Method Article

Study on Approximate Solution of Fractional Order Biological Population Model

Abstract

In this paper, we present an algorithm of the homotopy analysis Elzaki transform method (HAETM) which is a combination of Elzaki transform method and homotopy analysis method (HAM) to solve a more general biological population model. The fractional derivatives are described by Caputo sense. The proposed method presents a procedure of constructing the set of base functions and gives the high-order deformation equations in a simple form and provides the solution in the form of a convergent series. Three examples are used to illustrate the preciseness and effectiveness of the proposed method.

Keywords: fractional calculus; Elzaki transform; biological population model; homotopy analysis method

2010 Mathematics Subject Classification: 35C10; 35D99

1 Introduction

Fractional calculus have gained importance and popularity due to its various applications in fluid mechanics, visco-elasticity, biology, electrical network, optics and signal processing and so on [1-10]. Except in a limited number of these problems, we have difficulty to find their exact analytic solutions. An effective and easy method for solving such equations is needed.

Various powerful methods such as Differential Transform method (DTM) [11-13], Adomian Decomposition method (ADM) [14–16], Homotopy Perturbation method (HPM) [17-19], Variational Iteration method (VIM) [20-22] and other methods have been proposed to solve linear and nonlinear problems. Another analytical approach that can be applied to solve many types of nonlinear fractional differential equation is Homotopy Analysis method (HAM) [23-28], put forward by Liao in 1992. Very recently, the Homotopy Analysis method is combined with Elzaki transform [29,30] to produce a highly effective technique called Homotopy Analysis Elzaki Transform method (HAETM) for handling many nonlinear problems.

In this paper, we consider the nonlinear fractional order biological population model in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + f(u) \tag{1.1}$$

with the initial condition

$$u(x, y, 0) = f_0(x, y) \tag{1.2}$$

where u denotes the population density, and $f(u)$ represents the population supply due to birth and deaths. The derivatives in (1.1) are understood in the Caputo sense. In this paper, further we apply the homotopy analysis Elzaki transform method (HAETM) to solve the fractional biological population models.

The advantage of this method is its capability of combing two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. The fact that the HAETM solves nonlinear problems without using Adomian's polynomials and He's polynomials is a clear advantage over the Adomian decomposition method (ADM) and He's perturbation transform method (HPTM) [31,32]. The plan of our paper is as follows: Brief definition of fractional calculus are given in Section 2. Some theorems of Elzaki transform are given in Section 3. The homotopy analysis Elzaki transform method is presented in Section 4. In Section 5, three numerical examples are solved to illustrate the applicability of the considered method. Conclusions are presented in Section 6.

2 Basic Definitions

In this section, we mention the following basic definitions of fractional calculus.

Definition 2.1. *A real function $f(x)$, $x > 0$ is said to be in the space $C_\mu, \mu \in R$, if there exist a real number $p(> \mu)$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, +\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in N \cup \{0\}$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad \alpha > 0, x > 0 \tag{2.1}$$

$$J^0 f(x) = f(x) \tag{2.2}$$

For the Riemann-Liouville fractional integral we have:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \tag{2.3}$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \tag{2.4}$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \tag{2.5}$$

For $f \in C_\mu$, $\mu \geq -1$, α, $\beta \geq 0$ and $\gamma > -1$.

Definition 2.3. *The fractional derivatives of $f(x)$ in the Caputo sense is defined:*

$$D_t^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt \tag{2.6}$$

For $m-1<\alpha\leq m$, $m\in N$, $\mu\geq -1$, *and* $f\in C_\mu^m$.

Definition 2.4. *The Laplace transform of Caputo derivative is:*

$$L(D_t^\alpha f(x))=s^\alpha L[f(x)]-\sum_{k=0}^{m-1}s^{\alpha-k-1}f^{(k)}(0^+),\quad m-1<\alpha\leq m. \tag{2.7}$$

3 Elzaki Transform

Recently, Tarig Elzaki introduced a new integral transform, named Elzaki transform, and further applied it to the solution of ordinary and partial differential equations. The Elzaki transform is defined over the set of functions

$$A=\{f(t):\exists M,k_1,k_2>0,|f(t)|<Me^{\frac{|t|}{k_j}},t\in(-1)^j\times[0,+\infty)\} \tag{3.1}$$

by the following formula

$$T(v)=E[f(t)]=v\int_0^{+\infty}e^{\frac{-t}{v}}f(t)dt,\quad v\in[k_1,k_2]. \tag{3.2}$$

Theorem 3.1. *If* $f(t)=t^\alpha$,

$$E[t^\alpha]=v\int_0^{+\infty}e^{\frac{-t}{v}}t^\alpha dt=v^{\alpha+2}\Gamma(\alpha+1).$$

Theorem 3.2. *Elzaki transform on the Riemann-Liouville fractional integral operator of a function* $f(t)$*:*

$$E[J^\alpha f(t)]=v^{\alpha+1}T(v).$$

Theorem 3.3. *Elzaki transform on the Caputo fractional derivative of* $f(t)$*:*

$$E[D_x^{n\alpha}u(x,t)]=\frac{T(v)}{v^{n\alpha}}-\sum_{k=0}^{n-1}v^{2-n\alpha}u(0,t),\quad n-1<n\alpha\leq n$$

4 HAETM for Generalized Biological Population Model

To illustrate the basic idea of this method, let us consider the generalized biological population model:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2u^2}{\partial x^2}+\frac{\partial^2u^2}{\partial y^2}+ku^a(1-ru^b) \tag{4.1}$$

$t>0$, $x,y\in R$, $0<\alpha\leq 1$, with the initial condition

$$u(x,y,0)=f_0(x,y) \tag{4.2}$$

Now taking the Elzaki transform of both side of (4.1), we get

$$\frac{E[u(x,y,t)]}{v^{\alpha}} - v^{2-\alpha}u(x,y,0) - E[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + ku^a(1-ru^b)] = 0 \tag{4.3}$$

We define the nonlinear operator

$$\begin{aligned} N[\phi(x,y,t;q)] =& E[\phi(x,y,t;q)] - v^2u(x,y,0) - v^{\alpha}E[\frac{\partial^2}{\partial x^2}(\phi^2(x,y,t;q)) + \frac{\partial^2}{\partial y^2}(\phi^2(x,y,t;q)) + \\ & k\phi^a(x,y,t;q)(1-r)\phi^b(x,y,t;q)]. \end{aligned} \tag{4.4}$$

where $q \in [0,1]$ and $\phi(x,y,t;q)$ is a real function of x, y, t, q.

The zero-order deformation equation of the (4.3) has the form

$$(1-q)E[\phi(x,y,t;q) - u_0(x,y,t)] = \hbar q H(t) N[\phi(x,y,t;q)] \tag{4.5}$$

where E is the Elzaki transform, $q \in [0,1]$ is the embedding parameter, $H(t)$ denotes a nonzero auxiliary function, $\hbar \neq 0$ is an auxiliary parameter, $u_0(x,y,t)$ is an initial guess of $u(x,y,t)$ and $\phi(x,y,t;q)$ is an unknown function. Obviously,when the parameter $q = 0$ and $q = 1$, it holds

$$\phi(x,y,t;0) = u_0(x,y,t), \quad \phi(x,y,t;1) = u(x,y,t) \tag{4.6}$$

respectively. Thus, as q increase from 0 to 1, the solution $\phi(x,y,t;q)$ varies from the initial guess $u_0(x,y,t)$ to the solution $u(x,y,t)$. Expanding $\phi(x,y,t;q)$ in Taylor series with respect to q, we have

$$\phi(x,y,t;q) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t)q^m \tag{4.7}$$

where

$$u_m(x,y,t) = \frac{1}{m!}\frac{\partial^m \phi(x,y,t;q)}{\partial q^m}|_{q=0} \tag{4.8}$$

Differentiating the zero-order deformation equation (4.5) m times with respect to q and then dividing by $m!$ and finally setting $q = 0$ we get the following m^{th}-order deformation equation:

$$E[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = \hbar q H(t) R_m(\overrightarrow{u}_{m-1}(x,y,t)). \tag{4.9}$$

Define the vectors

$$\overrightarrow{u}_m = \{u_0(x,y,t), u_1(x,y,t), u_2(x,y,t), \cdots, u_m(x,y,t)\} \tag{4.10}$$

Applying the inverse Elzaiki transform on (4.9), we have

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + \hbar q E^{-1}[H(t)R_m(\overrightarrow{u}_{m-1}(x,y,t))] \tag{4.11}$$

where

$$R_m(\overrightarrow{u}_{m-1}(x,y,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,y,t;q)]}{\partial q^{m-1}}\ |_{q=0} \tag{4.12}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are properly chosen, the series (4.7) converges at $q = 1$, then we have

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t) \tag{4.13}$$

which must be one of the solutions of the original nonlinear equation.

5 Numerical results

In this section, we use homotopy analysis Elzaki transform method to solve nonlinear fractional biological population equation.

Example 5.1. Consider the equation (4.1) with $k = 1, a = 1, r = 0$, we have the following fractional biological population equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u \tag{5.1}$$

with the initial condition

$$u(x, y, 0) = \sqrt{\sin x \cdot \sinh y} \tag{5.2}$$

Applying the Elzaki transform to (5.1) and using (5.2), we have

$$\frac{E[u(x,y,t)]}{v^\alpha} - v^{2-\alpha}\sqrt{\sin x \cdot \sinh y} - E[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u] = 0 \tag{5.3}$$

The nonlinear operator is

$$\begin{aligned} N[\phi(x,y,t;q)] =& E[\phi(x,y,t;q)] - v^2\sqrt{\sin x \cdot \sinh y} - v^\alpha E[\frac{\partial^2}{\partial x^2}(\phi^2(x,y,t;q)) + \\ & \frac{\partial^2}{\partial y^2}(\phi^2(x,y,t;q)) + \phi(x,y,t;q)] \end{aligned} \tag{5.4}$$

and thus

$$\begin{aligned} R_m(\overrightarrow{u}_{m-1}) =& E[u_{m-1}] - (1-\chi_m)v^2\sqrt{\sin x \cdot \sinh y} - \\ & v^\alpha E[\frac{\partial^2}{\partial x^2}(\sum_{r=0}^{m-1} u_r u_{m-1-r}) + \frac{\partial^2}{\partial y^2}(\sum_{r=0}^{m-1} u_r u_{m-1-r}) + u_{m-1}] \end{aligned} \tag{5.5}$$

The m^{th}-order deformation equation is given by

$$E[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = \hbar R_m(\overrightarrow{u}_{m-1}(x,y,t)) \tag{5.6}$$

Applying the inverse Elzaki transform, we have

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + \hbar E^{-1}[R_m(\overrightarrow{u}_{m-1}(x,y,t))] \tag{5.7}$$

Solving the above equation (5.7), for $m = 1, 2, 3, \cdots$, we get

$$\begin{aligned} u_1(x,y,t) =& -\hbar\sqrt{\sin x \cdot \sinh y}\frac{t^\alpha}{\Gamma(\alpha+1)} \\ u_2(x,y,t) =& -\hbar(1+\hbar)\sqrt{\sin x \cdot \sinh y}\frac{t^\alpha}{\Gamma(\alpha+1)} \\ & + \hbar^2\sqrt{\sin x \cdot \sinh y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_3(x,y,t) =& -\hbar(1+\hbar)^2\sqrt{\sin x \cdot \sinh y}\frac{t^\alpha}{\Gamma(\alpha+1)} \\ & + 2\hbar^2(1+\hbar)\sqrt{\sin x \cdot \sinh y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & - \hbar^3\sqrt{\sin x \cdot \sinh y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{aligned} \tag{5.8}$$

and so on.

Substituting $u_1, u_2, u_3, \cdots$ into equation (4.13) gives the solution in series form, by $\hbar = -1$, we have

$$u(x,y,t) = \sqrt{\sin x \cdot \sinh y}[1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots] \tag{5.9}$$

If we put $\alpha = 1$, we have

$$u(x,y,t) = \sqrt{\sin x \cdot \sinh y}[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots] \tag{5.10}$$

$$u(x,y,t) = \sqrt{\sin x \cdot \sinh y} \cdot e^t \tag{5.11}$$

which is an exact solution.

Example 5.2. Consider the equation (4.1) with $a = 1, r = 0$, we have the following fractional biological population equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + ku \tag{5.12}$$

with the initial condition

$$u(x,y,0) = \sqrt{xy} \tag{5.13}$$

Applying the Elzaki transform to (5.12) and using (5.13), we have

$$\frac{E[u(x,y,t)]}{v^{\alpha}} - v^{2-\alpha}\sqrt{xy} - E[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + ku] = 0 \tag{5.14}$$

The nonlinear operator is

$$\begin{aligned} N[\phi(x,y,t;q)] =& E[\phi(x,y,t;q)] - v^2\sqrt{xy} - v^{\alpha}E[\frac{\partial^2}{\partial x^2}(\phi^2(x,y,t;q)) \\ &+ \frac{\partial^2}{\partial y^2}(\phi^2(x,y,t;q)) + k\phi(x,y,t;q)] \end{aligned} \tag{5.15}$$

and thus

$$\begin{aligned} R_m(\overrightarrow{u}_{m-1}) =& E[u_{m-1}] - (1-\chi_m)v^2\sqrt{xy} \\ &- v^{\alpha}E[\frac{\partial^2}{\partial x^2}(\sum_{r=0}^{m-1} u_r u_{m-1-r}) + \frac{\partial^2}{\partial y^2}(\sum_{r=0}^{m-1} u_r u_{m-1-r}) + ku_{m-1}] \end{aligned} \tag{5.16}$$

The m^{th}-order deformation equation is given by

$$E[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = \hbar R_m(\overrightarrow{u}_{m-1}(x,y,t)) \tag{5.17}$$

Applying the inverse Elzaki transform, we have

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + \hbar E^{-1}[R_m(\overrightarrow{u}_{m-1}(x,y,t))] \tag{5.18}$$

Solving the above equation (5.18), for $m = 1, 2, 3, \cdots$, we get

$$\begin{aligned} u_1(x,y,t) =& -k\hbar\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2(x,y,t) =& -k\hbar(1+\hbar)\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &+k^2\hbar^2\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_3(x,y,t) =& -k\hbar(1+\hbar)^2\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &+2k^2\hbar^2(1+\hbar)\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &-k^3\hbar^3\sqrt{xy}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{aligned} \tag{5.19}$$

and so on.

Substituting $u_1, u_2, u_3, \cdots$ into equation (4.13) gives the solution in series form, by $\hbar = -1$, we have

$$u(x,y,t) = \sqrt{xy}[1 + k\frac{t^{\alpha}}{\Gamma(1+\alpha)} + k^2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + k^3\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots] \tag{5.20}$$

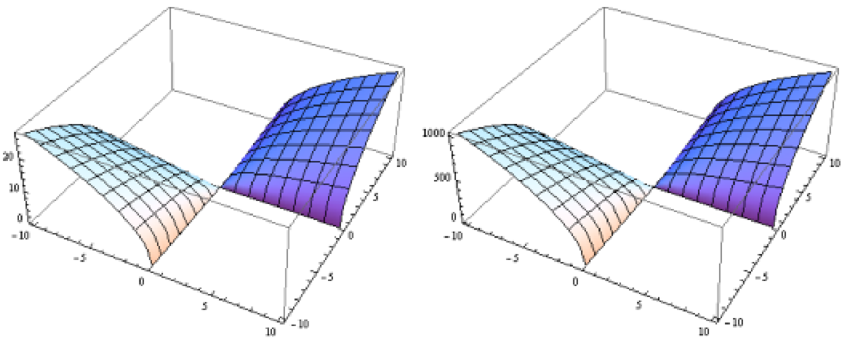
If we put $\alpha = 1$, we have

$$u(x,y,t) = \sqrt{xy}[1 + kt + \frac{k^2t^2}{2!} + \frac{k^3t^3}{3!} + \cdots] \tag{5.21}$$

$$u(x,y,t) = \sqrt{xy}\cdot e^{kt} \tag{5.22}$$

which is an exact solution.

The evolution results for the exact solution (5.22) and the approximate solution (5.20), for the case $\alpha = 1$, are shown in Figure 1. It can be seen from Figure 1 that the solution obtained by the HAETM is nearly identical with the exact solution.



(a) The surface shows the exact solution (5.22) when k=0.1, t=10

(b) The surface shows the numerical solution (5.20) when k=0.1, t=10

Figure 1:Exact and numerical solutions of Example 2

Example 5.3. Consider the equation (4.1) with $a = -1, b = 1$, we have the following fractional biological population equation:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \frac{\partial^2u^2}{\partial x^2} + \frac{\partial^2u^2}{\partial y^2} + ku^{-1} - kr \tag{5.23}$$

with the initial condition

$$u(x,y,0)=\sqrt{\frac{kr}{4}x^2+\frac{kr}{4}y^2+y+5} \tag{5.24}$$

Applying the Elzaki transform to (5.23) and using (5.24), we have

$$\frac{E[u(x,y,t)]}{v^{\alpha}}-v^{2-\alpha}\sqrt{\frac{kr}{4}x^2+\frac{kr}{4}y^2+y+5}-E[\frac{\partial^2u^2}{\partial x^2}+\frac{\partial^2u^2}{\partial y^2}+ku^{-1}-kr]=0 \tag{5.25}$$

The nonlinear operator is

$$\begin{aligned}N[\phi(x,y,t;q)]=&E[\phi(x,y,t;q)]-v^2\sqrt{\frac{kr}{4}x^2+\frac{kr}{4}y^2+y+5}-v^{\alpha}E[\frac{\partial^2}{\partial x^2}(\phi^2(x,y,t;q))\\&+\frac{\partial^2}{\partial y^2}(\phi^2(x,y,t;q))+k\phi^{-1}(x,y,t;q)-kr]\end{aligned} \tag{5.26}$$

and thus

$$\begin{aligned}R_m(\overrightarrow{u}_{m-1})=&E[u_{m-1}]-(1-\chi_m)v^2\sqrt{\frac{kr}{4}x^2+\frac{kr}{4}y^2+y+5}\\&-v^{\alpha}E[\frac{\partial^2}{\partial x^2}(\sum_{r=0}^{m-1}u_ru_{m-1-r})+\frac{\partial^2}{\partial y^2}(\sum_{r=0}^{m-1}u_ru_{m-1-r})+\frac{k}{u_{m-1}}-kr]\end{aligned} \tag{5.27}$$

The m^{th}-order deformation equation is given by

$$E[u_m(x,y,t)-\chi_mu_{m-1}(x,y,t)]=\hbar R_m(\overrightarrow{u}_{m-1}(x,y,t)) \tag{5.28}$$

Applying the inverse Elzaki transform, we have

$$u_m(x,y,t)=\chi_mu_{m-1}(x,y,t)+\hbar E^{-1}[R_m(\overrightarrow{u}_{m-1}(x,y,t))] \tag{5.29}$$

Solving the above equation (5.29), for $m=1,2,3,\cdots$, we get

$$\begin{aligned}u_1(x,y,t)&=\frac{-k\hbar t^{\alpha}}{\Gamma(1+\alpha)\sqrt{\frac{kr}{4}x^2+\frac{kr}{4}y^2+y+5}}\\u_2(x,y,t)&=\frac{-2k^2\hbar^2t^{2\alpha}}{\Gamma(1+2\alpha)(\sqrt{\frac{kr}{4}x^2+\frac{kr}{4}y^2+y+5})^3}\\u_3(x,y,t)&=\frac{-3k^3\hbar^3t^{3\alpha}}{\Gamma(1+3\alpha)(\sqrt{\frac{kr}{4}x^2+\frac{kr}{4}y^2+y+5})^5}\end{aligned} \tag{5.30}$$

and so on.

Substituting $u_1,u_2,u_3,\cdots$ into equation (4.13) gives the solution in series form, by $\hbar=-1$, we have

$$u(x,y,t)=u_0+\frac{kt^{\alpha}}{u_0}[\frac{1}{\Gamma(1+\alpha)}+\frac{2}{\Gamma(1+2\alpha)}\frac{-kt^{\alpha}}{u_0^2}+\frac{3}{\Gamma(1+3\alpha)}\frac{k^2t^{2\alpha}}{u_0^4}+\cdots] \tag{5.31}$$

namely

$$u(x,y,t)=u_0+\frac{kt^{\alpha}}{u_0}\sum_{n=0}^{\infty}\frac{n+1}{\Gamma(1+(n+1)\alpha)}(\frac{-kt^{\alpha}}{u_0^2})^n \tag{5.32}$$

If we put $\alpha=1$, we have

$$u(x,y,t)=u_0+\frac{kt}{u_0}e^{\frac{-kt}{u_0^2}} \tag{5.33}$$

which is an exact solution.

The evolution results for the exact solution (5.33) and the approximate solution (5.32), for the special case $\alpha = 1$, $k = 0.05$, $r = 45$ are shown in Table 1 for $t = 20$, and absolute errors are also calculated. It can be seen from Table 1 that the solution obtained by the HAETM has high accuracy.

Table 1: Comparison of the exact and approximate solutions of (5.23) by HAETM for t=20

(x,y)	Exact solution of (5.23)	Approximate solution by HAETM	Absolute error
(-20,-20)	20.9045	20.9045	5.684×10^{-14}
(-10,10)	10.4638	10.4638	3.004×10^{-11}
(0,0)	2.6022	2.6022	3.866×10^{-5}
(10,10)	11.3795	11.3795	1.394×10^{-11}
(20,20)	21.8403	21.8403	3.553×10^{-14}

6 CONCLUSIONS

In this paper, the homotopy analysis Elzaki transform method has been successfully applied to derive approximate solutions of the fractional order biological population equations subject to some initial conditions. The results obtained by this method agree well with the results obtained by ADM, VIM, HPM. The reliability of HAETM and reduction in computations give this method a wider applicability. Finally, we conclude that the HAETM is very powerful and efficient in finding analytical and numerical solutions for wider classes of linear and nonlinear fractional differential equations. It provides us with a simple way to adjust and control the convergence region of solution series by choosing proper values for auxiliary parameter $\hbar$ and auxiliary function $H(t)$. The corresponding solutions and 3D graphs are obtained according to the recurrence relation using Mathematica.

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