Two algorithms to determine the number pi (π)

ABSTRACT

Archimedes used the perimeter of inscribed and circumscribed regular polygons to obtain upper and lower bounds of π. Starting with two regular hexagons he doubled their side from 6 to 12, 24, 48 and 96. Using the perimeters of 96 side regular polygons, Archimedes showed that 3+10/71<π<3+1/7. His method can be realized as recurrence formula called the Borchardt-Pfaff algorithm. Heinrich Dörrie modified this algorithm to produce better approximations to π than these based on Archimedes' scheme. Lower bounds generated by this modified algorithm are the same as from the method discovered earlier by the cardinal Nicolaus Cusanus (XV century), and again re-discovered by Willebrord Snell (XVII century). Knowledge of Taylor series of the functions used in these methods allows to develop new algorithms.

Keywords: quadrature, circle, pi number, Archimedes, approximation, algorithm

1. INTRODUCTION

The first known rigours mathematical calculation of π was done by Archimedes. Archimedes' book "On the Measurements of a Circle", [1], written in the 3rd century B.C., contains three propositions. Proposition 3. represents the numerical computing of the number π. He used an algorithmic scheme based on doubling the number of sides in inscribed and circumscribed regular polygons. He started with the regular hexagons ($N = 6$) and doubled the number of their sides until $N = 96$. Archimedes obtained a series of two approximations, lower and upper, for length of the circumference of the circle with diameter equals to one ($d = 1$), thus consequently to the number π. Archimedes was able to determine the following bounds for π: $3 + 10/71 < \pi < 3 + 1/7$.

It's often suggested to combine these values by taking their arithmetic average. It's correct but it's possible to realize better combination (see Figure 1) than an arithmetical mean of these two limits. Archimedes' estimations can be improved using only information already generated by the constructed polygons. Here two such improvements are proposed and presented. New crated algorithms produce faster convergence to π than original techniques. Such approach already was realized for other methods. Figure 1 shows the results for the regular polygons ($N = 3, 4, \ldots, 12$) and their combinations proposed in XVII century [2]. Archimedes' approach is true real algorithm to obtain the value of π. The method is capable to generate an arbitrarily precision of the number π. The process is relatively slow in its convergence. It is also difficult to use in direct calculations for large number of sides. It is a similar situation as with Turing's machine and a modern computer. Theoretically all computable problems can be realized on both types of machines. It's only matter of time. There were many attempts to improve Archimedes' method. One such approach resulted in Pfaff-Borchardt-Schwab's method developed in XIX century.

The method is defined by the following formulas: $a' = 2ab/(a + b)$, $b' = \sqrt{a'b}$, new values a', b' are determined by old values a, b - the values from previous step. It's an iterative process and it's easy to realize on a computer. Starting with $a = 2\sqrt{3}$ and $b = 3$; the values for circumscribed and inscribed 6-gons, we can generate the sequence of the intervals [b, a], $b < a$. The intervals contain π. It's π for the circle of the diameter one ($d = 1$), or for a unit circle ($r = 1$), and in this case it's half of its perimeter, which, of course, it's also π.

$x = 180^{o}/N \quad c = a + (b-a)/3$

N	a=N*sin(x)	b=N*tan(x)	c
3	2.598076211	5.19615242	3.46410161
4	2.828427124	4.00000000	3.21895141
5	2.938926263	3.63271264	3.17018839
6	3.000000002	3.46410162	3.15470054
7	3.037186177	3.37102234	3.14846490
8	3.061467461	3.31370850	3.14554781
9	3.078181287	3.27573210	3.14403156
10	3.090169940	3.24919696	3.14317895
11	3.099058124	3.22989142	3.14266922
12	3.105828544	3.21539031	3.14234913

Figure 1: π estimations based on: inscribed polygons (a), circumscribed polygons (b), and their combination (c).

Ludolph van Ceulen (1540-1610) was the last person who performed great Archimedean calculation. He used 2^{62}-gons and obtained 39 places with 35 correct digits. The number is still called Ludolph's number in some parts of Europe. For example, in Poland it is called in Polish "liczba ludolfina". Archimedes' method may be interpreted as a rectification problem. Its goal is to find the length of the arc of the circle. In this case the method estimates the circumference of the circle (i.e. full arc for the angle 2π). Very simple and beautiful rectification method was developed by the Polish mathematician Adam Adamandy Kochański [3]. His construction results with π estimation equals to 3.141533. Kochański's geometrical construction can be done with only one opening of a compass. In this case the process is not iterative. It is no iterative process but one-time construction.

2. MATERIAL AND METHODS

We consider here two basic methods, Snells' rectification method and Dörrie's method [2, 4]. Both methods were developed to accelerate Archimedes' process. Here, we are doing the next step further. Our two approaches use the values generated by Snell's and Dörrie's method to construct better approximation for the number π. We listed all used methods in this work in Table 1. In our notation we added X (after M) to indicate that the method (M) is the result of combinations. We assumed that combination occurred, when the composite method is defined by elements already calculated in its components, [5-7]. Consider three of the following methods: MX4: Snell-P based on perimeter of the circle, MX5: Snell-A based on area of the circle, (Huygens, 1654)) and MX6: Ch-H based on the methods M1, M2 and M3, [5]. Table 1 represents the applied methods, their descriptions, and the results for using n=3 and 6. (π=3.14159265358979...). The method M8 was invented by Cusanus (XV), Snell-Huygens (XVII), and again by Dörrie (XX century).

Table 1. Methods, their descriptions and the results for π using n=3 and 6.

Method: pi~n*M{X}	Description	n=3	n=6
M1	sin(x)	2.598076	3.000000
M2	tan(x)	5.196152	3.464101
M3	sin(2x)/2	1.299038	2.598076
MX4=M1+(M2-M1)/3	Snell-P	3.464101	3.154700
MX5=M2+(M3-M2)/3	Snell-A	3.897114	3.175426
MX6=(32M1+4M2-6M3)/30	Ch-H	3.204293	3.142264
M7=(2 cos (x/3)+1) tan (x/3)	Snell-ArcU	3.144031	3.141740
M8=3M1/(2+cos (x))	Snell-ArcL	3.117691	3.140237
MX9=(M2*M1*M1)^1/3	A- Dörrie	3.273370	3.147345
MX10=M8+(MX9-M8)/5	Szysz-Dörrie	3.148827	3.141658
MX11=M7+(M8-M7)/10	Szyszkowicz	3.141397	3.141589

Note: M8 was invented by Cusanus, Snell-Huygens, and Dörrie (B).

2.1 Snell's rectification

Cardinal Nicolaus Cusanus (1401-1464) has elaborated the following rectification of the arc in the circle for the corresponding angle x: arc = 3sin(x)/(2 + cos(x)). This formula was once more again proposed two hundred years later by the Dutch mathematician and physicist Snell (Willebrord Snellius (1580-1626)). We don't know it was an original invention or using the known result obtained by the cardinal. Snell developed two approximations for the length of the arc, lower (M8: Snell-ArcL) and upper (M7: Snell-ArcU), Huygens (1654). We combine these two methods to define better approximation (MX11; Szyszkowicz, 2015, [6]). To develop such approach, we used Taylor series for the corresponding methods (Figure 2 and 3), in this case M7 and M8, and generated the new method as MX11=u*M7+v*M8. The coefficients u and v are determined by the following system of the equations (see Figure 3): u + v = 1, u/1620 − v/180 = 0. The solution allows us to define better method of the form MX11=M7+(M8-M7)/10. Figure 3 shows that in its Taylor series the next term after x is x to power 7. We keep the element x but eliminate x to power 5. Here x = π/n and as n is growing n*MX11 goes to π.

M1 sin(x): $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800} + O(x^{12})$

M2 tan(x): $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + O(x^{12})$

M3 sin(2x)/2: $x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \frac{2x^9}{2835} - \frac{4x^{11}}{155925} + O(x^{12})$

.Figure 2. Taylor series of the methods related to Archimedes' algorithm.

The Ch-H (MX6, [5]) method can be developed differently than originally presented by its authors. The method can be determined as the results of a linear combination $MX6 = a * M1 + b * M2 + c * M3$. Using their Taylor representation, it's possible to keep the term with x (a+b+c=1) and to eliminate the terms with x in power 3 and 5. The new formula will have the term with x in power 7. With new set of the parameters a and b, the method is also defined as $MX6 = a*MX2+b*MX4$, with the conditions on the parameters a+b=1 and a/20+2b/15=0.

Below is presented the program in R. It realizes some of the presented methods. The results are listed for N=64.

```
#Program realizes the following methods:M1, M2, M8, MX9, and MX10
options(digits=15)
N=4; b=2*sqrt(2); a=4 #square
N=6; b=3; a=2*sqrt(3) #hexagon
for (k in 1:5){
cn=c(k-1,N); print(cn)
arch = c(b,a) #Archimedes' results
# Dörrie:
B=3*a*b/(2*a + b)
A=(a*b*b)^(1/3)
dor = c(B,A) # Dörrie's results
#Szyszkowicz
S=B+(A-B)/5 # Szyszkowicz's method
res=c(arch,dor,S)
print(res)
#Next Archimedes:
a=2*a*b/(a+b)
b=sqrt(a*b)
N=N+N}
method=c("M1","M2","M8","MX9","MX10")
print(method)
print(pi); #The end
#The results for 96-gon
M1: 3.14103195089051; M2: 3.14271459964537; M8: 3.14159263357057
MX9: 3.14159273368372; MX10: 3.14159265359320; pi: 3.14159265358979
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$$\text{M7: } x+\frac{x^5}{1620}+\frac{x^7}{40824}+\frac{7x^9}{6298560}+\frac{3931x^{11}}{78568237440}+O(x^{12})$$

$$\text{M8: } B=\frac{3ab}{2a+b}=\frac{3\tan(x)\sin(x)}{2\tan(x)+\sin(x)}=\frac{3\sin(x)}{2+\cos(x)}$$

$$\text{M8: } x-\frac{x^5}{180}-\frac{x^7}{1512}-\frac{x^9}{25920}+\frac{x^{11}}{3991680}+O(x^{13})$$

$$\text{MX9: } A=\sqrt[3]{ab^2}=\sqrt[3]{\tan(x)\sin^2(x)}$$

$$\text{MX9: } x+\frac{x^5}{45}+\frac{4x^7}{567}+\frac{x^9}{405}+\frac{248x^{11}}{280665}+\frac{4808x^{13}}{14926275}+\frac{1504x^{15}}{12629925}+\frac{39129572x^{17}}{879232228875}+O(x^{18})$$

$$\text{MX10: } x+\frac{x^7}{1134}+\frac{x^9}{2160}+\frac{227x^{11}}{1283040}+O(x^{13})$$

$$\text{MX11: } x-\frac{x^7}{22680}-\frac{x^9}{349920}+\frac{437x^{11}}{6235574400}+O(x^{13})$$

Figure 3. Taylor series of the presented methods.

2.2 Dörrie's sequence

In his book ("100 Great Problems of Elementary Mathematics") a German mathematician Heinrich Dörrie in the problem No. 38 presented another approach to improve Archimedes method, [4]. He constructed two new series B and A, ([B, A] interval) which give better approximation for the length of the circumference (C) of the circle. For a given values b, a (the [b, a] interval) are generated B=3ab/(2a + b) and A = √3 ab2. He proved that the following inequalities hold b < B < C < A < a. The sequence of Bs increases to C, and the sequence of As decreases to C. Always the interval [b, a] contains the interval [B, A]. For example, starting with a regular hexagon d = 1, a = 2√3, b = 3 we have the following values from Dörrie's method B = 3.140237343, A = 3.14734519, a precision achieved by the Archimedes method first with a 96-gon. It's interesting that the method used to generate the sequence B is the same formula as proposed by the cardinal Cusanus and Snell (M8) also see Figure 3. In a similar way as the method MX11 was obtained the method MX10 was determined. The method is constructed as follows MX10=M8+(MX9-M8)/5.

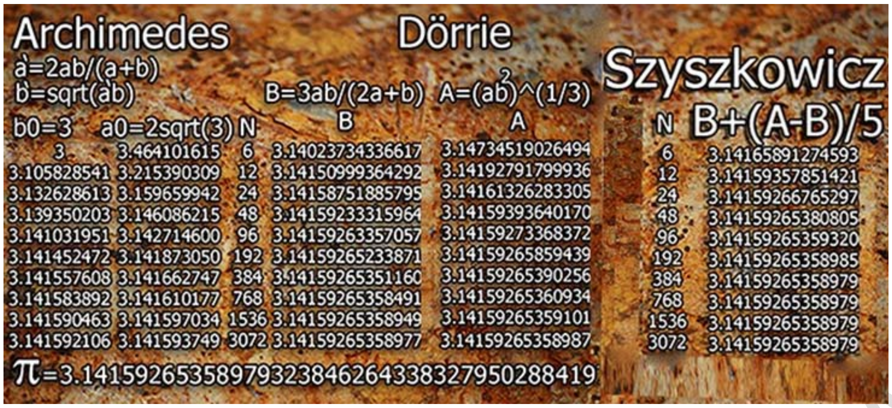

Figure 4. The approximations generated by Archimedes, Dörrie’s method, and Szyszkowicz’s method(MX10).

3. RESULTS AND DISCUSSION

The main results of this paper are two methods (MX10 and MX11), where we used Taylor series to justify their correctness and accuracy. The methods are very easy to program. Some calculations were executed. Table 2 presents the obtained results for the method MX11 and a few other methods known in literature.

Table 2. Estimated values of pi for various methods and N.

Size	(M1+M2)/2	MX4	MX5	MX6	M7	M8	MX11
N	Archimedes	Snell-P	Snell-A	Ch-H	Snell-ArcU	Snell-ArcL	Szyszkowicz
6	3.89711432	3.46410162	3.89711432	3.20429399	3.14403156	3.11769145	3.14139755
8	3.41421356	3.21895142	3.33333333	3.15032227	3.14234913	3.13444650	3.14155887
10	3.28581945	3.17018839	3.21435552	3.14368811	3.14189972	3.13874170	3.14158392
12	3.23205081	3.15470054	3.17542648	3.14226497	3.14174002	3.14023734	3.14158975
14	3.20410425	3.14846489	3.15948495	3.14185286	3.14167196	3.14086739	3.14159151
16	3.18758798	3.14554781	3.15194804	3.14170766	3.14163906	3.14116990	3.14159214
18	3.17695670	3.14403156	3.14800282	3.14164881	3.14162159	3.14132974	3.14159240
20	3.16968345	3.14317895	3.14577340	3.14162228	3.14161162	3.14142063	3.14159252
22	3.16447477	3.14266922	3.14443578	3.14160929	3.14160560	3.14147540	3.14159258
24	3.16060943	3.14234913	3.14359354	3.14160249	3.14160179	3.14150999	3.14159261

The main results of this paper are two methods (MX10 and MX11), where we used Taylor series to justify their correctness and accuracy. The methods are very easy to program. Some numerical calculations were executed. Figure 4 shows the results for the Pfaff-Borchardt-Schwab algorithm (a, b values), Dörrie's method (A, B values) and the method MX10 proposed in this paper. Table 2 presents the obtained results for the method MX11 and a few other methods known in literature.

Figure 5 has two panels. On the left panel we see rectification process for the arc corresponding to the angle x = 135 degree. It's relatively large angle.

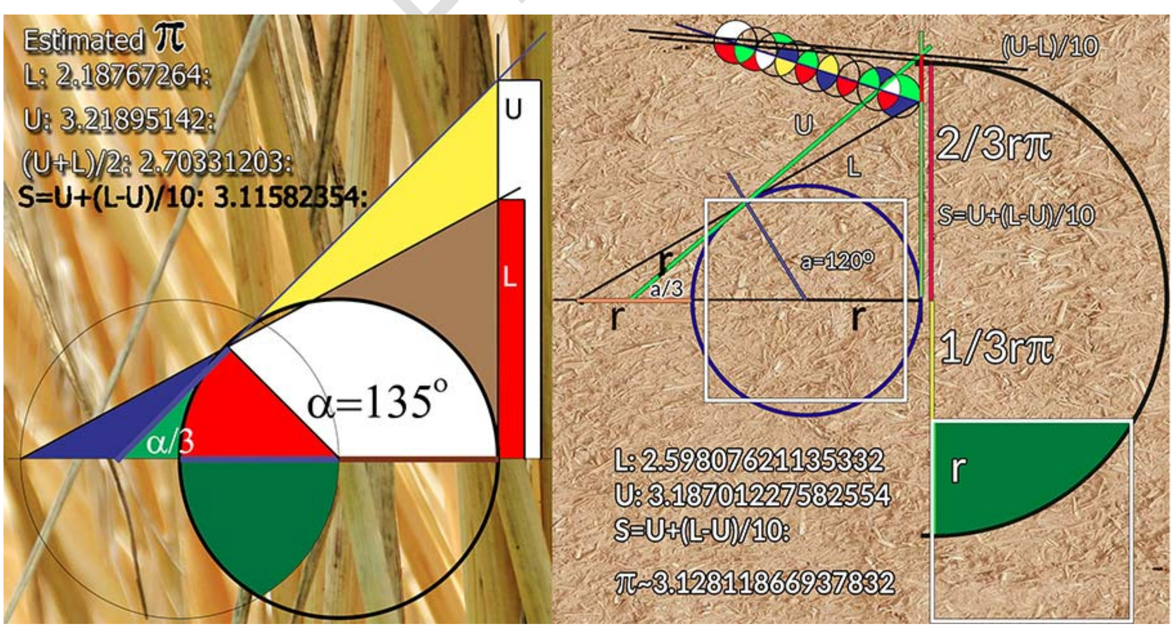

Figure 5: Rectification/quadrature - Szyszkowicz's method (MX11).

As the method needs also the angle x/3, we should be able to do trisection of the angle x. In this case it is possible to do this by a pure geometrical construction. It's easy to obtain the angle x/3. It's half of the right angle (90/2 = 45 = 125/3). The lower (L) and upper (U) estimations are generated by the methods M8 and M7, respectively. They have geometrical interpretations: the angle's vertex has the distance r (radius) to the circle for L, and to the cutting point on the circumference for the angle x. We are using the method MX11 to obtained better approximation for the number π. On the right-hand side panel we have more difficult situation. The angle of 120 degrees can't be trisected. We need the angel of 40 degrees. We may use other sources of such angle, not from the geometrical construction process. In this case a used graphic software was asked to rotate horizontal segment by 40 degrees. The method MX11 is applied and determines the segment S = U + (L − U)/10. The obtained segment (2/3πr) is extended by 1/3πr and r. It allows us to perform the squaring of the rectangle of the side πr and r. By the consequence we did approximated quadrature of our circle with estimated value of the number π. In the geometrical process Thales theorem on proportion is applied to divide the segment U −L into 10 equal parts.
The presented results summarize obtained approximations by various methods. As the values show the best approximation is produced by the MX10 method. The method is the result of the combination of two sequences generated by Dörrie's algorithm.

4. CONCLUSION

Well know methods to approximate the number pi are realized. Taylor series of these method (and Richardson extrapolation) allow to produce new methods with better convergence properties. Two methods are proposed: (i) combined Dörrie's sequence (MX10 method), (ii) combined Snell's sequence (MX11).
Two methods presented here improve Archimedes' technique. The method MX11 can be used geometrically for an angle x, if x=3 can be constructed.

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