R statistical software with different sample sizes. The results indicate that the most appropriate method for the scale parameter is precautionary loss function under either uniform or Jeffrey's prior irrespective of the sample sizes allocated and the values taken by the other parameters.

Keywords: Weimal distribution; Bayesian Methods; Prior distributions; Loss functions; Maximum likelihood Estimation; Mean Square Error; Sample size.

1. Introduction

Ieren and Yahaya[1] developed a new distribution namedWeimal distribution as an extension of the Normal distributionwith two additional parameters for the scale and shape of the new distribution.The maximum likelihood estimates of parameters wereobtained by the method of maximum likelihood in [2]. The fitness of Weimaldistribution was tested by using two lifetime datasets and it was discovered that the new distribution provides a better fit for the skewed datasets when compared to other existing generalizations of the normal distribution including Kumaraswamy-Normal and Beta-Normal as well as the normal distribution.

In statistics, we have two basicmethods ofparameter estimationand these are the classical and the non-classical methods. In the classical theory of estimation, the parameters are taken to be fixed but unknown whereas we consider the parameters to be unknown and random just like variables. The most popular and unique method under classical theory is the method of maximum likelihood estimation while the Bayesian estimation method is considered under non-classical theory. But, in commonreal-lifeproblemsdescribed by life time distributions, the parameters cannot be treated as fixed in all the life testing periodaccording to [3] as well as[4] and [5].Based on this fact, it becomes obvious the frequentist or classical approach can no longer handle adequately problems of parameter estimationin life time models and therefore the need for non-classical or Bayesian estimation in life time models.

In order to achieve the gap above,many researchers have used Bayesian estimation methodfor parametersof different probability distributions and a list of some of these studies is as follows:Bayesian estimation for the extreme value distribution using progressive censored data and asymmetric loss by [6], Bayesian estimators of the shape and scale parameters of modified Weibull distribution using Lindley's approximation under the squared error loss function, LINEX loss function and generalized entropy loss function by [7],comparison of Bayesian estimates of the shape parameter of Generalized Exponential Distribution based on a class of non-informative prior under the assumption of quadratic loss function, squared log-error loss function and general entropy loss function (*GELF*) and maximum likelihood estimates by [8], Bayesian Survival Estimator for Weibull distribution with censored data by [9] as well as [10], [11]. Similarly, [12] studied the shape parameter of generalized Rayleigh distribution under non-informative priors with a comparison to the method of maximum likelihood. Besides, a good number of loss functions have been shown to be performing during estimation under Bayesian method in so many studies including[13], [14], [15], [16], [17], [18]and [19]etc.

Since the approach of estimating a parameter differs from one parameter of a distribution to another, this studyaims at estimating the scale parameter of the Weimal distribution using Bayesian approach and making a comparison between the Bayesian approach and the method of maximum likelihood estimation approach. The rest of this paper organizedin sections as follows: section 1 presents the introduction, Section 2 gives the materials and methods used in the article beginning with the distribution and likelihood function in sub-Section 2.1, estimation under uniform prior in 2.2, estimation under Jeffrey's prior in 2.3 and estimation using method of maximum likelihood in subsection 2.4. In section 3 we present the results and discussions and finally the conclusion in Section 4.

2. Materials and Methods

2.1 PDF and Likelihood function

The *pdf* of the Weimal distributionwith unknown parameter vector $\theta = (\alpha, \beta, \mu, \sigma)^T$ isgiven as:

$$f(x;\theta) = \frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\}, \quad (2.1.1)$$

respectively, where $-\infty < X < \infty$ represent any continuous random variable, $\sigma > 0$ is the dispersion parameter, $-\infty < \mu < \infty$ is the location parameter, $\alpha > 0$ is the scale parameter and $\beta > 0$ isthe shape parameterof the distribution.

The total log-likelihood function for θ is obtained from *f(x)* as follows:

$$L(X_1, X_2, \ldots\ldots, X_n / \alpha, \beta, \mu, \sigma) = \left(\frac{\alpha\beta}{\sigma}\right)^n \sum_{i=1}^{n}\phi\left(\frac{x_i-\mu}{\sigma}\right)\frac{\sum_{i=1}^{n}\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\beta-1}}{\sum_{i=1}^{n}\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\beta+1}}\exp\left\{-\alpha\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta}\right\}. \quad (2.1.2)$$

The likelihood function for the scale parameter, α, is given by;

$$L(\underline{X} \mid \alpha) \propto (\alpha)^n \exp\left\{-\alpha\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta}\right\}.$$

Hence, for simplicity and ease of derivation and computation, we let

$$\omega := \sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta},$$

such that the above likelihood function becomes

$$L(\underline{X} \mid \alpha) \propto (\alpha)^n \exp\{-\alpha\omega\}.\quad(2.1.3)$$

where

$$\omega = \sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta}.$$

2.2 Bayesian Analysis under the Assumption of Uniform Prior Using Three Loss Functions

One crucial aspect when dealing with Bayesian approach is the selection of a prior distribution for the parameter of interest. Most at times priors are chosen according to one's subjective knowledge and beliefs. Another important aspect of it is the choice of a loss function.

To derive the posterior distribution of a parameter given some sample observations, we apply Bayes' Theorem which is stated as follows:

$$p(\alpha \mid \underline{X}) = \frac{p(\alpha) L(\underline{X} \mid \alpha)}{\int_0^{\infty} p(\alpha) L(\underline{X} \mid \alpha)\, d\alpha}, \qquad (2.2.1)$$

where $p(\alpha)$ and $L(\underline{X} \mid \alpha)$ are the prior distribution and the Likelihood function respectively.

The uniform prior is defined as:

$$p(\alpha) \propto 1, \quad 0 < \alpha < \infty.$$

The posterior distribution of the scale parameter α under uniform prior is obtained from equation (2.2.1) using integration by substitution method as

$$p(\alpha \mid \underline{X}) = \frac{\alpha^n e^{-\alpha\omega}}{\omega^{-(n+1)}\Gamma(n+1)}.\quad(2.2.2)$$

The Bayes estimators and posterior risksunder uniform prior using *SELF*, *QLF* and *PLF* are given respectively as follows:

$$\alpha_{SELF} = \frac{n+1}{\omega},\quad(2.2.3)$$

$$P(\alpha_{SELF}) = \frac{(n+2)(n+1)-((n+1))^2}{(\omega)^2},\quad(2.2.4)$$

$$\alpha_{QLF} = \frac{n-1}{\omega}, \quad (2.2.5)$$

$$P(\alpha_{QLF}) = \frac{1}{n}, \quad (2.2.6)$$

$$\alpha_{PLF} = \frac{\left[(n+2)(n+1)\right]^{\frac{1}{2}}}{\omega}, \qquad (2.2.7)$$

$$P(\alpha_{PLF}) = 2\left\{\frac{\left\{(n+2)(n+1)\right\}^{\frac{1}{2}} - (n+1)}{\omega}\right\}. (2.2.8)$$

2.3 Bayesian Analysis under the Assumption of Jeffrey's Prior Using Three Loss Functions

Also, the Jeffrey's prior is defined as:

$$p(\alpha) \propto \frac{1}{\alpha}, \quad 0 < \alpha < \infty. (2.3.1)$$

The posterior distribution of the scale parameter α for a given data under Jeffrey prior is obtained from equation (2.2.1) using integration by substitution method as

$$p(\alpha \mid \underline{X}) = \frac{\alpha^{n-1}\omega^{n}e^{-\alpha\omega}}{\Gamma(n)}. (2.3.2)$$

The Bayes estimators and posterior risks under uniform prior using *SELF*, *QLF* and *PLF* are given respectively as follows:

$$\alpha_{SELF} = \frac{n}{\omega}, \quad (2.3.3)$$

$$P(\alpha_{SELF}) = \frac{n}{\omega^{2}}, \qquad (2.3.4)$$

$$\alpha_{QLF} = \frac{n-2}{\omega}, \quad (2.3.5)$$

$$P\left(\alpha_{QLF}\right) = \frac{1}{n-1}, \quad (2.3.6)$$

$$\alpha_{PLF} = \frac{\left[n(n+1)\right]^{\frac{1}{2}}}{\omega}, \quad (2.3.7)$$

$$P\left(\alpha_{PLF}\right) = 2\left\{\frac{\left[n(n+1)\right]^{\frac{1}{2}} - n}{\omega}\right\}. (2.3.8)$$

2.4 Maximum Likelihood Estimation

This part of the article estimates the scale parameter of the Weimal distribution using the method of maximum likelihood estimation. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Weimal distribution with unknown parameter vector $\theta = (\alpha, \beta, \mu, \sigma)^T$. The overall log-likelihood function for **θ** is obtained from *f(x)* as follows:

$$L(X_1, X_2, \ldots\ldots, X_n / \theta) = \left(\frac{\alpha\beta}{\sigma}\right)^n \sum_{i=1}^{n} \phi\left(\frac{x_i-\mu}{\sigma}\right) \frac{\sum_{i=1}^{n}\left[\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\beta-1}}{\sum_{i=1}^{n}\left[1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\beta+1}} \exp\left\{-\alpha\sum_{i=1}^{n}\left[\frac{\Phi\left(\frac{x_i-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x_i-\mu}{\sigma}\right)}\right]^{\beta}\right\}. (2.4.1)$$

The likelihood function for the scale parameter, α, is given by;

$$L(\underline{X} \mid \alpha) \propto (\alpha)^n \exp\{-\alpha\omega\}. (2.4.2)$$

Let the log-likelihood function, $l = \log L(\underline{X} \mid \alpha)$, therefore

$$l = n\log\alpha - \alpha\omega. (2.4.3)$$

Differentiating l partially with respect to α, the scale parameter and solving for $\hat{\alpha}$ gives;

$$\frac{\partial l(\theta)}{\partial \alpha} = \frac{n}{\alpha} - \omega = 0,$$

$$\hat{\alpha} = \frac{n}{\omega}. \quad (2.4.4)$$

3. Results and Discussions

3.1 Simulation and Comparison

In this section, a package in R software named "newdistr" developed by [20]has been usedto generate random samples of sizes n = (5, 10, 15, 20, 25,30, 35, 55, 75, 100, 150) from Weimal distribution by using different values for the distribution parameters as stated in the headings of the tables below. These tables present the results of our simulation study by providing the Mean Square Errors (*MSEs*) for the estimators of the scale parameter of the Weimal distribution under the some of the concern estimation methods or loss functions such as Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey prior.

Table 3.1: Mean Square Errors (*MSEs*) forestimate of thescale parameter based ondifferent sample sizes for $\alpha = 0.5$, $\beta = 3.5$, $\mu = 1.0$ and $\sigma = 1.0$.

Sample sizes	***MLE***	**Uniform Prior**			**Jeffrey's Prior**		
		SELF	***QLF***	***PLF***	***SELF***	***QLF***	***PLF***
5	0.4504	0.6854	0.2803	0.8216	0.4504	0.1752	0.5544
10	0.1297	0.1501	0.1152	0.1622	0.1297	0.1066	0.1389
15	0.0899	0.0924	0.0890	0.0943	0.0899	0.0897	0.0909
20	0.0819	0.0811	0.0835	0.0809	0.0819	0.0859	0.0814
25	0.0814	0.0796	0.0836	0.0789	0.0814	0.0862	0.0805
30	0.0817	0.0796	0.0840	0.0786	0.0817	0.0866	0.0806
35	0.0835	0.0814	0.0857	0.0805	0.0835	0.0880	0.0824
55	0.0913	0.0897	0.0930	0.0889	0.0913	0.0948	0.0905

75	0.0978	0.0965	0.0991	0.0959	0.0978	0.1004	0.0972
100	0.1037	0.1027	0.1047	0.1022	0.1037	0.1057	0.1032
150	0.1116	0.1109	0.1122	0.1106	0.1116	0.1129	0.1112

From Table3.1, it is observed that *MSEs* of the estimates increases as we increase the sample sizes and we also found that for all the samples the *PLF* has a minimum bias under both priors irrespective of the variation in the samples indicating that the *PLF*under both priors is the best method for the scale parameter of the Weimaldistribution.

Table 3.2: Mean Square Errors (*MSEs*) for estimate of the scale parameter based on different sample sizes for $\alpha = 1.0$, $\beta = 0.5$, $\mu = 1.5$ and $\sigma = 2.5$.

Sample sizes	***MLE***	**Uniform Prior**			**Jeffrey's Prior**		
		SELF	***QLF***	***PLF***	***SELF***	***QLF***	***PLF***
5	0.5882	0.7009	0.5406	0.7782	0.5882	0.5579	0.6339
10	0.4647	0.4436	0.4917	0.4354	0.4647	0.5246	0.4537
15	0.4938	0.4732	0.5159	0.4637	0.4938	0.5398	0.4835
20	0.5206	0.5041	0.5377	0.4963	0.5206	0.5556	0.5124
25	0.5441	0.5308	0.5577	0.5243	0.5441	0.5718	0.5374
30	0.5616	0.5505	0.5730	0.5451	0.5616	0.5845	0.5561
35	0.5746	0.5651	0.5842	0.5605	0.5746	0.5939	0.5699
55	0.6155	0.6098	0.6213	0.6069	0.6155	0.6271	0.6126
75	0.6401	0.6360	0.6442	0.6340	0.6401	0.6483	0.6380
100	0.6596	0.6567	0.6625	0.6552	0.6596	0.6655	0.6581
150	0.6841	0.6823	0.6860	0.6813	0.6841	0.6878	0.6832

In the Table 3.2, it is also clear that *MSEs*for all the estimators gets larger as sample size is increased.The PLF has alsothe minimum *MSEs* independent of the sample size and prior distribution which still indicates that it is a perfect estimator for the scale parameter of the Weimal distribution irrespective of the value of the shape, location and dispersion parameter.

Table 3.3: Mean Square Errors (*MSEs*) for estimate of the scale parameter based on different sample sizes for $\alpha = 1.5$, $\beta = 0.5$, $\mu = 2.5$ and $\sigma = 1.5$.

Sample sizes	*MLE*	Uniform Prior			Jeffrey's Prior		
		SELF	*QLF*	*PLF*	*SELF*	*QLF*	*PLF*
5	1.2261	1.2163	1.3009	1.2347	1.2261	1.4407	1.2133
10	1.2998	1.2372	1.3683	1.2087	1.2998	1.4427	1.2685
15	1.3976	1.3540	1.4429	1.3332	1.3976	1.4898	1.3760
20	1.4592	1.4272	1.4919	1.4116	1.4592	1.5254	1.4433
25	1.5067	1.4819	1.5319	1.470	1.5067	1.5574	1.4944
30	1.5415	1.5214	1.5619	1.5115	1.5415	1.5824	1.5315
35	1.5656	1.5488	1.5826	1.5405	1.5656	1.5998	1.5573
55	1.6397	1.6298	1.6496	1.6250	1.6397	1.6595	1.6348
75	1.6824	1.6756	1.6892	1.6722	1.6824	1.6961	1.6790
100	1.7155	1.7107	1.7204	1.7082	1.7155	1.7253	1.7131
150	1.7566	1.7536	1.7597	1.7521	1.7566	1.7627	1.7551

From Table 3.3,it is obvious that *PLF* (under uniform and Jeffrey priors) method yielded the best estimate for the scale parameter despite the changes in the sample sizes. Besides, the *MSEs* still increase as sample sizes becomes larger and there is no change even with the different parameter values.

Table 3.4: Mean Square Errors (*MSEs*) for estimate of the scale parameter based on different sample sizes for $\alpha = 2.0$, $\beta = 0.5$, $\mu = 0.5$ and $\sigma = 0.5$.

Sample sizes	*MLE*	Uniform Prior			Jeffrey's Prior		
		SELF	*QLF*	*PLF*	*SELF*	*QLF*	*PLF*
5	2.3640	2.2318	2.5612	2.1913	2.3640	2.8234	2.2928
10	2.6348	2.5307	2.7448	2.4819	2.6348	2.8607	2.5832
15	2.8015	2.7348	2.8698	2.7026	2.8015	2.9398	2.7685
20	2.8978	2.8503	2.9461	2.8270	2.8978	2.9952	2.8743
25	2.9693	2.9330	3.0060	2.9152	2.9693	3.0430	2.9513
30	3.0214	2.9923	3.0508	2.9780	3.0214	3.0803	3.0070
35	3.0567	3.0325	3.0811	3.0205	3.0567	3.1057	3.0447
55	3.1639	3.1499	3.1779	3.1430	3.1639	3.1919	3.1569
75	3.2246	3.2150	3.2342	3.2103	3.2246	3.2439	3.2199
100	3.2714	3.2646	3.2783	3.2612	3.2714	3.2851	3.2680
150	3.3292	3.3250	3.3334	3.3229	3.3292	3.3376	3.3271

More so the result fromTable3.4corresponds with the previous results showing that uniform and Jeffrey's priors with *PLF*have the smallest *MSEs* which by comparison produces the best estimates for the scale parameter, and looking at all the results presented in the tables, we can conclude that Bayes estimates under precautionary loss function (*PLF*) using uniform prior and Jeffrey's prior are associated with minimum *MSEs* when compared to those obtained using *MLE*, *SELF*, and *QLF* under both uniform and Jeffrey's priors irrespective of the assumed parametric values and allocated sample sizes of n=5, 10, 15, 20, 25, 30, 55, 75, 100 and 150.

4. Summary and Conclusion

In summary, we obtained Bayesian estimators of the scale parameter of the Weimal distribution under Posterior distributions assuming Uniform and Jeffrey's priors. Bayes estimators and their posterior risks have been derived and presented using three loss functions, namely:Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*) and Precautionary Loss Function (*PLF*). The performance of these estimators is assessed based on theMean Square Errors (*MSEs*) of the estimates. A simulation study is carried out in R statistical software to compare the performance of the estimators from the two methods considered in this paper and it is discovered that the *PLF* (under uniform and Jeffrey priors) produces estimates with minimum *MSEs* consistently irrespective of the parameter values and differences in sample size. Therefore, we conclude that Bayesian Method under both uniform and Jeffrey's priors using precautionary loss function (*PLF*) is better compared to Maximum Likelihood Estimation and should be considered when estimating the scale parameter of the Weimal distribution irrespective of the differences in sample sizes and the parameter values.

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