Original Research Article

On Properties and Application of Lomax-Gompertz Distribution

Abstract

This article introduces a new distribution called the Lomax-Gompertz distribution developed through a Lomax Generator proposed in an earlier study. Some statistical properties of the proposed distribution comprising moments, moment generating function, characteristics function, quantile function and the distribution of order statistics were derived. The plots of the probability density function revealed that it is positively skewed. The model parameters have been estimated using the method of maximum likelihood. The plot the of survival function indicates that the Lomax-Gompertz distribution could be used to model time or age-dependent data, where probability of survival is believed to be decreasing with time or age. The performance of the Lomax-Gompertz distribution has been compared to other generalizations of the Gompertz distribution using three real-life datasets used in earlier researches

Keywords:- Gompertz distribution; Lomax generator; moments; Maximum Likelihood Estimation.

1.0 Introduction

The Gompertz distribution (*GD*) as a generalization of exponential distribution can handle both positively and negatively skewed datasets and is commonly used in many applied problems, particularly in lifetime data analysis [1]. The *GD* is applied in the survival analysis, in some sciences such as Gerontology [2]; Computer[3]; Biology (Economos 1982); and Marketing science [4]. The hazard rate function of GDis an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers [5].[6] discussed the stress-strength reliability problem in Gompertz case andbased on the exact central moments, higher accuracy approximations can be defined for them. In demographic or actuarial applications, maximum-likelihood estimation is often used to determine the parameters of the GD. The GD with parameters θ>0 andγ>0has cumulative distribution function (*cdf*) and probability density function (*pdf*) respectively given by:

$$G(x) = 1 - e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)} \tag{1.1}$$

and

$$g(x) = \theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)} \tag{1.2}$$

For $x \ge 0, \theta > 0, \gamma > 0$, where θ and γ are the respective shape and locationparameters.

Recently, [7] proposed a generalization of the GD which was based on an idea of [8] and the resultant distribution is known as generalized Gompertz distribution(GGD)since it combines the features of the Gompertz distributions, exponential distribution(E), and the generalized exponential (GE). Other generalized Gompertz distributions include the Beta Gompertz distribution [9];odd generalized Exponential-Gompertz distribution (*OGEGD*) [10] and the Transmuted Gompertz distribution (*TGD*) [11].

In this article, we introduce a new four parameter Lomax-Gompertz distribution (LGD) with the aid of a Lomax G generator proposed by Cordeiro *et al.*(2014).

2.0 Material and methods

2.1 Introduction of Lomax-Gompertz Distribution

[12] defined the *cdf* and *pdf* of the Lomax-G family of distributions for any continuous distribution as follows:

$$F(x) = \int_{0}^{-\log[1-G(x)]} \alpha\beta^{\alpha} \frac{dt}{(\beta+t)^{\alpha+1}} = 1 - \left\{\frac{\beta}{\beta - \log[1-G(x)]}\right\}^{\alpha} \qquad (2.1.1)$$

and

$$f(x) = \alpha\beta^{\alpha} \frac{g(x)}{[1-G(x)]\{\beta - \log[1-G(x)]\}^{\alpha+1}}, \qquad (2.1.2)$$

Where g(x) and G(x) are the respective *pdf* and *cdf* of any continuous distribution to be generalized,whileα>0 and β>0 are the two additional new parameters responsible for the scale and shape of the distribution respectively. We now define the cdf and pdf of the proposed Lomax Gompertz distribution (LGD) by introducing the *cdf* and corresponding *pdf* of the Gompertz distribution into equationsn 2.1.1 and 2.1.2 as follows;

$$F(x) = 1 - \beta^{\alpha}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)\right\}^{-\alpha} \qquad (2.1.3)$$

$$f(x) = \alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x} - 1\right)\right\}^{-(\alpha+1)} \qquad (2.1.4)$$

The plot of the respective *pdf* and *cdf* of the LGD using some chosen values of the shape and scale parameters are presented below.

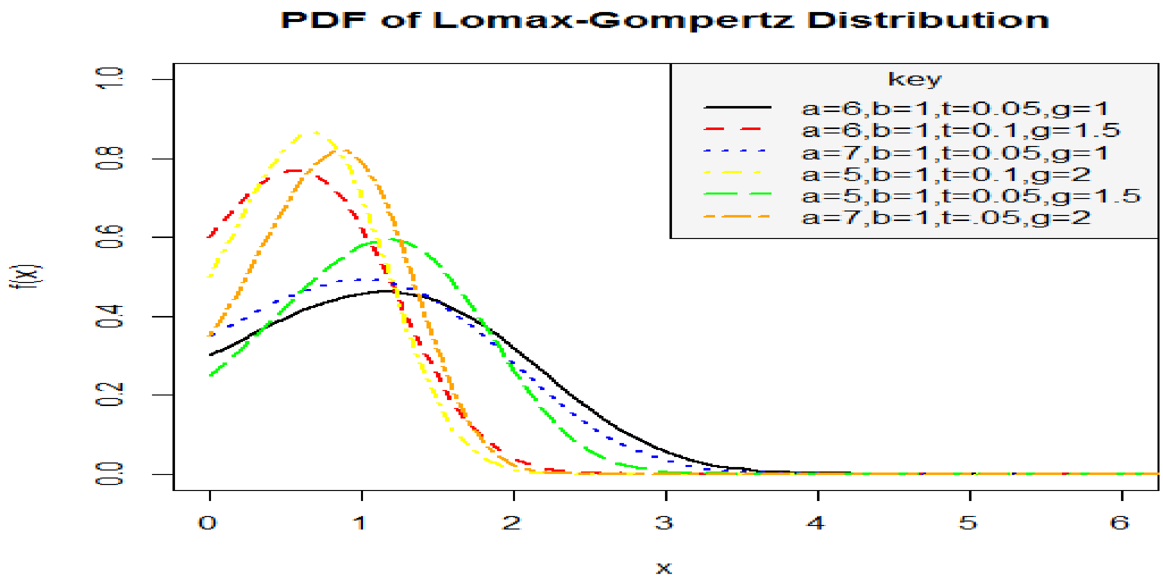

Fig. 2.1.1: *pdf plot* of the *LGD* for different values of $a = \alpha, b = \beta, t = \theta$ *and* $g = \gamma$.

Fig 2.1.1 indicates that the *LGD*is a skewed distribution which is skewed to the right. This means that distribution can be very useful for datasets that are positively skewed.

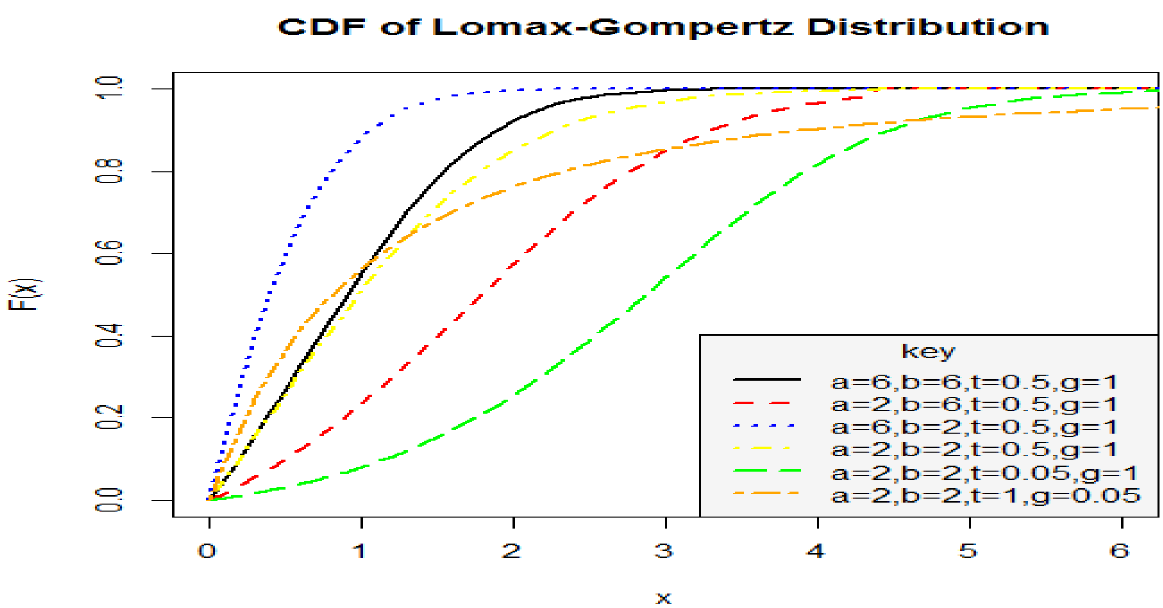

Fig. 2.1.2: *cdf plot* of the *LGD* for different values of $a = \alpha, b = \beta, t = \theta$ *and* $g = \gamma$.

From the above *cdf* plot, the *cdf* increases as*X*increases, and approaches 1 when *X* becomes large as expected.

Some properties of the LGD are presented below.

2.2 Moments

Moments of a random variable are very important in distribution theory, because they are used to study some of the most important features and characteristics of a random variable comprising mean, variance, skewness and kurtosis.

The nth moment of a continuous random variable $X$is given by:

$$\mu_n^{'} = E\left[X^n\right] = \int_0^\infty x^n f(x)dx \tag{2.2.1}$$

Expansion and simplification the pdf of Lomax-Gompertz distribution in equation (2.1.4) yields

$$f(x) = \frac{\alpha\beta^\alpha \theta e^{\gamma x} e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\left(\beta - \log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}}{\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]}$$

$$f(x) = \alpha\beta^\alpha \theta e^{\gamma x} \beta^{-(\alpha+1)}\left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{-(\alpha+1)}$$

$$f(x) = \frac{\alpha\theta e^{\gamma x}}{\beta\left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{\alpha+1}} \tag{2.2.2}$$

Let

$$A = \left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{\alpha+1} \tag{2.2.3}$$

Using the generalized binomial theorem on A yield

$$\left(1-\beta^{-1}\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^{(\alpha+1)} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha+1}{i}\beta^{-i}\left(\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^i \tag{2.2.4}$$

Now, consider the following formula which holds for $i\geq 1$ (http://function.wolfram.com/Elementaryfunctions/log/06/01/04/03/), and then we can write the last term in equation (2.2.4) as

$$\left(\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right)^i = \sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]^l \tag{2.2.5}$$

Where for (for $j\geq 0$) $P_{j,0}=1$ and (for k=1,2,…..)

$$P_{j,k} = k^{-1}\sum_{m=1}^{k}(-1)^m \frac{[m(j+1)-k]}{(m+1)} P_{j,k-m} \qquad (2.2.6)$$

Combining equations (2.2.4) and (2.2.5) and inserting in equation (2.2.2), we have:

$$f(x) = \frac{\alpha\theta e^{\gamma x}}{\beta\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]^l}$$

$$f(x) = \frac{\alpha\theta e^{\gamma x}}{\beta\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\left[e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right]^l}$$

$$f(x) = \frac{\alpha\theta e^{\gamma x} e^{\frac{\theta l}{\gamma}\left(e^{\gamma x}-1\right)}}{\beta\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}}$$

$$f(x) = \frac{\alpha\theta e^{-\frac{\theta l}{\gamma}} e^{\gamma x} e^{\frac{\theta l}{\gamma}e^{\gamma x}}}{\beta\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}} \qquad (2.2.7)$$

Using power series expansion on the last term in the numerator part of equation (2.2.6) yields:

$$e^{\frac{\theta l}{\gamma}e^{\gamma x}} = \sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}e^{r\gamma x} \qquad (2.2.8)$$

Now, substituting equation (2.2.8) into equation (2.2.7) yields:

$$f(x) = \frac{\alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}e^{\gamma x}e^{r\gamma x}}{\beta\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}}$$

$$f(x) = \frac{\alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}e^{\gamma(1+r)x}}{\beta\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}} \qquad (2.2.9)$$

Simplifying

$$f(x) = \alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}\left(\beta\sum_{i=0}^{\infty}(-1)^i \binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\right)^{-1} e^{\gamma(1+r)x}$$

$$f(x) = W_{i,j,k,l,r}\, e^{\gamma(1+r)x} \qquad (2.2.10)$$

Where

$$W_{i,j,k,l,r}=\alpha\theta e^{-\frac{\theta l}{\gamma}}\sum_{r=0}^{\infty}\frac{\theta^r l^r}{\gamma^r r!}\left(\beta\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+1}{i}\beta^{-i}\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{i}{(i-j)}\binom{k-i}{k}\binom{k}{j}\binom{i+k}{l}P_{j,k}\right)^{-1}$$

Hence,

$$\mu_n^{'}=E\left[X^n\right]=\int_0^{\infty}x^n f(x)dx=\int_0^{\infty}W_{i,j,k,l,r}\,x^n e^{\gamma(1+r)x}dx \qquad (2.2.11)$$

Using integration by substitution:

Let $-u=\gamma(1+r)x \Rightarrow x=-\frac{u}{\gamma(1+r)}$

$$-\frac{du}{dx}=\gamma(1+r)$$

$$dx=\frac{-du}{\gamma(1+r)}$$

Substituting for u and dx in equation (2.2.11) and simplifying; we have:

$$\mu_n^{'}=E\left[X^n\right]=\int_0^{\infty}x^n f(x)dx=W_{i,j,k,l,r}\left[\frac{-1}{\gamma(1+r)}\right]^{n+1}\int_0^{\infty}u^{n+1-1}e^{-u}du \qquad (2.2.12)$$

Again recall that $\int_0^{\infty}t^{k-1}e^{-t}dt=\Gamma(k)$ and that $\int_0^{\infty}t^k e^{-t}dt=\int_0^{\infty}t^{k+1-1}e^{-t}dt=\Gamma(k+1)$

Thus we obtain the n^{th} ordinary moment of a Lomax-Gompertz distributed random variable as:

$$\mu_n^{'}=E\left[X^n\right]=W_{i,j,k,l,r}\left[\frac{-1}{\gamma(1+r)}\right]^{n+1}\Gamma(n+1) \qquad (2.2.13)$$

The mean, median, kurtosis and skewness can be obtained from equation 2.2.13.

2.2.1 The Mean

The mean of the *LGD* can be obtained from the n^{th} moment of the distribution when $n=1$ as follows:

$$\mu_1^{'}=E\left[X^1\right]=\frac{W_{i,j,k,l,r}}{\left[\gamma(1+r)\right]^2} \qquad (2.2.14)$$

Also the second moment of the *LGD* is obtained from the n^{th} moment of the distribution when $n=2$ as

$$E\left[X^2\right]=\frac{-2W_{i,j,k,l,r}}{\left[\gamma(1+r)\right]^3} \qquad (21)$$

2.2.2 The Variance

The n^{th} central moment or moment about the mean of X, say μ_n, can be obtained as

$$\mu_n = E\left[X-\mu_1^{'}\right]^n = \sum_{i=0}^{n}(-1)^i \binom{n}{i} \mu_1^{'i}\mu_{n-i}^{'}$$

(2.2.15)

The variance of X for *LGD* is obtained from the central moment when *n=2,* that is,

$$Var(X) = E[X^2] - \{E[X]\}^2 \quad (2.2.16)$$

$$Var(X) = \frac{-2W_{i,j,k,l,r}}{\left[\gamma(1+r)\right]^3} - \left\{\frac{W_{i,j,k,l,r}}{\left[\gamma(1+r)\right]^2}\right\}^2 \quad (2.2.17)$$

2.3 Moment Generating & Characteristics Functions

The moment generating function (*mgf*) is a simple way of arranging all the respective moments in a single function. It produces all the moments of the random variable by way of differentiation i.e., for any real number say k, the k^{th} derivative of $M_X(t)$ evaluated at $t = 0$ is the k^{th} moment $\mu_k^{'}$ of X.

The *mgf* of a random variable X can be obtained by

$$M_x(t) = E\left[e^{tx}\right] = \int_0^{\infty} e^{tx} f(x)dx \quad (2.3.1)$$

Recall that by power series expansion,

$$e^{tx} = \sum_{n=0}^{\infty}\frac{(tx)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}x^n \quad (2.3.2)$$

Using the result in equation (2.3.2) and simplifying the integral in (2.3.1), therefore we have;

$$M_x(t) = E\left[e^{tx}\right] = \sum_{n=0}^{\infty}\frac{(tx)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_n^{'} \quad (2.3.3)$$

where n and t are constants, t is a real number and μ_n' denotes the n^{th} ordinary moment of X and can be obtained from equation (2.2.13) as stated previously.

The characteristics function has many useful and important properties which give it a central role in statistical theory. Its approach is particularly useful for generating moments, characterization of distributions and in analysis of linear combination of independent random variables.

The characteristics function of a random variable X is given by;

$$\varphi_x(t) = E\left[e^{itx}\right] = E\left[\cos(tx) + i\sin(tx)\right] = E\left[\cos(tx)\right] + E\left[i\sin(tx)\right] \quad (2.3.4)$$

Using power series expansion and simplifying the algebra above gives

$$\phi_x(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mu'_{2n}+i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\mu'_{2n+1} \qquad (2.3.5)$$

Where μ'_{2n} and μ'_{2n+1} are the moments of *X* for *n=2n* and *n=2n*+1 respectively and can be obtained from μ'_n in equation (2.2.13)

2.4 Quantile Function

This function is derived by inverting the cdf of any given continuous probability distribution. It is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question. Let $Q(u) = F^{-1}(u)$ be the quantile function (qf) of $F(x)$ for $0 < u < 1$.
Taking *F(x)* to be the *cdf* of the Lomax-Gompertz distribution and inverting it as above will give us the quantile function as follows.

$$F(x)=1-\beta^{\alpha}\left\{\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right\}^{-\alpha}$$

Inverting $F(x) = u$

$$F(x)=1-\beta^{\alpha}\left\{\beta-\log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)}\right)\right]\right\}^{-\alpha}=u \qquad (2.4.1)$$

Simplifying equation (3.29) above, we obtain:

$$Q(u)=X_q=\frac{1}{\gamma}\left\{\log\left[1+\frac{\gamma}{\theta}\left(\frac{\beta}{(1-u)^{\frac{1}{\alpha}}}-\beta\right)\right]\right\} \qquad (2.4.2)$$

The quantile based measures of skewness and kurtosis are employed due to non-existence of the classical measures in some cases. The Bowley's measure of skewness (Kennedy and Keeping, 1962.) based on quartiles is given by;

$$SK=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \qquad (2.4.3)$$

And the Moores' (1998) kurtosis is on octiles and is given by;

$$KT=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{1}{4}\right)} \qquad (2.4.4)$$

2.5 Order Statistics

Order statistics are used in a wide range of problems including robust statistical estimation and detection of outliers, characterization of probability distributions and goodness of fit tests, entropy estimation, analyses of censored samples, reliability analysis, quality control and strength of materials. Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this sample. The *pdf*, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1} \left[1 - F(x)\right]^{n-i} \qquad (2.5.1)$$

where *f(x)* and *F(x)* are the *pdf* and *cdf* of the LGD respectively.

Using equations (2.1.3) and (2.1.4), the *pdf* of the i^{th} order statistic $X_{i:n}$, can be expressed from equation (2.5.1) as;

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \left[\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}\right] * \left[1 - \beta^{\alpha}\left\{\beta + \frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}\right]^{i+k-1}$$

(2.5.2)

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the *LGD* are respectivelygiven by:

$$f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}\right] * \left[1 - \beta^{\alpha}\left\{\beta + \frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}\right]^{k}$$

(2.5.3)

and

$$f_{n:n}(x) = n\left[\alpha\beta^{\alpha}\theta e^{\gamma x}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-(\alpha+1)}\right]\left[1 - \beta^{\alpha}\left\{\beta + \frac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}\right]^{n-1}$$

(2.5.4)

2.6 Reliability Analysis

2.6.1 Survival Function

Survival function is the likelihood that a system or an individual will not fail after a given time. It tells us about the probability of success or survival of a given product or component. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \qquad (2.6.1)$$

Where *F(x)* is *cdf* of the Lomax-Gompertz distribution, we have:

$$S(x) = \beta^{\alpha}\left\{\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-\alpha}$$

(2.6.2)

Below is a plot of the survival function at chosen parameter values in figure **2.6.1**

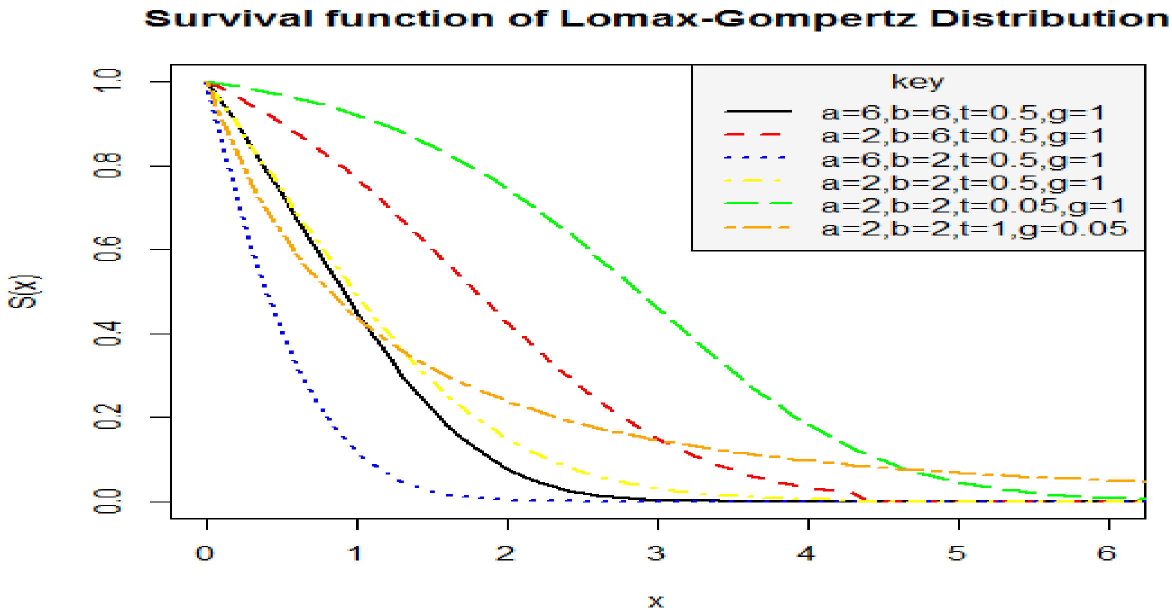

Figure 2.6.1: The survival function of the *LGD* for different values of $a=\alpha, b=\beta, t=\theta and g=\gamma$ as shown on the key in the plot above.

Interpretation: The figure above revealed that the probability of survival for any random variable following a Lomax-Gompertz distribution drops as the values of the random variable increases, that is, as time or age grows, probability of life or survival decreases. This implies that the Lomax-Gompertz distribution can be used to model random variables whose survival rate decreases as their age grows.

2.6.2 Hazard Function

Hazard function as the name implies is also called risk function, it gives us the probability that a component will fail or die for an interval of time. The hazard function is defined mathematically as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{2.6.3}$$

Taking *f(x)* and *F(x)* to be the *pdf* and *cdf* of the proposed Lomax-Gompertz distribution and

Substituting for *f(x)* and *F(x)* in equation (2.6.3) and simplifying gives the following results.

$$h(x)=\alpha\theta e^{\gamma x}\left\{\beta+\tfrac{\theta}{\gamma}\left(e^{\gamma x}-1\right)\right\}^{-1} \tag{2.6.4}$$

The following is a plot of the hazard function at chosen parameter values in figure **2.6.2**

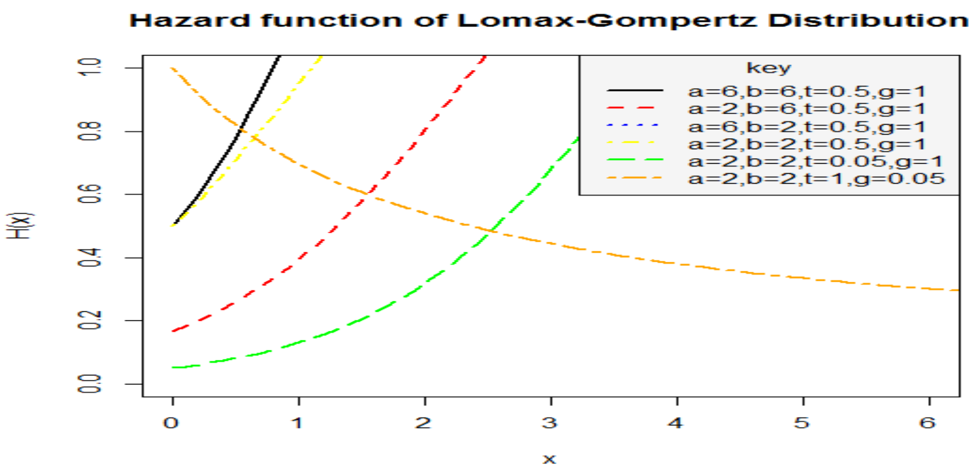

Figure 2.6.2: The hazard function of the *LGD* for different values of $a=\alpha, b=\beta, t=\theta and g=\gamma$ as shown on the key in the plot above.

Interpretation: the figure above revealed that the probability of failure for any random variable following a Lomax-Gompertz distribution increases as the values of the random variable increases, that is, as the values of the variable gets larger or increases, the probability of death increases.

2.7 Estimation of Parameters

Let $X_{1,...,}X_n$ be a sample of size *n* independently and identically distributed random variables from the *LGD* with unknown parameters *α, β, θ and γ* as defined previously.

The likelihood function of the random sample is given by:

$$L(X_1, X_2, \ldots\ldots, X_n / \theta, \gamma, \alpha, \beta) = \left(\alpha\beta^{\alpha}\theta\right)^n e^{\gamma\sum_{i=1}^{n} x_i} \sum_{i=1}^{n}\left(\beta - \log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)\right]\right)^{-(\alpha+1)} \qquad (2.7.1)$$

Taking the natural logarithm of the likelihood function, i.e., Let, $L(n) = \log L(X_1, X_2, \ldots\ldots, X_n / \theta, \gamma, \alpha, \beta)$:

$$l(n) = n\log\alpha + n\alpha\log\beta + n\log\theta + \gamma\sum_{i=1}^{n} x_i - (\alpha+1)\sum_{i=1}^{n}\log\left(\beta - \log\left[1-\left(1-e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)\right]\right) \qquad (2.7.2)$$

Differentiating $l(n)$ partially with respect to *Θ, γ, α* and *β* respectively gives;

$$\frac{\partial l(n)}{\partial\theta} = \frac{n}{\theta} - \frac{(\alpha+1)}{\gamma}\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\left(e^{\gamma x_i}-1\right)}{\left(\beta+\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)\right)\left(e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)}\right\} \qquad (2.7.3)$$

$$\frac{\partial l(n)}{\partial\gamma} = \sum_{i=1}^{n} x_i - \frac{\theta(\alpha+1)}{\gamma^2}\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\left(1-e^{\gamma x_i}\right)}{\left(\beta+\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)\right)\left(e^{-\frac{\theta}{\gamma}\left(e^{\gamma x_i}-1\right)}\right)}\right\} \qquad (2.7.4)$$

$$\frac{\partial l(n)}{\partial \alpha} = \frac{n}{\alpha} + n \log \beta - \sum_{i=1}^{n} \log\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right) \quad (2.7.5)$$

$$\frac{\partial l(n)}{\partial \beta} = \frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\left\{\frac{1}{\left(\beta + \tfrac{\theta}{\gamma}\left(e^{\gamma x_i} - 1\right)\right)}\right\} \quad (2.7.6)$$

Equating equations (2.7.3), (2.7.4), (2.7.5) and (2.7.6) to zero and solving for the solution of the non-linear system of equations will give us the maximum likelihood estimates of parameters $\theta, \gamma, \alpha, and, \beta$ respectively. However, the solution cannot be obtained analytically except with the aid of suitable statistical software like Python, R, SAS, etc. when data sets are given.

3.0 Results and Discussion

The three data sets, their descriptive statistics, graphics and applications are presented here. We have compared the performance of the proposed distribution, Lomax-Gompertz distribution to other generalizations of the Gompertz distribution such as Generalized Gompertz distribution (*GGD*), odd generalized Exponential-Gompertz distribution (*OGEGD*), Transmuted Gompertz distribution (*TGD*) and the Gompertz distribution (*GD*).

The following are the data sets used for analysis and applications in this paper. These are:

Dataset I: This data set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by [13] for fitting the Lindley distribution. This dataset has been used previously by [14] and [15]. It is as follows: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5.

Dataset II: This data set is the strength data of glass of the aircraft window reported by [16]. This data has also been used by [17]. This data is as follows: 18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Dataset III: This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by [18] and has been used by [19] and [20]. It is as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

The following table gives the summary descriptive statistics for the three data sets above.

Table 3.1: Summary Statistics for the three data sets

Parameters	N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values for data set I	100	0.80	4.675	8.10	13.020	9.877	38.500	52.3741	1.4728	5.5403
Values for data set II	31	18.83	25.51	29.90	35.83	30.81	45.38	52.61	0.4054	2.2866
Values for data set III	20	1.10	1.475	1.70	2.05	1.90	4.10	0.4958	1.7198	5.9241

We also provide some histograms and densities for the three datasets as shown in **Figures 3.1, 3.2** and 3.**3** below respectively.

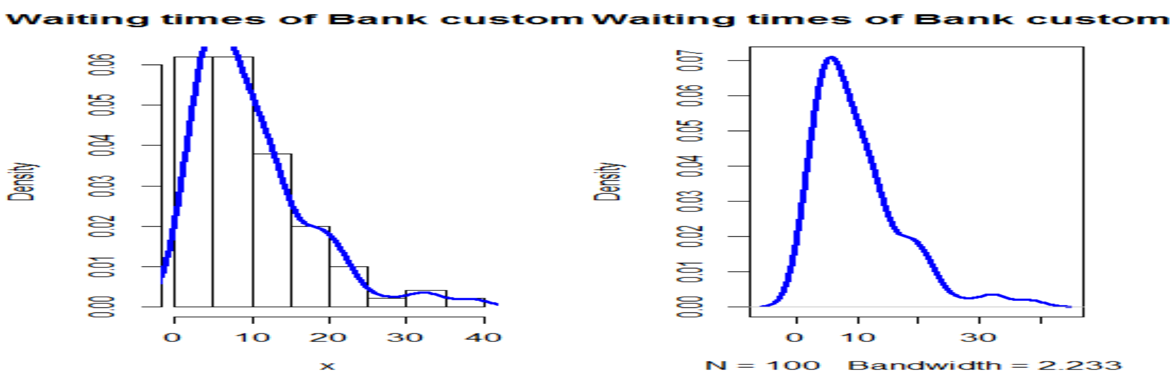

Figure 3.1: A histogram and density plot for waiting times of bank customers (Data set I)

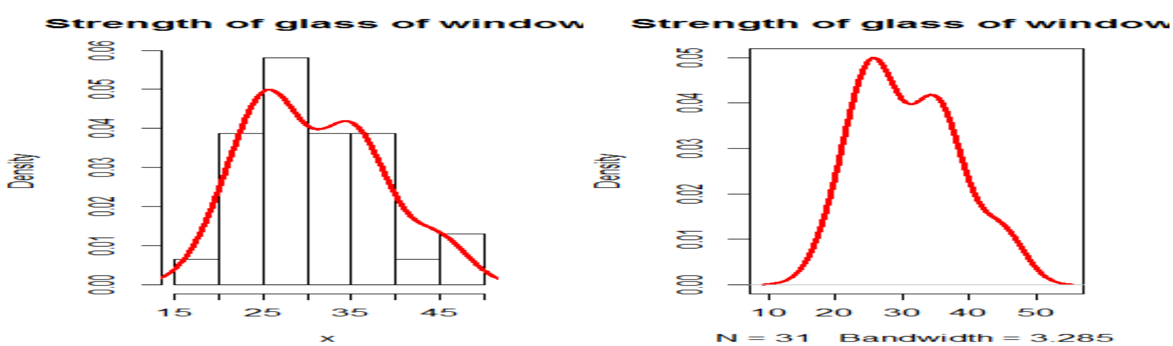

Figure 3.2: A Histogram and density plot for the strength data of glass of aircraft window (Data set II)

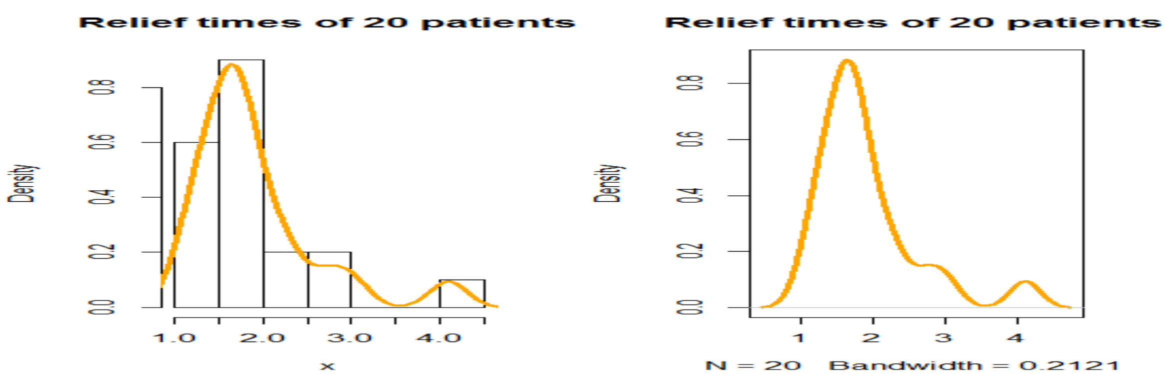

Figure 3.3: A Histogram and density plot for the Relief times of 20 patients (Data set III)

From the descriptive statistics in tables 3.1 and the histograms and densities shown above in figures 3.1, 3.2 and 3.3 for the three data sets respectively, we observed that the three data sets are positively skewed; however, the third data set has a higher skewness coefficient followed by the first and then the second with a very low peak.

For us to fit and assess the performance of the models listed above, we made use of some criteria: the *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion) and *HQIC* (Hannan Quin information criterion). The formulas for these statistics are given as follows:

$$AIC = -2ll + 2k \quad CAIC = -2ll + \tfrac{2kn}{(n-k-1)} \text{ and } HQIC = -2ll + 2k\log\left[\log(n)\right]$$

Where $ll = L$ and it denotes the log-likelihood function evaluated at the *MLEs*, k is the number of model parameters and n is the sample size.

Decision bench mark: The model with the lowest values of these statistics would be chosen as the best model to fit the data.

Table 3.2: Performance evaluation of the Lomax-Gompertz distribution with some generalizations of the Gompertz distribution using the *AIC*, *CAIC and HQIC* values of the models evaluated at the *MLEs* based on data set I.

Distributions	**Parameter estimates**	**-ll=(-log-likelihood value)**	***AIC***	***CAIC***	***HQIC***	**Ranks of models performance**
LGD	$\boldsymbol{\hat{\theta}}$**=0.2593** $\boldsymbol{\hat{\gamma}}$**=0.4411** $\boldsymbol{\hat{\alpha}}$**=2.3755** $\boldsymbol{\hat{\beta}}$**=3.1367**	**347.8476**	**703.6952**	**704.1162**	**707.9126**	**1**
GGD	$\hat{\theta}$=0.2215 $\hat{\gamma}$=0.0932 $\hat{c}$=0.3262	739.5045	1485.0090	1485.2590	1488.1720	**4**

TGD	$\hat{\theta}$=0.1950 $\hat{\gamma}$=0.0217 $\hat{\lambda}$=-0.1190	365.8488	737.6975	737.9475	740.8606	**2**
OGEGD	$\hat{\theta}$=0.0347 $\hat{\gamma}$=0.0063 $\hat{\alpha}$=7.5647 $\hat{\beta}$=1.5793	659.9827	1327.9650	1328.3870	1332.1830	**3**
GD	$\hat{\theta}$=2.0907 $\hat{\gamma}$=0.0433	2894.2880	5792.575	5792.6990	5794.6840	**5**

Using the values of the parameter MLEs and the corresponding values of -*ll*, *AIC*, *CAIC* and *HQIC* for each model as shown in table 3.2, we can understand that the *LGD* performs better with smaller values of the information criteria compared the other models. The above performance can be traced to the fact that the proposed distribution is heavily skewed to the right with a high peak and the first data set is also positively skewed with a large coefficient of kurtosis.

Table 3.3: Performance evaluation of the Lomax-Gompertz distribution with some generalizations of the Gompertz distribution using the *AIC*, *CAIC*, *and HQIC* values of the models based on dataset II.

Distributions	**Parameter estimates**	**-ll=(-log-likelihood value)**	***AIC***	***CAIC***	***HQIC***	**Ranks of models performance**
LGD	$\boldsymbol{\hat{\theta}}$**=0.1808** $\boldsymbol{\hat{\gamma}}$**=0.0108** $\boldsymbol{\hat{\alpha}}$**=7.0269** $\boldsymbol{\hat{\beta}}$**=8.2813**	**193.1088**	**394.2177**	**395.7562**	**396.0875**	**1**
GGD	$\hat{\theta}$=0.2824 $\hat{\gamma}$=0.0019 $\hat{c}$=3.0485	281.3734	568.7469	569.6358	570.1492	**2**
TGD	$\hat{\theta}$=0.5276 $\hat{\gamma}$=0.0122 $\hat{\lambda}$=0.7111	665.7328	1337.4060	1338.3540	1338.8680	**4**
OGEGD	$\hat{\theta}$=0.0545 $\hat{\gamma}$=0.0373 $\hat{\alpha}$=2.0383 $\hat{\beta}$=0.2229	443.9031	895.8062	897.3447	897.6760	**3**
GD	$\hat{\theta}$=2.0907 $\hat{\gamma}$=0.0433	780.4185	1564.837	1564.961	1566.946	**5**

Table 3.3 also shows the parameter estimates to each of the five fitted distributions for the second data set (data set II), the table also provide the values of -*ll*, *AIC*, *CAIC* and *HQIC* of the fitted models evaluated at their corresponding *MLEs*. The values in **Table 3.3** indicate that the *LGD* has better performance with the lowest values of *AIC*, *CAIC* and *HQIC* followed by the *GGD*, *TGD, OGEGD* and *GD*. Again the reason behind this outperformance is that, the second data set has a low degree of kurtosis and skewness to the right meanwhile,

our proposed model has various shapes with both moderate and higher peak all skewed to the right.

Table 3.4: Performance evaluation of the Lomax-Gompertz distribution with some generalizations of the Gompertz distribution using the *AIC*, *CAIC and HQIC* values of the models based on data set III.

Distributions	Parameter estimates	-ll=(-log-likelihood value)	*AIC*	*CAIC*	*HQIC*	Ranks of models performance
LGD	$\hat{\theta}$=**0.2646** $\hat{\gamma}$=**1.0598** $\hat{\alpha}$=**2.9677** $\hat{\beta}$=**8.5964**	**25.1072**	**58.2143**	**60.8809**	**58.9918**	**4**
GGD	$\hat{\theta}$=0.9839 $\hat{\gamma}$=0.3899 $\hat{c}$=7.1231	19.2364	44.4729	45.9729	45.0559	**1**
TGD	$\hat{\theta}$=0.1472 $\hat{\gamma}$=0.8821 $\hat{\lambda}$=0.1998	24.6575	55.3151	56.8151	55.8982	**2**
OGEGD	$\hat{\theta}$=0.1094 $\hat{\gamma}$=0.3918 $\hat{\alpha}$=2.9711 $\hat{\beta}$=4.4035	186.5786	381.1572	383.8238	381.9347	**5**
GD	$\hat{\theta}$=0.2765 $\hat{\gamma}$=0.5845	25.8436	55.6873	56.3932	56.0760	**3**

Table 3.4 also presents the parameter estimates and the values of -*ll*, *AIC*, *CAIC* and *HQIC* for the five fitted models for the third data set. However, the values in the above table show that the *GGD* has better performance with the lowest values of *AIC*, *CAIC* and *HQIC* compared to the other four models including the proposed distribution. The proposed model performed poorly, closely following the baseline distribution. This poor performance could be attributed to the smaller sample size.

4.0 Summary and Conclusions

This article introduced a new distribution called Lomax-Gompertz distribution. It studied some mathematical and statistical properties of the proposed distribution with some graphical demonstration appropriately. The derivations of some expressions for its moments, moment generating function, characteristics function, survival function, hazard function, quantile function and ordered statistics has been done effectively. The pdf plot of the distribution revealed that it is positively skewed and its degree of kurtosis depends on the values of the parameters. The model parameters have been estimated using the method of maximum likelihood estimation. The implications of the plots for the survival function indicate that the Lomax-Gompertz distribution could be used to model time-dependent events or variables whose survival decreases as time grows or where survival rate decreases with time. The results of the three applications showed that the proposed distribution performs better than some extensions of the Gompertz distribution however, depending on the nature of the data sets. It was revealed that this new distribution has better performance for positively skewed data sets with larger sample sizes.

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