Analytic Travelling Wave Solutions and Numerical Analysis of Fisher's Equation via Explicit-Implicit FDM

Abstract

We study the nonlinear parabolic Fisher's equations for travelling wave solutions. The analyses focus on to describe the analytic solution in the spatial pattern of travelling wave solutions; especially the solutions are characterized in invariant with respect to translation in space. There are two phases in the work; first one is the analysis of our considered equation while the second phase studies the numerical investigations to turn the result in a robust vocation. For numerical treatment, we select the implicit-explicit finite difference method (FDM) using different values of time steps which are matching with exact solution.

1. Introduction

Various types of natural processes which entail mechanisms through reaction-diffusion equations and one of the most important examples of nonlinear partial reaction-diffusion equation is Fisher's equations. This equation has been used for designating several types of physical case like as heat and mass transfer, flame propagation, chemical reactions etc. The necessity of Fisher's equation is in gene technology discussed by travelling wave solutions which has been studied at first in the propagation of a gene within a population. Ronald Fisher presented this fisher's model in [1] and his paper consisted of population dynamics to describe the spatial spread of an advantageous allele and Andrey Nikolaevich Kolmogorov, the Russian mathematician (1903-1987) took a part on this equation also known as Kolmogorov-Petrovsky-Piskunov or KPP or Fisher-KPP equation [2].

Many researchers worked on this topics or equations. An analytic method to construct explicitly exact and approximate solutions for nonlinear evolution equations is suggested by Feng [3, 4]. These solutions included solitary wave solutions, singular traveling wave solutions, and periodical wave solutions. After that, Demina studied the meromorphic solutions (including rational, periodic, elliptic) of autonomous nonlinear ordinary differential equations and gave an algorithm for constructing meromorphic solutions. Next Yuan [5] introduced the complex method for solving nonlinear Fisher's Kolmogorov equation of degree three. Tyson and Brazhnik [6] discussed about travelling wave solution of this types of nonlinear equation in two spatial dimensions. A numerical scheme to solve this equation was developed by Tang and Weber [7]. George Adomian introduced us another powerful technique known as Adomian decomposition method (ADM) [8] which is useful for solving nonlinear problems like fisher's equations. Fisher's equation is one of the simplest semi-linear reaction diffusion equation. It can exhibits traveling wave solutions that switch between equilibrium states. To execute the behavior of neutron population in a nuclear reactor, Canosa [9, 10] used a particular case of equation which is given in [1]. Further, Haar wavelet was utilized by Hariharan et al. [11]. Ablowitz and Zepetella [12] used Laurent series expansion to solve Fisher's equation. Since travelling wave plays an important role in biology too, Murray's [13] authoritative work 'Mathematical Biology' is dedicated to biological waves.

In this paper, we consider Fisher's equation to analyze both analytically and numerically using implicit-explicit finite difference methods. The paper is organized as follows: in Section 2, we consider the general Fisher's equation via logistic type growth function and after that we translated the equation into a dimensionless form in Section 3. In the next Section 4, we explore the travelling wave solutions of a special case of Fisher's equation analytically. The numerical solutions are presented graphically to validate the theoretical results in Section 5 while comparing with the exact solution. The stability analysis of the equilibrium points are studied in Section 6 and describe the error analysis. Finally, Section 7 concludes the summary and discussion of the paper.

2. Fisher's Equation

Let us first consider the reaction dispersal equation of the form

$$\frac{\partial p}{\partial t} = \mathrm{d}\frac{\partial^2 p}{\partial x^2} + g(p) \tag{1.1}$$

Where g is a nonlinear function of p and p is described as a population of organisms, particles of chemicals, insect population, population density, or a colonial bacteria. By considering the logistic type of reaction term, the Fisher's equation now can be written in the form of

$$\frac{\partial p}{\partial t} = \mathrm{d}\frac{\partial^2 p}{\partial x^2} + rp\left(1 - \frac{p}{k}\right) \tag{1.2}$$

Here d is the diffusion coefficient or constant, r is the intrinsic growth rate, k is the carrying capacity, t is time, x is the spatial location and $p = p(t, x)$ is the state variable of the diffusion species at location x and time t while the reaction term is given by the logistic law.

3. Dimensional Analysis

For acquiring the dimensionless form of Fisher's equation, at first we have to consider the dimensionless variables

$$u = \frac{p}{M} \text{ and } l = \frac{t}{N} \tag{1.3}$$

Where $M$and N are scaling parameters for this equation. Applying chain rule, we get

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial l} \tag{1.4}$$

Taking the values of u and l from equation (1.3) and using them in equation (1.4), we obtain from (1.4)

$$\frac{\partial u}{\partial l} = \frac{N}{M}\frac{\partial p}{\partial t} \tag{1.5}$$

Now we can re-write the equation (1.5) such that

$$\frac{\partial p}{\partial t} = \frac{M}{N}\frac{\partial u}{\partial l} \tag{1.6}$$

Again,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{M}\frac{\partial p}{\partial t}\frac{\partial t}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{M}\frac{\partial p}{\partial x}$$

Then we can write

$$\frac{\partial p}{\partial x} = M\frac{\partial u}{\partial x} \tag{1.7}$$

And also

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{M}\frac{\partial p}{\partial x}\right) = \frac{1}{M}\frac{\partial^2 p}{\partial x^2}$$

Hence we can get,

$$\frac{\partial^2 p}{\partial x^2} = M\frac{\partial^2 u}{\partial x^2} \tag{1.8}$$

Using equation (1.3), (1.6) and (1.8) in equation (1.2), we obtain

$$\frac{M}{N}\frac{\partial u}{\partial l} = \mathrm{d}M\frac{\partial^2 u}{\partial x^2} + ruM\left(1 - \frac{uM}{k}\right)$$

$$\Rightarrow \frac{\partial u}{\partial l} = \mathrm{d}N\frac{\partial^2 u}{\partial x^2} + \frac{N}{M}ruM\left(1 - \frac{uM}{k}\right)$$

$$\Rightarrow \frac{\partial u}{\partial l} = \mathrm{d}N\frac{\partial^2 u}{\partial x^2} + Nru\left(1 - \frac{u}{k/M}\right)$$

The relation $\frac{k}{M} = 1,\ Nr = 1$ implies that

$$N = \frac{1}{r} \text{ and } M = k$$

So we can say that N is the reciprocal of the intrinsic growth rate and M is the carrying capacity. After setting $\frac{k}{M} = 1,\ Nr = a$ and $\mathrm{d}N = W$ we can re-consider the Fisher's equation in a new form such that

$$\frac{\partial u}{\partial l} = W\frac{\partial^2 u}{\partial x^2} + au(1-u) \tag{1.9}$$

where a is the reactive factor and W is a diffusion constant.
Let us now suppose that

$$l^* = al \text{ and } x^* = x\left(\frac{a}{W}\right)^{\frac{1}{2}}$$

Now we can write these in this way such that

$$l = \left(\frac{1}{a}\right)l^* \text{and } x = x^*\left(\frac{W}{a}\right)^{\frac{1}{2}}$$

These non-dimensionalized variables gives us

$$\frac{\partial u}{\partial l} = a\frac{\partial u}{\partial l^*}$$

$$\frac{\partial u}{\partial x} = \left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x^*}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x^*}\right] = \left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial}{\partial x^*}\left[\left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x^*}\right] = \frac{a}{W}\frac{\partial^2 u}{\partial x^{*2}}$$

This additionally yields

$$a\frac{\partial u}{\partial l^*} = au(1-u) + W\left(\frac{a}{W}\right)\frac{\partial^2 u}{\partial x^{*2}}$$

$$\Rightarrow \frac{\partial u}{\partial l^*} = u(1-u) + \frac{\partial^2 u}{\partial x^{*2}}$$

For ignoring the superscript star "*" notation and let $l^* = t$ and $x^* = x$, we find the required dimensionless form of Fisher's equation such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$$

which introducing us a mutation occurring in a species distributed in a linear habitat. In this equation,$u = u(t,x)$ is density of population, x is spatial variable and t is representing the time.

4. Solution and Exploration of Fisher's Equation

For searching the solution and exploration of Fisher's equation, we use phase portrait to describe the behavior of the roots and also use implicit-explicit finite difference method for solving it. At first we discuss travelling wave solution about this nonlinear equation. Since dimensionless form of the Fisher's equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$$

Let us consider a particular case of this equation, see, for example in [14]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u(1-u) \tag{1.10}$$

Now we have to search for wave solution of this equation. At first we let a wave transformation which is of the form

$$u(t,x) = G(s), \qquad s = x - ct \tag{1.11}$$

At $s \to \pm\infty$ the function G approaches to the constant values. The function G to be determined should be twice differentiable. Here c is the unknown wave speed which must be determined as a part of the solution of the problem. We have to use ordinary differential equation for finding travelling wave solution [15] of Fisher's equation [14]. We can find a second order ordinary differential equation for G from (1.10) and (1.11) such that

$$-c\frac{dG}{ds} = \frac{d^2 G}{ds^2} + 6G(1-G) \tag{1.12}$$

According to the phase plane analysis, we have to analyze the equation (1.12) which cannot be solved in a closed form. In a standard way we write (1.12) as a simultaneous system of first order equations by defining $H = \frac{dG}{ds}$ and hence we obtain

$$\begin{cases} \dfrac{dG}{ds} = H \\ \dfrac{dH}{ds} = -6G(1-G) - cH \end{cases} \tag{1.13}$$

By solving this system for equilibrium points,

$$\begin{cases} 0 = H \\ 0 = -6G(1-G) - cH \end{cases} \tag{1.14}$$

There are two stationary equilibrium points such that (0,0) and (1,0). The system is then linearized near the stationary points. For obtaining the eigenvalues corresponding to the equilibrium points, we have to use Jacobian matrix as well as characteristic equations. Now Jacobian matrix of the system (1.13) is

$$J(G,H) = \begin{pmatrix} 0 & 1 \\ 12G-6 & -c \end{pmatrix}$$

At (0,0), we obtain

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -6 & -c \end{pmatrix}$$

After that we have to use the characteristic equation for finding the eigenvalues. The characteristic equation is

$$\begin{aligned} & |J - \lambda.I| = 0 \\ & \Rightarrow \left| \begin{pmatrix} 0 & 1 \\ -6 & -c \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \\ & \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -6 & -c-\lambda \end{vmatrix} = 0 \\ & \Rightarrow (-\lambda)(-c-\lambda) + 6 = 0 \\ & \Rightarrow \lambda^2 + \lambda.c + 6 = 0 \end{aligned}$$

So the eigenvalues corresponding to the point (0,0) is

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 24}}{2}$$

Similarly, using characteristic equation, we can obtain the eigenvalues corresponding to the equilibrium point (1,0) such that

$$\lambda_{3,4} = \frac{-c \pm \sqrt{c^2 + 24}}{2}$$

We can mainly observe the behavior of the system from these roots.

- If $c \geq 2\sqrt{6}$ then $\lambda_{1,2}$ are both real and negative. Here (0,0) is a stable node for the linearized system.
- If $c \in (0, 2\sqrt{6})$ then $\lambda_{1,2}$ are complex with negative real part. For this situation (0,0) is a stable focus.
- On the other hand, $\lambda_{3,4}$ are real and opposite sign and in this case (1,0) is a saddle point. There exists finite limits of $G(s)$as $s \to \pm\infty$. At this time equilibrium points are the limit points of solutions.

For $s \to \pm\infty$, we can find the travelling wave solutions of (1.12) which is equivalent to searching for orbits of (1.13). If they join separate equilibrium points, then such orbits are known as heteroclinic orbits. If the orbit returns to the same equilibrium point from which it started known as homoclinic. There are two orbits giving rise, together with the equilibrium point (1,0) to the unstable manifold defined at least in some neighborhood of the saddle point (1,0) such that each orbit $\mu(s) = \big(G(s), H(s)\big)$ satisfies $\mu(s)$ to (1,0) as $s \to -\infty$. At least one of these orbits can be continued till $s \to +\infty$ and reaches then (0,0) in a monotonic way and we can get an exact solution of equation (1.10) using initial and boundary conditions such that

$$G(s) = u(t,x) = \frac{1}{(1 + e^{(x-5t)})^2}$$

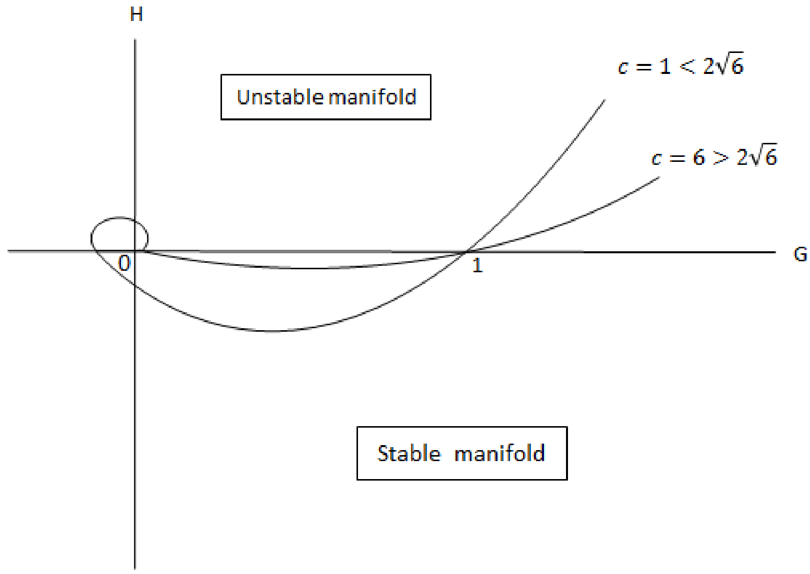

Figure-1: Phase portrait with $c = 1$ and $c = 6$.

For any $c > 0$, ther exists a unique right-going travelling wave with speed c connecting the state $u = 1,\ u_s = 0$ for $x \to -\infty$ to the state $u = 0,\ u_s = 0$ for $x \to +\infty$. Then we will find faster waves.

- For $c \geq 2\sqrt{6}$, the wave monotonically decreasing function of x, while for $c < 2\sqrt{6}$, it is oscillatory.
- That is, the critical points in the G, H plane are (1,0), a saddle point and (0,0), a stable node for $c \geq 2\sqrt{6}$ and a spiral for $c < 2\sqrt{6}$.
- So, the orbit is globally defined for $s = x - ct \in (-\infty, \infty)$ joining equilibrium points (1,0) and (0,0). Hence G is monotonically decreasing and becomes flat at "$\pm\infty$" giving a travelling wave front solution.

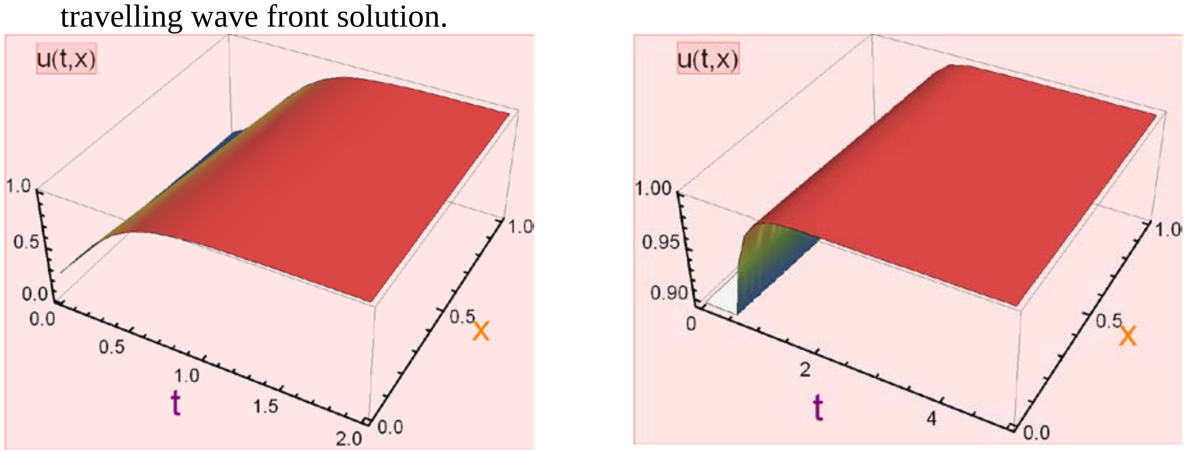

Figure-1(a): Exact solution $G(s) = u(t, x)$ for different times over the domain.

5. Implicit-Explicit Finite Difference Method

We use numerical methods for solving this model and comparing its approximate solution with travelling wave solutions of Fisher's equations. Various types of implicit-explicit finite difference method are important for solving nonlinear partial differential equations.

The main motivation to introduce implicit-explicit finite difference method is to compare the approximate solution with the exact one and polynomial fit data; a new dimension. Now we introduce implicit-explicit method [16] for solving the governing equations as recalled here

$$\frac{\partial p}{\partial t}=\frac{\partial^2 p}{\partial x^2}+rp\left(1-\frac{p}{k}\right) \tag{1.15}$$

Since we have obtained equation (1.10) which is a dimensionless form of the equation (1.15) as defined in the earlier section and hence we can write

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+6u(1-u) \tag{1.16}$$

Where

the domain, $\sigma=(0,1)$

the initial condition, $u(0,x)=\frac{1}{(1+e^{x})^2}$ and

the boundary condition, $u(t,0)=\dfrac{1}{(1+e^{-5t})^2}$

$$u(t,1)=\frac{1}{(1+e^{1-5t})^2}$$

Obtaining the difference method of the equation (1.16), at first, we have to use the Taylor series in t to form the difference quotient

$$\frac{\partial u}{\partial t}(t_j,x_i)=\frac{u(t_j+\Delta t,x_i)-u(t_j,x_i)}{\Delta t}-\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j,x_i) \tag{1.17}$$

for some $\tau_j\in(t_j,t_{j+1})$ and $\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j,x_i)$ is the error term.

Now using central-difference method to form the difference quotient by Taylor series in x, we have

$$\frac{\partial^2 u}{\partial x^2}(t_j,x_i)=\left[\frac{u(t_j,x_i+\Delta x)-2u(t_j,x_i)+u(t_j,x_i-\Delta x)}{(\Delta x)^2}\right]-\frac{(\Delta x)^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j,\gamma_i) \tag{1.18}$$

Where $\gamma_i\in(x_{i-1},x_{i+1})$ and (t_j,x_i) is the interior gridpoint and $\frac{(\Delta x)^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j,\gamma_i)$ is the error.

Suppose that, $\Delta x=h,\ \Delta t=K$. Then (1.17) becomes

$$\frac{\partial u}{\partial t}(t_j,x_i)=\frac{u(t_j+K,x_i)-u(t_j,x_i)}{K}-\frac{K}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j,x_i) \tag{1.19}$$

and (1.18) becomes

$$\frac{\partial^2 u}{\partial x^2}(t_j,x_i)=\left[\frac{u(t_j,x_i+h)-2u(t_j,x_i)+u(t_j,x_i-h)}{h^2}\right]-\frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j,\gamma_i) \tag{1.20}$$

Putting (1.19) and (1.20) in (1.16) and ignoring the local truncation error of order O$(K+h^2)$ consisting of $-\frac{K}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j,x_i)$ and $-\frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j,\gamma_i)$ and next discretizing the equation (1.16) by implicit and explicit scheme, we have

$$\frac{u_i^{j+1}-u_i^j}{K}=\left[\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{h^2}\right]+6u_i^j\left(1-u_i^j\right)$$

which yileds

$$\Rightarrow u_i^{j+1}=R_u\left[u_{i+1}^j-2u_i^j+u_{i-1}^j\right]+6Ku_i^j\left(1-u_i^j\right)+u_i^j \tag{1.21}$$

Where the new parameter is defined as $R_u = \frac{K}{h^2}$. To get the numerical solutions, we need to employ the boundary conditions (1.21). The algorithm is developed in FORTRAN 90/95 languages and the version is Plato. In the rest of the section, the results are presented graphically for further discussion.

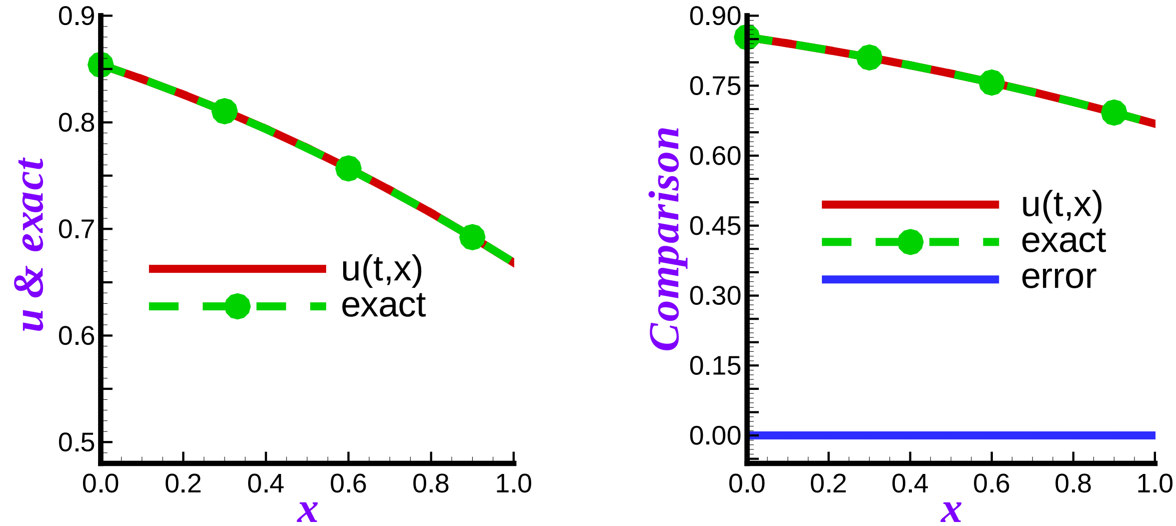

Figure-2: Comparison of $u(t, x)$ and exact solution with error over the domain at time $t = 0.5$.

The density has been normalized at value taken over the domain at different times. From this graphical structure, we able to see that the solution of $u(t, x)$ is decreasing which means $u(t, x)$ lessens over the domain at time $t = 0.5$.

The exact solutions of equation (1.16) using travelling wave scheme which is also represented in Figure-2 (left) and the error term are visible in Figure-2 (right). The graph shows that travelling wave solution is also monotonically decreasing while the mesh time step is $\Delta t = K = 0.001$ at time $t = 0.5$ over the domain. It is seen that the solution obtained by implicit-explicit FDM is visually coincides with the exact solution.

Since we have discussed about the nature of the solutions graphically, one additional numerical solutions are presented in Figure-3 at time $t = 10$ over the domain $\Omega \in (0,1)$.

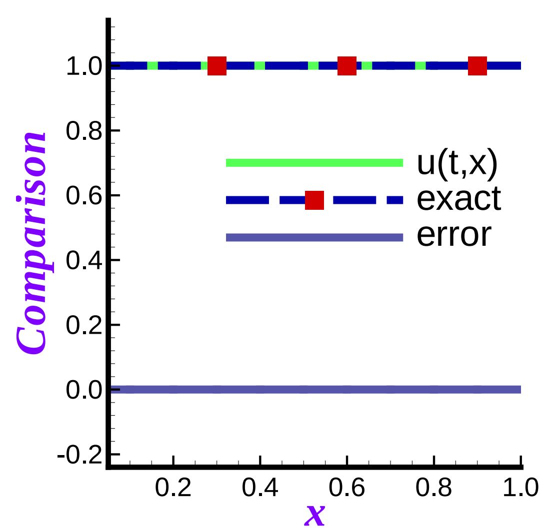

Figure-3: The solution of $u(t, x)$ and exact with error over the domain at time $t = 10.0$

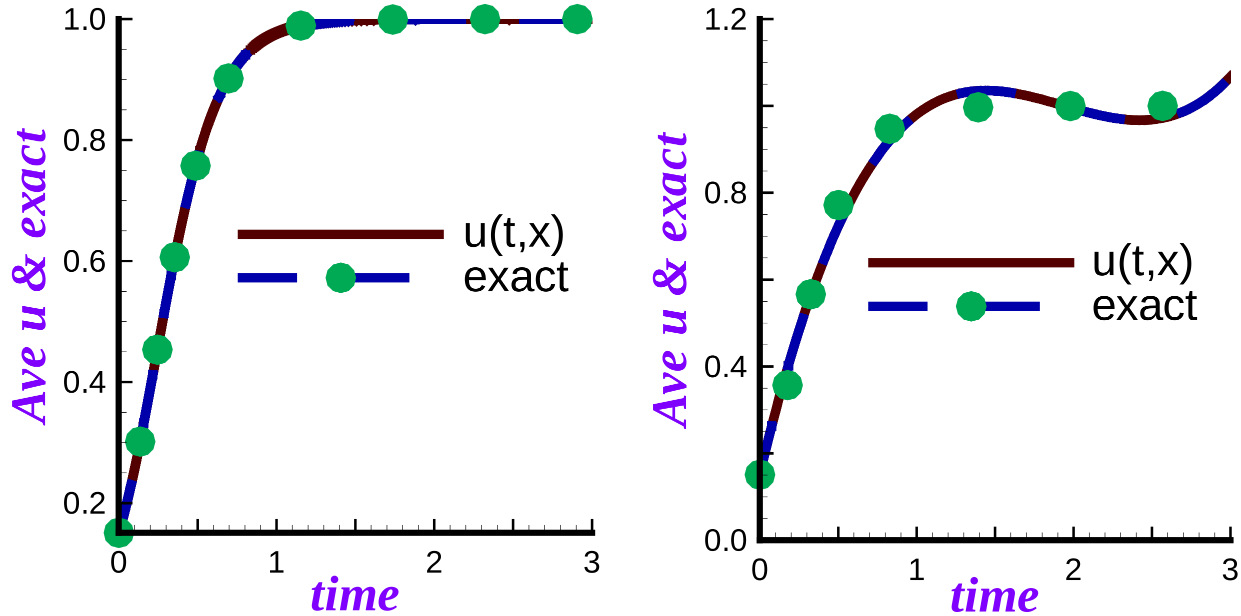

Figure-4: Graphical representation of time vs. average solution of $u(t,x)$ (a) for implicit-explicit FDM and (b) by polynomial-fit and compared with wave solution over the domain at time $t = 3.0$

At this stage, we are interested to discuss about average solution produced by implicit-explicit finite-difference method. The solution depicted in Figure-4 (left) reasonably accurate since there is a better agreement with the exact solution. If we consider the polynomial fit approximations of our available data, the numerical solution is very close to the travelling wave solution over the space at time $t = 3.0$.

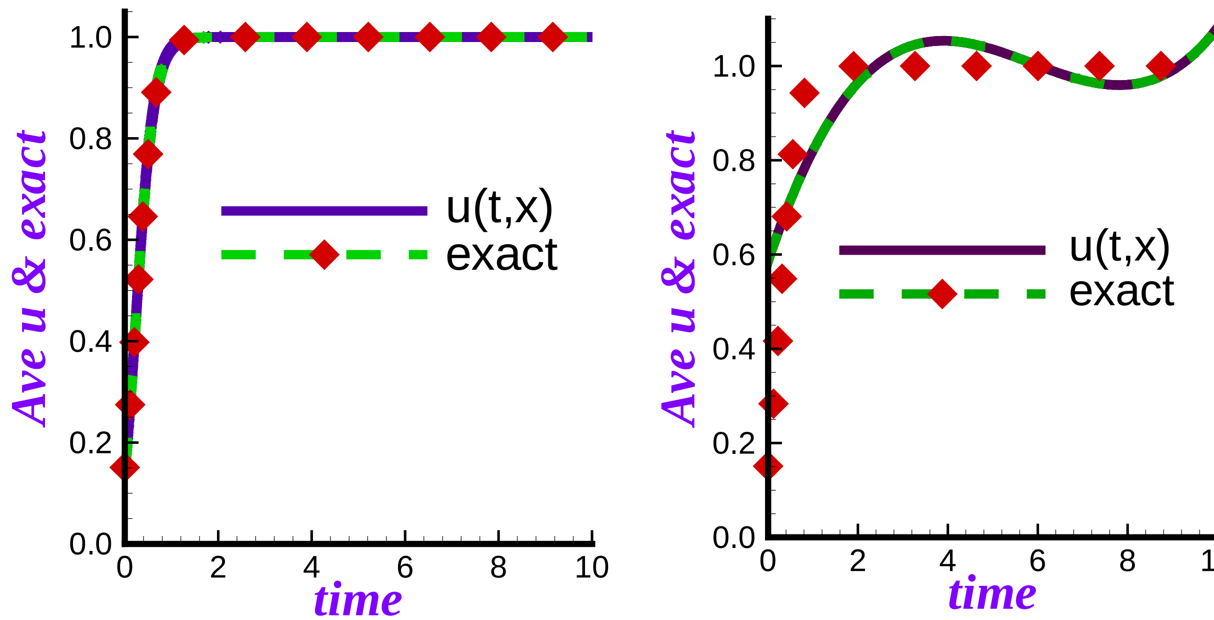

Figure-5: Graphical representation of time vs. average solution of $u(t,x)$ (a) for implicit-explicit FDM and (b) by polynomial-fit and compared with wave solution over the domain at time $t = 10.0$

If t varying, we can illustrate the figures as decorated in Figure-5, where we illustrate average of $u(t,x)$ and compare it with wave or exact solution for time $t = 10.0$. The behavior of the solutions is similar to the solutions in Figure-4. The total illustration of average implicit-explicit solution with wave solution vs time is given below:

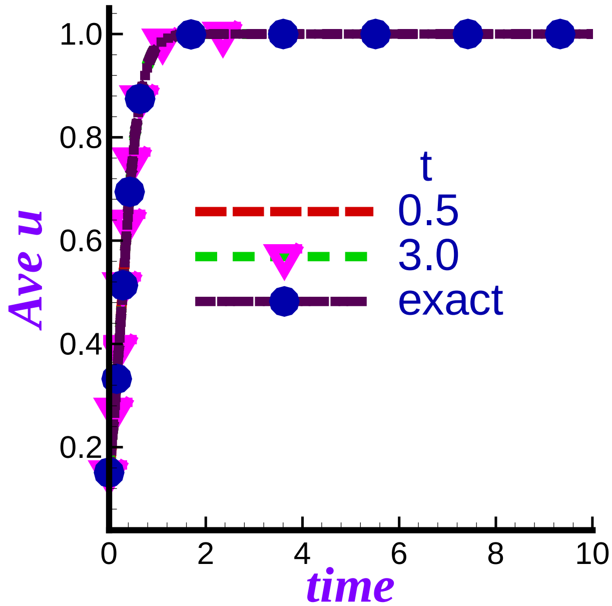

Figure-6: Average solutions of $u(t,x)$ at time $t = 0.5, t = 3.0$ and compared with exact solution for time $t = 10.0$

After using analytic and numerical solutions, we have obtained the Figure-6 at $t = 0.5$, $t = 3.0$ respectively. Time increases from 0.5 to 3.0 and corresponding average of $u(t,x)$ or implicit-explicit finite difference solutions are shown simultaneously in Figure-6. Finally, we introduce multiple plot, see Figure-7, using numerical data for various times in one diagram using polynomial-fit illustration. While the time is increasing the solutions are closer to the exact solutions which is more meaningful.

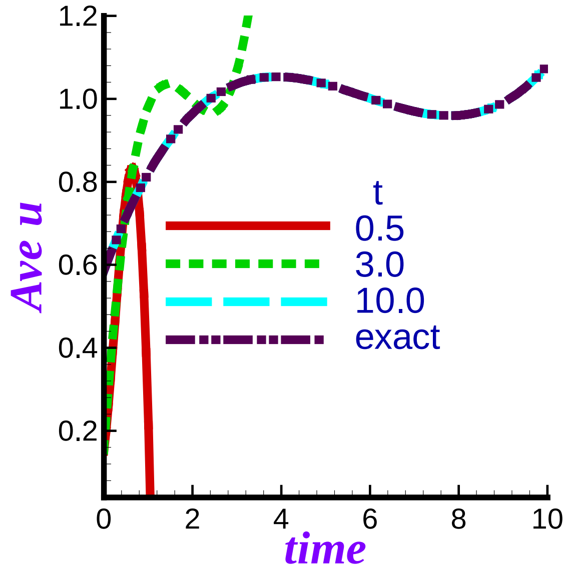

Figure-7: Multiple plot of $u(t,x)$ at different times and the exact solution for $t = 10.0$

6. Stability and Error

The following sections are concerned about the stability and error test. Implicit-explicit method is valid for the condition $0 < R_u \leq \frac{1}{2}$ only. If the solution of the finite difference equations is to be reasonably accurate approximation to the solution of the corresponding nonlinear partial differential equation, then the condition must be satisfied. For this, let us denote E as an exact solution of partial differential equation and the exact solution of finite difference implicit-explicit scheme is denoted by u. Then we consider $e = E - u$, where e is discretization error. Since the simplest implicit-explicit finite difference approximation of equation (1.16) can be written as

$$\frac{u_i^{j+1} - u_i^j}{K} = \left[\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}\right] + 6u_i^j\left(1 - u_i^j\right) \tag{1.22}$$

The simplification of equation (1.22) is given in equation (1.21). Let us consider now $u_i^{j+1} = E_i^{j+1} - e_i^{j+1}$ and $u_i^j = E_i^j - e_i^j$ at the mesh points. Then putting these in equation (1.22), we obtain

$$e_i^{j+1} = E_i^{j+1} - R_u\left[E_{i+1}^j - e_{i+1}^j - 2(E_i^j - e_i^j) + E_{i-1}^j - e_{i-1}^j\right]$$
$$-6K(E_i^j - e_i^j)\left(1 - (E_i^j - e_i^j)\right) - (E_i^j - e_i^j) \tag{1.23}$$

After using Taylor's theorem [17] in equation (1.23), we can see that $\left|E_i^j - u_i^j\right| \leq e_j$ where e_j presents the maximum value of $\left|e_i^j\right|$ which proves that u converges to E when $R_u \leq \frac{1}{2}$ and t is finite.

The implicit-explicit method of finite difference scheme is unstable when $R_u > \frac{1}{2}$ and conditionally stable if $0 < R_u \leq \frac{1}{2}$. Graph of error using approximate and exact solutions is given below:

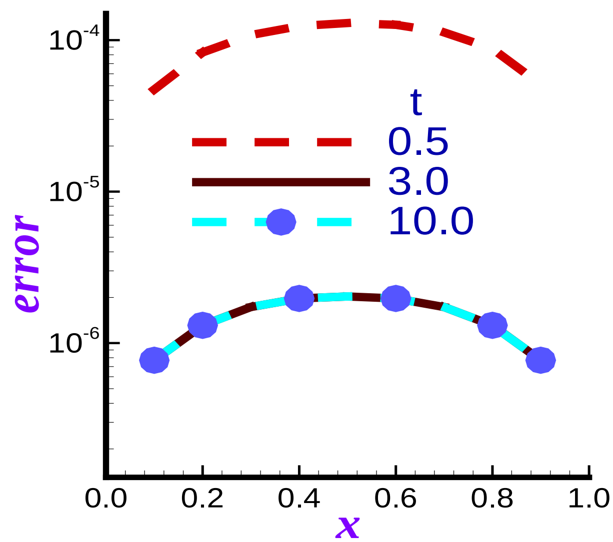

Figure-8: Error over the domain at time $t = 0.5, t = 3.0$ and $t = 10.0$.

We show that the errors consisting of difference between exact and approximate solutions using different patterns over the domain at increasing time.

7. Conclusion

It is observed that the density of population diminishes over the domain at certain time and average solutions are coincided for increasing of times. In this paper, we have find that travelling wave solutions exists for $c \geq 2\sqrt{6}$ in the selected Fisher's equation and wave develops with speed $c = 2\sqrt{6}$ in the governing equation. Nonlinear problems like fisher's equations can be solved by implicit-explicit schemes. We have generally used implicit-explicit method and compared it with travelling wave solutions to justify our solutions. For solving Fisher's equation, this method is a simply powerful technique. The approximate solutions obtained by using this method produce better results as compared with travelling wave solution. However, in neurophysiology, chemical kinetics and population dynamics such types of modeling phenomena like Fisher's equation can be extended for future study.

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