EXACT ARITHMETIC OF ZERO AND INFINITY BASED ON THE CONVENTIONAL DIVISION BY ZERO $(\mathbf{0}/\mathbf{0}) = 1$

ABSTRACT. The chief object of this work is to create an exact and consistent arithmetic of zero, denoted $\mathbf{0}$, and infinity (zero divisor), written as $1/\mathbf{0}$ and denoted $\boldsymbol{\infty}$, based on the conventional division by zero

$$\frac{\mathbf{0}}{\mathbf{0}} = 1.$$

Manifold and undeniable applications of this arithmetic are given in this work in order to show its usefulness.

1. INTRODUCTION

The beginners in mathematics, on passing from the rudimentary operations of addition, subtraction, and multiplication involving zero to the operation of division by zero find themselves in a completely alien department of reasoning [25]. They have not ascended by painless and gentle steps. They are excruciatingly stamped with the established opinion that the continuity of arithmetic and algebra has been broken by the very stern commandment: *Thou shalt not divide by zero* [29].

They find themselves required to skip the approach that has hitherto guided their calculations and to put in its stead a notion that is utterly repugnant to their preconceived idea of numbers [1], [7]. When they are told that they cannot divide by zero, they restrict themselves to only division by finite quantities. They are instructed that the ratio of two zeros is indeterminate.

In the study of indeterminate forms, the beginners are drawn far away from their beloved faculty of arithmetic and algebra to the dreadful department of limit. For it seems very unfortunate that we should teach the beginners that $\lim_{x\to\mathbf{0}} x = \mathbf{0}$, suggesting that x reaches its limit $\mathbf{0}$, and then in the case

$$\lim_{x\to\mathbf{0}} \frac{x}{x} = 1$$

which suggests the conventional division by zero [12],[17]

$$\frac{\mathbf{0}}{\mathbf{0}} = 1,$$

instruct them that "it does not matter if x does not reach its limit $\mathbf{0}$ ".

Many mathematicians and scientists seek a renewal of the idea of division by zero. Thus, the notion of division by zero has been revisited by such excellent modern scholars as the two German brothers I. Barukcic and J. Barukcic [7], the American mathematics educationist J. Czajko [12] [13], the Japanese mathematician S. Saitoh

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[27] [28], the British computer scientist J. Anderson [1], the African mathematician and computer scientist W. P. Mwangi [17], professor J. Bergstra [8], H. Okumura [28], T. Matsuura [28], H. Michiwaki [27], M. Yamada [27] and many others.

The chief aim of this paper is the creation of a precise and consistent arithmetic of zero and infinity, basing it on the conventional division by zero $\mathbf{0/0} = 1$.

Now I have taken this work as thinking it will appear useful to all the mathematicians and the physicists worthy of their study, for it will contain the solutions to some of our questions which often arise in the arithmetic of infinities. Because this work would take up a great compass for a full justification of division by zero, I separated it into sections, subsections, and subsubsections but in the process of time, as usually happens to such as undertake great things, I grew weary, and went on slowly, it being a profuse and cumbersome subject, and a difficult thing to complete. Having been encouraged to go on by Mr. Agun Ikhile, I was ashamed to permit any laziness of disposition to have a greater influence upon me than the delight of taking pains in such studies as were very useful; I therefore stirred up myself, and went on more cheerfully.

The order in which the work is developed and many of the explanations offered, though novel in many respects, are believed to be well calculated to meet the difficulties and secure the interest of the reader.

The remainder of this paper is divided into 5 sections. Section 2 deals with the concept of zero, the number standing for absolute nothing. Section 3 is concerned with the multiplicative inverse of zero, the number called infinity. The true meaning of the operation of division is furnished in section 4. Section 5 discusses the indeterminate forms of the calculus. Section 6, the last section, gives undeniable applications of the arithmetic of zero and infinity.

2. ZERO

For the consummate comprehension of the arithmetic of zeros and infinities (zero divisors), it is expedient to fathom the profundity of the notion of zero, a concept which has been wrapped in obscurity by so many writers. Here, I shall do my utmost to set forth an ingenuous and non-perplexing theory of zero, leaving controversy as far as possible on one side. To start with, it may be well to point out that I shall use the word zero to mean generally *nothing* so that no number will be called zero unless it is truly *absolute nothing.*

2.1. **Basic Rules of Zero.**

In attempting to reach a precise arithmetic of infinities, it is useful to establish two rules of zero–reflexive and substitution.

2.1.1. *Reflexive Rule.* This rule is stated as follows. Let A be any quantity whatsoever–constant, variable or function. If $A = A$, then $A - A = 0$ and $-A + A = 0$ where 0 stands for *absence of the quantity* A. The **elimination expression** $A - A$ or $-A + A$ is a **take away subtraction** and means the removal or taking away of the quantity A completely from A.

The reflexive rule applies whenever we come across a number, variable or function being subtracted from itself in the process of simplifying a mathematical expression or solving a mathematical equation. Thus the expression

$$(x + 1)^2 - x^2 - 1$$

which equals

$$x^2 + 2x + 1 - x^2 - 1,$$

becomes, collecting like terms, the expression

$$x^2 - x^2 + 2x + 1 - 1.$$

This expression is rewritten as

$$0 + 2x + 0.$$

This is irreversibly equal to $2x$; the omission of the sign 0 is justified because 0 merely represents the absence of the quantity cancelled or removed.

Consider the problem of solving the equation

$$2x - 10 = x.$$

To remove x from the right–hand side of the equation, add $-x$ to both sides. The equation becomes

$$2x - 10 - x = x - x$$

which becomes

$$x - 10 = 0.$$

To remove -10 from the left–hand side, add $+10$ to both sides of the equation. Thus, we have

$$x - 10 + 10 = 0 + 10$$

which becomes

$$x + 0 = 0 + 10.$$

Omitting the symbol 0 as it merely represents absence of the quantity cancelled or removed, we get $x = 10$.

2.1.2. *Substitution Rule.* This rule goes thus. Let c be a constant and x a variable. If c is put in place of x in the expression $x - c$, the result is $\boldsymbol{c} - c = \mathbf{0}$ or if $-c$ is substituted for x in the expression $x + c$, the answer is $\boldsymbol{-c} + c = \mathbf{0}$. The bold–faced c, that is $\boldsymbol{c}$, is used to indicate an evaluation process. The bold–faced 0, namely $\mathbf{0}$, is use to represent *the null difference of the two different species of quantities x and c when $x = c$.* Here, the **evaluation expression $\boldsymbol{c} - c$** is a **comparison subtraction** and means that there is no quantitative difference $x - c$ when $x = c$.

Let it be noted that the expression $c - c$ and $f(x) - f(x)$ signify the elimination or removal or cancellation of the constant c and the function $f(x)$ from a mathematical expression or one side of an equation respectively. On the other hand, the expression $\boldsymbol{c} - c$ signifies that the quantitative difference $x - c$ has been made absolutely nothing when x has been made equal to c. It does not imply the removal or elimination or cancellation of x or c.

Since it is easy to differentiate an elimination expression from an evaluation expression, we shall soon drop the use of the bold–faced c. We shall however maintain throughout this work the use of the bold–faced 0.

2.2. **Zero in Concrete Arithmetic.** The zero 0 is the sign associated, as already noted, with the removal of a quantity from an expression or one side of an equation. In concrete arithmetic, it is the symbol for the emptiness of a group of objects when all the objects are removed from the group.

On the other hand, **0**, represents the null difference of the variable quantity x and the constant quantity c when the variable is made equal to the constant. In concrete arithmetic, it is the symbol for the nullity of the difference of the numbers of objects in two different groups of objects when the variable number of objects in one group is made equal to the fixed number of objects in the other group.

2.3. **The Equality of 0 and** 0**.** The zero **0** can be shown to be equal to the zero 0. For if we start with the equation $x = c$, and subtract c from both sides, we shall have $x - c = c - c$ which becomes $x - c = 0$. For the equation $x - c = 0$ to hold good, x must equal c. Putting c in place of x in the equation furnishes $\boldsymbol{c} - c = 0$ which becomes $\mathbf{0} = 0$.

Whatever weight is given to the sign '='in the result $\mathbf{0} = 0$, it is clear that it comes close to identifying **0** with 0. But one should not speak merely of **0** playing the role of 0 or of **0** only representing 0.

Again, we consider the equation $x^2 = 1$. If we remove 1 from the right-hand side by adding -1 to both sides, we have

$$x^2 - 1 = 0$$

which becomes

$$(x + 1)(x - 1) = 0.$$

If we set $x = 1$, we obtain

$$(\mathbf{1} + 1)(\mathbf{1} - 1) = 0$$

which becomes $2 \cdot \mathbf{0} = 0$. Similar arguments show that for any finite number a and positive number n,

$$a \cdot \mathbf{0}^n = 0.$$

Our inability to distinguish between the two different species of zero, **0** and 0, is the reason why mathematicians have to complain that there are paradoxes in the analysis of the infinite. This is not to be wondered at, for if only we comprehend the difference between these two species of zero, we shall be in the position to resolve all the paradoxes and contradictions which reveal themselves in the analysis of the infinities.

2.4. **Properties of** 0**.** Here we provide the properties of 0, the symbol we have used to represent the *absolute nothing* which arises from the cancellation of a quantity by itself. If a is a finite quantity, then

$$\begin{aligned} a + 0 &= 0 + a = a \\ a - 0 &= a \\ 0 - a &= -a \\ a \times 0 &= 0 \times a = 0 \end{aligned}$$

2.5. **Properties of 0.** We have seen the properties of 0; let us now ascend to see some properties of **0**. From the equality $\mathbf{0} = 0$, it follows that all the properties of 0 are also the properties of **0**. Thus

$$a + \mathbf{0} = a + 0 = a$$
$$a - \mathbf{0} = a - 0 = a$$
$$\mathbf{0} - a = 0 - a = -a$$
$$a \times \mathbf{0} = a \times 0 = 0$$

2.6. **0 as an Integer.** Consider the sequence of variables

$$\ldots, \quad n-4, \quad n-3, \quad n-2, \quad n-1, \quad n, \quad n+1, \quad n+2, \quad \ldots.$$

If we let $n = 1$, we obtain the sequence of integers

$$\ldots, \quad -3, \quad -2, \quad -1, \quad \mathbf{0}, \quad 1, \quad 2, \quad 3, \quad \ldots.$$

From this it follows that **0** is an integer.

2.7. **Additive Inverse of 0.** Consider the sequence of variables

$$\ldots, \quad -2-n, \quad -1-n, \quad -n, \quad -(n-1), \quad 2-n, \quad 3-n, \quad 4-n, \quad \ldots.$$

If we set $n = 1$, we get the sequence of integers

$$\ldots, \quad -3, \quad -2, \quad -1, \quad -\mathbf{0}, \quad 1, \quad 2, \quad 3, \quad \ldots.$$

Matching the sequence of integers in the previous subsection with the above sequence, we see at once that **0** and its additive inverse $-\mathbf{0}$ coincide as they occupy the same position in both sequences. It logically follows that since **0**'s position in the first sequence is that of magnitude of absolute nothing, its additive inverse $-\mathbf{0}$ must be of magnitude of absolute nothing.

2.8. **0 as an Infinitely Small Quantity.** The term *quantity* refers to how many or how much of something is present. It may be zero (nothing), finite (definite) or infinite (endless).

The zero **0** is an infinitely small quantity. This is so because **0** is the reciprocal of an infinitely large quantity.

Consider the recursive formula of the factorial of $n + 1$:

$$(n+1)! = (n+1) \cdot n!$$

If we set $n = -1$, we get $\mathbf{0}! = \mathbf{0} \cdot (-1)!$ which we rewrite as

$$\mathbf{0} = \frac{\mathbf{0}!}{(-1)!}.$$

Taking $\mathbf{0}! = 1$ we write

$$\mathbf{0} = \frac{1}{(-1)!}.$$

Now,

$$(-1)! = (-1) \cdot (-2)! = (-1) \cdot (-2) \cdot (-3)! = \cdots = (-1) \cdot (-2) \cdot (-3) \cdots .$$

It follows immediately that

$$\mathbf{0} = \frac{1}{(-1) \cdot (-2) \cdot (-3) \cdots};$$

that is, **0** is the ratio of unity to the infinite product $(-1) \cdot (-2) \cdot (-3) \cdots$.

But though it is not my objective to treat the infinitely small, yet because it ought to be pellucid and incontrovertible to all, that infinitely small quantities are genuinely zeros, that is absolute nothing, it seems apt to advert to that identity which unfold the link between infinitesimals and zeros.

The quotient of the fraction

$$\frac{1}{(-1)\cdot(-2)\cdot(-3)\cdots}$$

may be taken as an infinitely small quantity as it is the ratio of two other quantities in which the numerator is finite and denominator is infinite. If we want the product of all the numbers that make up the product $(-1)\cdot(-2)\cdot(-3)\cdots$, since these numbers progress with no end, and the product increases, it certainly cannot be finite. By this fact it becomes an infinitely large quantity. Indeed, the larger the denominator of a fraction with a fixed and finite numerator becomes, the smaller the value of the fraction becomes, and if the denominator becomes an infinitely large quantity, then necessarily the value of the fraction becomes an infinitely small quantity.

A strong proof of the fact that the infinitely small are zeros is the already adduced identity

$$\frac{1}{(-1)\cdot(-2)\cdot(-3)\cdots} = \mathbf{0}.$$

For though the fraction

$$\frac{1}{(-1)\cdot(-2)\cdot(-3)\cdots}$$

is a number infinitely small as it is the ratio of unity to an infinitely large quantity, the infinite product of all negative integers, we are forced somewhat to assert that the infinitely small quantity is is equal to the zero $\mathbf{0}$. For philosophical mathematicians, being unacquainted with results of this sort, which clearly betoken that infinitesimals are all zeros, erroneously confound infinitesimals with zeros. Let us, therefore, hold, for the intent of the present work, that every infinitesimal number is a zero or absolute nothing.

The great Euler was the first to notice this essence of the infinitesimals. In his famous book[], we read his most quoted statement:

> To anyone who asks what an infinitely small quantity in mathematics is, we can respond that it is really equal to zero.

I give few remarks on the zeros $\mathbf{0}$ and 0, and close this subsection. The zero 0 cannot be expressed as a ratio of two numbers and therefore unrelated to infinite quantities and hence infinitely small quantities. This is so because the mathematical expression which gives rise to it is not related to any other mathematical expression. For instance, the expression $x-x$ which equals 0 is not equal to any other expression.

On the other hand, the expression $x-c$ which gives rise to the zero $\mathbf{0}$ when we let $x=c$ [19], [20]is equal to and hence related to another expression [21], viz

$$x-c=\frac{(x-c)!}{(x-c-1)!}.$$

So if we set $x=c$, we get

$$\mathbf{0}=\frac{\mathbf{0}!}{(\mathbf{0}-1)!}=\frac{1}{(-1)!}.$$

This relation makes the zero $\mathbf{0}$ acquire the following unique properties:

- The zero **0** divided by itself equals unity.
- The zero **0** obeys the principle of impending operation on zero, discussed in subsection (2.10).

2.9. **Conventional Division by Zero: $\frac{\mathbf{0}}{\mathbf{0}} = \mathbf{1}$.** The infinitesimal quantity

$$\frac{1}{(-1)\cdot(-2)\cdot(-3)\cdots}$$

is really equal to the zero **0**. Since this is true, nothing hinders us from dividing **0** by **0**; for

$$\frac{\mathbf{0}}{\mathbf{0}} = \frac{(-1)\cdot(-2)\cdot(-3)\cdots}{(-1)\cdot(-2)\cdot(-3)\cdots} = 1.$$

The equation

$$\frac{\mathbf{0}}{\mathbf{0}} = 1$$

is the conventional division of zero by zero, an idea vehemently promoted by Czajko and Barukcic [1].

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[1] In the abstract of his excellent paper entitled *On Cantorian Spacetime over Number Systems with Division by Zero* Czajko gives the purpose of his work: "Division by zero is proposed in order to open new horizons for abstract mathematics and physics. In his introduction, he speaks of the significance of division by zero in physics:

> Some problems of physics could be traced to hidden, unresolved issues in pure mathematics (PM), some of which are almost as old as the, allegedly impossible and therefore prohibited, division by zero. There is really no need and no compelling reason for the prohibition, however. Quite on the contrary, we actually need the banned operation in order for us to advance physics.

On page 264, Czajko disapproves of the common identity $0 \cdot 0 = 0$:

> However, I just could not accept the almost 5000 years prescription for multiplication of natural zeros:
>
> $$0 \cdot 0 = 0. \tag{0b}$$

Then on page 265, he includes infinity in his system and furnishes an amendment $0^2 = 1/\infty^2$:

> Whether one likes it or not we need both zero and infinity to obtain an operationally number system. Because the natural zero belongs to integers and reals ($0 \in Z \subset \mathbb{R}$), the zero needs to be invertible under division too. Hence, we must expand the reals and by including the integer infinity ∞ :
>
> $$0 = \frac{1}{\infty} \tag{0d}$$
>
> which makes the incorrect Eq. (0b) only approximately true, since we get:
>
> $$0 \cdot 0 = \frac{1}{\infty^2} \approx 0 \tag{0e}$$
>
> where the result of multiplication of natural zeros does not belong to integers, but to reals of course.

On introducing integer infinity to his number system, he proposes

$$\frac{0}{0} = \frac{\infty}{\infty} = 1.$$

The excellent German mathematician, I. Barukcic, whose writings exhibit a true ingenuity, claims that

$$+0 \times +\infty = 1, \quad \frac{+0}{+0} = 1, \quad \frac{+\infty}{+\infty} = 1$$

2.10. **Principle of Impending Operation on 0.** The problem with which we shall be occupied in the present talk is that of a closer investigation of the principle of impending operation on $\mathbf{0}$. Here we shall give a perfect discussion of this principle and in doing so, shall endeavour to confine our remarks on what strictly relates to it.

The idea of treating $\mathbf{0}$ as an infinitely small quantity in calculations was first moved by the excellent wisdom of the illustrious Indian mathematician and astronomer, Bhaskara II. In his *Lilavati* we read his principle of impending operation on $\mathbf{0}$ [3], [4][22]:

> The product of zero is nought, but it must be retained as a multiple of zero if any operation impend. Zero having become a multiplier, should nought afterward become a divisor, the definite (finite) quantity must be understood to be unchanged.

Bhaskara says that

$$a \cdot \mathbf{0} = 0$$

only if no further operation on $\mathbf{0}$ is approaching. He seems to identify $a \cdot \mathbf{0}$ and 0 on the ground that the calculation which gives rise to $a \cdot \mathbf{0}$ has ended or there arise no operation requiring the result $a \cdot \mathbf{0}$.

Suppose, after the operation $a \cdot \mathbf{0}$, there is an approaching operation in which $\mathbf{0}$ is a divisor. According to Bhaskara, we should use the form $a \cdot \mathbf{0}$ (and not merely 0 to which $a \cdot \mathbf{0}$ equals) in this new operation such that when it is divided by $\mathbf{0}$, the result gives a, that is

$$\frac{a \cdot \mathbf{0}}{\mathbf{0}} = a.$$

The above equation is often called Bhaskara's identity.

Here is a simple demonstration of the above point. If we set $x = 1$ in the expression

$$\frac{x^2 - 1}{x - 1},$$

we get

$$\frac{1^2 - 1}{1 - 1}$$

which is written as

$$\frac{0}{0}$$

and termed *indeterminate*. The mystery is easily resolved with a little algebra:

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1}.$$

Setting $x = 1$ gives

$$\frac{1^2 - 1}{1 - 1} = \frac{(1 + 1)(1 - 1)}{1 - 1} = \frac{2 \cdot \mathbf{0}}{\mathbf{0}} = 2.$$

The evaluation of

$$\frac{x^2 - 1}{x - 1}$$

at $x = 1$ is so obvious as not to need technical definition. If x is nearly 1, then $x - 1$ is nearly $\mathbf{0}$ and $x^2 - 1$ is nearly $2 \cdot \mathbf{0}$ as can be seen in the Table 1. Eventually, when $x = 1$, $x - 1 = \mathbf{0}$ and $x^2 - 1 = 2 \cdot \mathbf{0}$. This simple example shows that Bhaskara's identity holds true.

x	$x-1$	x^2-1
1.1	0.1	2.1×0.1
1.01	0.01	2.01×0.01
1.001	0.001	2.001×0.001
⋮	⋮	⋮

TABLE 1. Table of values

2.11. **The Unit Zero 0.** By definition, the *unit zero*, denoted $\mathbf{0}$, is the result of the evaluation of $x-c$ at $x=c$ or $x+c$ at $x=-c$.

2.12. **Powers of the Unit Zero.** The mth power of the unit zero is written as

$$\mathbf{0}^m.$$

Thus, $\mathbf{0}^2, \mathbf{0}^3, \mathbf{0}^4$ and so on are integral powers of zero while $\mathbf{0}^{1/2}, \mathbf{0}^{1/3}, \mathbf{0}^{2/3}$ and so forth are fractional powers of zero.

2.13. **Orders of Zeros.** Any expression of the form

$$a \cdot \mathbf{0}^m,$$

where a is a finite number and m is a positive number, is called a zero of mth order. The number a is called the finite part of the zero and the number m is the order part. The zero $-5 \cdot \mathbf{0}^3$ is a 3rd order zero with a finite part of -5 and an order part of 3.

2.14. **Operations on Zeros.** To create an exact and consistent arithmetic of zero has acquired an extraordinary celebrity from the fact that none has been created, but there is no reason to doubt that it is possible.

2.14.1. *Equality of Zeros.* The zeros $a \cdot \mathbf{0}^m$ and $b \cdot \mathbf{0}^n$ are equal if and only if $a = b$ and $m = n$.

2.14.2. *Addition and subtraction.* For the zeros $a \cdot \mathbf{0}^m$ and $b \cdot \mathbf{0}^m$ of the same order m,

$$a \cdot \mathbf{0}^m + b \cdot \mathbf{0}^m = (a+b) \cdot \mathbf{0}^m = 0$$

and

$$a \cdot \mathbf{0}^m - b \cdot \mathbf{0}^m = (a-b) \cdot \mathbf{0}^m = 0.$$

For instance, take the third order zeros $2 \cdot \mathbf{0}^3$ and $7 \cdot \mathbf{0}^3$. Then,

$$2 \cdot \mathbf{0}^3 + 7 \cdot \mathbf{0}^3 = 9 \cdot \mathbf{0}^3 = 0$$

and

$$2 \cdot \mathbf{0}^3 - 7 \cdot \mathbf{0}^3 = -5 \cdot \mathbf{0}^3 = 0.$$

The addition or subtraction of zeros of different orders cannot be reduced to any simpler form, and the combination is therefore called *compound zeros*. For example,

$$2 \cdot \mathbf{0}^3 + 7 \cdot \mathbf{0}^5$$

consists of zeros of different orders. All we know of this compound zero is that it is equal to the zero 0, that is

$$2 \cdot \mathbf{0}^3 + 7 \cdot \mathbf{0}^5 = 0.$$

2.14.3. *Scalar multiplication.* The product of any zero, say $a \cdot \mathbf{0}^m$, and any scalar c (number c), is the zero $ca \cdot \mathbf{0}^m$ obtained by multiplying c by the finite part of the zero. For instance, $7 \cdot \mathbf{0}^5$ multiplied by 2 is $14 \cdot \mathbf{0}^5$.

2.14.4. *Multiplication of Zeros.* For the zeros $a \cdot \mathbf{0}^m$ and $b \cdot \mathbf{0}^n$, we get the multiplication operation

$$a \cdot \mathbf{0}^m \times b \cdot \mathbf{0}^n = ab \cdot \mathbf{0}^{m+n} = 0.$$

For instance,

$$2 \cdot \mathbf{0}^3 \times 7 \cdot \mathbf{0}^4 = 14 \cdot \mathbf{0}^7 = 0.$$

Addition, subtraction and multiplication of zeros give a zero.

2.14.5. *Division of Zeros.* For the zeros $a \cdot \mathbf{0}^m$ and $b \cdot \mathbf{0}^n$, we have the division operation

$$a \cdot \mathbf{0}^m \div b \cdot \mathbf{0}^n = \frac{a}{b} \cdot \mathbf{0}^{m-n}.$$

If $m > n$, the quotient is a zero and if $m < n$, the quotient is infinity. If, however, $m = n$, the quotient is the finite number a/b.

2.14.6. *Improved Principle of Impending Operation on Zero.* Here, we improve the principle of impending operation on zero: "If in some mathematical calculations, the zero $\mathbf{0}$ is likely to occur frequently, then, though $a \cdot \mathbf{0}^n = 0$ where a is a finite number and n is any positive number, one should maintain the form $a \cdot \mathbf{0}^n$ in the rest of the operations until the final operation with $\mathbf{0}$ is reached. This is because if a finite number is multiplied by zero and divided by the same zero, then the result is the finite number".

3. On Infinity

In the previous section we discussed zero in detail. Here, however, it is with infinity we are concerned, and while zero is frequently mentioned, yet we do not pause on every page to insist on it; instead, we have sought to stress that part of mathematics which is not usually considered. We grant that the discussion is one–sided, for it only seek to deal with one side of analysis, the neglected side–*infinity as a number*.

There are many mathematicians who protest violently against division by zero being a number and heap up arguments against it; for they see in it only a sort of meaninglessness. Very interestingly, they seldom present arguments that show the ratio of a finite number to zero is infinity. This is a very poor way to study mathematics. I say this because if we are to present an honest study of any subject, then we must present all arguments relevant to what we are studying and not just some of them.

In the face of these misunderstandings, I should like to discuss this number with the readers, not in conformity with what is known of modern mathematics and even less with philosophical objections; not as an adversary of the theory of zero and infinity, but as one who wishes to submit himself to the school of arithmetic of zero and infinity, and learn from it all that it is necessary to know.

3.1. **Infinity as a Number.** Infinity is larger than any finite number. It is a number which counts endless number of things or measures magnitudes larger than finite magnitudes. To put it simply, infinity counts things whose number is without end or measures things whose magnitudes or sizes are without bound. There is no better way to sum it up than to say with the acute Bhaskara, "This fraction is termed *anantha rashi*".

Though infinities signify *without end*, they are, nevertheless, numbers of which we can form an accurate idea, since, however, hidden the meaning of the fraction $\frac{2}{\mathbf{0}}$, for instance, we are not ignorant that it must be truly a number, when multiplied by $\mathbf{0}$ would exactly produce 2; and this property is sufficient to give us an idea of the number $\frac{2}{\mathbf{0}}$.

Take the mirror problem of finding the number n of images of an object placed between two plane mirrors inclined at an angle θ to each other. In Physics, the number is obtained from the experimental formula

$$n = \frac{360}{\theta} - 1.$$

If the two mirrors face each other, the number of images seen is without end or limit. This number must therefore be an infinite number, the ratio of a finite number to zero. Now, since the two mirrors are facing each other, $\theta = \mathbf{0}$ and so the number of images is

$$n = \frac{360}{\mathbf{0}} - 1$$

which is clearly an infinite number.

3.2. **The Unit Infinity.** By definition, the unit infinity, denoted $\boldsymbol{\infty}$, is the result of the evaluation of

$$\frac{1}{x-c}$$

at $x = c$ or

$$\frac{1}{x+c}$$

at $x = -c$. It is, therefore, the reciprocal of zero $\mathbf{0}$ viz.

$$\boldsymbol{\infty} = \frac{1}{\mathbf{0}}.$$

The above identity shows a connection among the three classes of numbers–zeros, finite numbers and infinite numbers– for the head of zeros is $\mathbf{0}$, of finite numbers is 1, and of infinite numbers is $\boldsymbol{\infty}$. We can switch from any two given classes to the third by employing the switching formulas:

$$\frac{1}{\mathbf{0}} = \boldsymbol{\infty}, \quad \boldsymbol{\infty} \cdot \mathbf{0} = 1, \quad \frac{1}{\boldsymbol{\infty}} = \mathbf{0}.$$

3.3. **Unit Infinity as a Limit.** It is well known that the limit of the sequence of the multiplicative inverses of the natural numbers

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \dots$$

is the number $\mathbf{0}$. In as far as this is true, one is inevitably by way of correspondence as in

$$\begin{aligned}\frac{1}{1} &\leftrightarrow 1\\ \frac{1}{2} &\leftrightarrow 2\\ \frac{1}{3} &\leftrightarrow 3\\ \frac{1}{4} &\leftrightarrow 4\\ &\vdots\end{aligned}$$

to suppose that the limit of the sequence of the natural numbers

$$1, \quad 2, \quad 3, \quad 4, \ldots$$

is the unit infinity

$$\boldsymbol{\infty}.$$

These limits, $\mathbf{0}$ and $\boldsymbol{\infty}$, can be placed respectively at the two extremities of the sequence of numbers with whole numbers and their multiplicative inverses extending endlessly in opposite directions:

$$\mathbf{0}, \quad \ldots, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad 1, \quad 2, \quad 3, \quad \ldots, \quad \boldsymbol{\infty}.$$

Any two numbers in the sequence that are equidistant from the number 1 are multiplicative inverses of each other, thus producing 1 when multiplied together:

$$n \cdot \frac{1}{n} = 1,$$

such that for the two limits or extremities or final terms, $\mathbf{0}$ and $\boldsymbol{\infty}$, one could have as well

$$\mathbf{0} \cdot \frac{1}{\mathbf{0}} = 1$$

or simply

$$\mathbf{0} \cdot \boldsymbol{\infty} = 1.$$

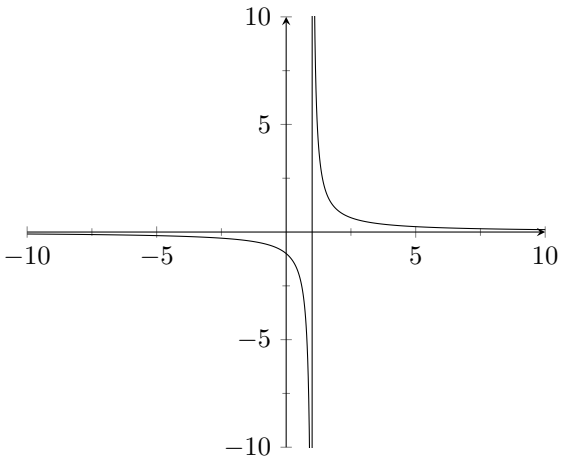
3.4. **Additive Inverse of $\boldsymbol{\infty}$.** We have shown that $\mathbf{0}$ and $-\mathbf{0}$ coincide at the same place in the sequence of integers. It logically follows that their respective multiplicative inverses

$$\boldsymbol{\infty} \quad \text{and} \quad -\boldsymbol{\infty}$$

are reached simultaneously in calculations.

The fact that the negative infinity $-\boldsymbol{\infty}$ is reached at the same time with the positive infinity $\boldsymbol{\infty}$ in calculations can be used to explain the graph (see the figure below) of the function

$$f(x) = \frac{1}{x-1}.$$



To study the behaviour of the curve, let us move gradually along the $x-$axis from right to left. We notice that as we approach $x = 1$, the curve approaches the vertical asymptote $x = 1$ and the function value which has been positive and finite approaches a positive and infinite number. When we reach $x = 1$, the function value equals the positive infinite number $\frac{1}{\mathbf{0}} = \boldsymbol{\infty}$, the quantity beyond which the function value will never go. At this stage, starting from any finite x, the curve is said to have extended without bound. Since, as we already noted, the zero number $+bm0$ coincides with the zero number $-\mathbf{0}$, it turns out that the function value reaches the negative infinite number $-\boldsymbol{\infty}$ when it has reached the positive infinite number $+\boldsymbol{\infty}$ at $x = 1$. Hence, as we move away from $x = 1$, the curve returns from the negative infinite number already mentioned and the function value becomes negative and finite. As we move farther and farther away from $x = 1$, the curve bends more and more away from the vertical asymptote $x = 1$.

3.5. **Powers of the Unit Infinity.** The mth power of unit infinity is written as

$$\boldsymbol{\infty}^m.$$

Thus, $\boldsymbol{\infty}^2, \boldsymbol{\infty}^3, \boldsymbol{\infty}^4$, and so forth are integral powers of infinity and $\boldsymbol{\infty}^{1/2}, \boldsymbol{\infty}^{1/3}$, $\boldsymbol{\infty}^{2/3}$, are fractional powers of infinity.

3.6. **Orders of Infinity.** Any number written in the form

$$a \cdot \boldsymbol{\infty}^m,$$

where a is a finite number and m is a positive number, is called an infinity of mth order. The number a is called the finite part and m is the order part. Thus, $2 \cdot \boldsymbol{\infty}^3$, which has a finite part of 2 and an order part of 3, is a 3rd order infinity

If a be a finite constant, the expressions

$$\frac{a}{\mathbf{0}^n} = a \cdot \boldsymbol{\infty}^n \quad \text{and} \quad \frac{a}{\boldsymbol{\infty}^n} = a \cdot \mathbf{0}^n$$

are rigidly exact. The first asserts that if the numerator of a fraction is a constant and the denominator is a zero of nth order, the fraction equals an infinity of nth order. The second asserts that if the denominator is an infinity of n order, the fraction equals an nth order zero.

We shall use the usual sign ∞ (not bold–faced) to represent in general any infinite number. Thus, for instance,

$$4 \cdot \boldsymbol{\infty}^3 = \infty.$$

3.7. **Infinities as Transfinite Numbers.** All infinities or better still infinite numbers are suprafinites or transfinites (*supra* or *trans*=above,beyond) , that is, they are all larger than any finite number proposed. To show this, take the valid statement

$$a > b \cdot \mathbf{0}^m$$

where a and b are any finite numbers and m is a positive number. Since this statement holds good, it follows logically that

$$\frac{a}{\mathbf{0}^m} > b,$$

which becomes

$$a \cdot \boldsymbol{\infty}^m > b.$$

Thus, the infinite number $a \cdot \boldsymbol{\infty}^m$ is suprafinite or transfinite, larger than any finite number b. It is, therefore, a number bigger than all finite numbers no matter the greatness of their sizes.

3.8. **Operations on Infinity.** Though we can never count infinite (endless) number of things, we can form a true and accurate arithmetic of it. Infinite numbers occur in mathematics, engineering and the sciences and therefore the knowledge of the operations on them is essential. Like finite numbers, infinite numbers can be added, subtracted, multiplied, and divided. Our desire here is, therefore, to put infinity on equal footing as ordinary numbers. Once this has been done, infinity is a perfectly acceptable candidate for mathematical analysis.

3.8.1. *Equality of Infinities.* Two infinite numbers $a \cdot \boldsymbol{\infty}^m$ and $b \cdot \boldsymbol{\infty}^n$ are equal if $a = b$ and $m = c$, that is, their finite parts are equal and their order parts are equal.

3.8.2. *Addition and Subtraction.* The sum and difference of two infinite numbers $a \cdot \boldsymbol{\infty}^m$ and $b \cdot \boldsymbol{\infty}^m$ of the same order are defined by adding or subtracting their finite parts:

$$a \cdot \boldsymbol{\infty}^m + b \cdot \boldsymbol{\infty}^m = (a + b) \cdot \boldsymbol{\infty}^m = \infty$$

and

$$a \cdot \boldsymbol{\infty}^m - b \cdot \boldsymbol{\infty}^m = (a - b) \cdot \boldsymbol{\infty}^m = \infty.$$

For instance,

$$3 \cdot \boldsymbol{\infty}^4 + 2 \cdot \boldsymbol{\infty}^4 = 5 \cdot \boldsymbol{\infty}^4 = \infty$$

and

$$3 \cdot \boldsymbol{\infty}^4 - 2 \cdot \boldsymbol{\infty}^4 = \boldsymbol{\infty}^4 = \infty.$$

The sum or difference of infinite numbers of different powers can be reduced to no simpler form and so called *compound infinity*. For example,

$$3 \cdot \boldsymbol{\infty}^7 + 2 \cdot \boldsymbol{\infty}^4$$

is a compound infinity as it consists of two infinite numbers of difference orders 7 and 4. All that should be said of this infinity is that it is equal to ∞, that is

$$3 \cdot \boldsymbol{\infty}^7 + 2 \cdot \boldsymbol{\infty}^4 = \infty.$$

3.8.3. *Scalar Multiplication.* The product of the infinite number $a \cdot \boldsymbol{\infty}^m$ and the scalar c (number c) is obtained as follows:

$$c \times a \cdot \boldsymbol{\infty}^m = ca \cdot \boldsymbol{\infty}^m = \infty.$$

For example,

$$3 \times 5 \cdot \boldsymbol{\infty}^2 = 15 \cdot \boldsymbol{\infty}^2 = \infty.$$

3.8.4. *Multiplication of Infinite Numbers.* For the infinities $a \cdot \boldsymbol{\infty}^m$ and $b \cdot \boldsymbol{\infty}^n$, we have

$$a \cdot \boldsymbol{\infty}^m \times b \cdot \boldsymbol{\infty}^n = ab \cdot \boldsymbol{\infty}^{m+n} = \infty.$$

As an instance, we have

$$4 \cdot$$

3.8.5. *Division of Infinite Numbers.* For the infinite numbers $a \cdot \boldsymbol{\infty}^m$ and $b \cdot \boldsymbol{\infty}^n$, we have

$$a \cdot \boldsymbol{\infty}^m \div b \cdot \boldsymbol{\infty}^n = \frac{a}{b} \cdot \boldsymbol{\infty}^{m-n}.$$

If $m > n$, the quotient is an infinity and it $m < n$, the quotient is a zero. If, however, $m = n$, the quotient is a finite number (a/b). As an instance, we have

$$4 \cdot \boldsymbol{\infty}^5 \div 2 \cdot \boldsymbol{\infty}^6 = 8 \cdot \boldsymbol{\infty}^{-1} = 8 \cdot \mathbf{0} = 0.$$

4. The True Meaning of Division

Division, in reality, is the act of finding what the dividend is equivalent to when the divisor is made equivalent to unity. For instance, take the division of 10 by 2. If the divisor

$$2 \equiv 1$$

then the dividend

$$10 = 5 \times 2 \equiv 5,$$

that is if 2 is equivalent to 1, then 10 is equivalent to 5. Thus $10 \div 2 = 5$.

Again, we consider the division of -8.8 by 8. If the divisor

$$8 \equiv 1$$

then the dividend

$$-8.8 = -1.1 \times 8 \equiv -1.1,$$

that is, if 8 is equivalent to 1, then -8.8 is equivalent to -1.1. Thus, $-8.8 \div 8 = -1.1$.

Take the division of 80 by -4. If the divisor

$$-4 \equiv 1$$

then the dividend

$$80 = (-20) \times (-4) \equiv -20.$$

It follows that $80 \div (-4) = -20$.

Consider the division of -10 by -100. If the divisor

$$-100 \equiv 1$$

then the dividend

$$-10 = 0.1 \times (-100) \equiv 0.1;$$

-10 is equivalent to 0.1 when -100 is made equivalent to unity. So, $(-10) \div (-100) = 0.1$.

By this definition of the operation of division, zero is a sure candidate of division without restriction. Let us, as an illustrative example, consider the division of $2 \cdot \mathbf{0}$

by $\mathbf{0}$. To arrive at the quotient of this division, we apply the definition of division already adduced. If the divisor

$$\mathbf{0} \equiv 1$$

then the dividend

$$2 \cdot \mathbf{0} \equiv 2 \cdot 1 = 2;$$

the zero $2 \cdot \mathbf{0}$ is equivalent to 2 when the zero $\mathbf{0}$ is made equivalent to 1. Thus, we get

$$\frac{2 \cdot \mathbf{0}}{\mathbf{0}} = 2.$$

We take, as a further illustration, the division of the zero $-6 \cdot \mathbf{0}^3$ by the zero $3 \cdot \mathbf{0}^2$. If the divisor

$$3 \cdot \mathbf{0}^2 \equiv 1$$

then the dividend

$$-6 \cdot \mathbf{0}^3 = (-2 \cdot \mathbf{0}) \cdot (3 \cdot \mathbf{0}^2) \equiv -2 \cdot \mathbf{0}.$$

Therefore, we write

$$\frac{-6 \cdot \mathbf{0}^3}{3 \cdot \mathbf{0}^2} = -2 \cdot \mathbf{0}.$$

Finally, we consider the division of 5 by $\mathbf{0}$. If the divisor

$$\mathbf{0} \equiv 1$$

then the dividend

$$5 = 5 \cdot \mathbf{0} \cdot \boldsymbol{\infty} \equiv 5 \cdot \boldsymbol{\infty}.$$

If the zero $\mathbf{0}$ is made equivalent to the finite number 1, the finite number 5 becomes equivalent to the infinite number $5 \cdot \boldsymbol{\infty}$. Hence, we have

$$\frac{5}{\mathbf{0}} = 5 \cdot \boldsymbol{\infty} = \infty.$$

5. Indeterminate Forms

This section deals with the indeterminate forms.

A given function can assume the indeterminate form

$$\frac{0}{0}$$

for a certain value of the independent variable. It can also assume the forms:

$$0 \times \infty, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad \infty^0, \quad 0^0, \quad 0^\infty, \quad 1^\infty$$

These can be reduced to the form $0/0$. Any of these forms can be equal to any number which is the true value of the given function.

5.1. **Zero Divided by Zero.** Suppose we were asked to determine the value of

$$f(x) = \frac{x^2 + x - 6}{x + 3}$$

where $x = -3$. Direct substitution shows that both numerator and denominator must equal zero numbers. To obtain the expressions for these, we first factor $x^2 + x - 6$ and set $x = -3$. Thus, $x^2 + x - 6$ becomes $(x - 2)(x + 3)$ which on setting $x = -3$ becomes $(-3-2)(-3+3) = -5 \cdot \mathbf{0}$. We must retain this zero number because of the further operation of division by the denominator zero number $(-3 + 3) = \mathbf{0}$. Thus,

$$f(-3) = \frac{-5 \cdot \mathbf{0}}{\mathbf{0}} = -5.$$

Suppose we wish to evaluate the function

$$f(x) = \frac{\sqrt{x+1} - 1}{x}$$

at $x = \mathbf{0}$. We accomplish this as follow:

$$f(\mathbf{0}) = \frac{\sqrt{\mathbf{0}+1} - 1}{\mathbf{0}}.$$

To simplify this, we apply the method of rationalization, viz.

$$\begin{aligned} f(\mathbf{0}) &= \frac{\sqrt{\mathbf{0}+1} - 1}{\mathbf{0}} \times \frac{\sqrt{\mathbf{0}+1} + 1}{\sqrt{\mathbf{0}+1} + 1} \\ &= \frac{(\sqrt{\mathbf{0}+1})^2 - 1^2}{\mathbf{0}(\sqrt{\mathbf{0}+1} + 1)} \\ &= \frac{\mathbf{0} + 1 - 1}{\mathbf{0}(\sqrt{\mathbf{0}+1} + 1)} \\ &= \frac{\mathbf{0}}{\mathbf{0}(\sqrt{\mathbf{0}+1} + 1)} \\ &= \frac{1}{\sqrt{\mathbf{0}+1} + 1} \\ &= \frac{1}{2} \end{aligned}$$

We desire to evaluate

$$\frac{2x^7 \sin x}{2 - x^4 - 2\cos x^2}$$

at $x = \mathbf{0}$. First we evaluate the numerator at $x = \mathbf{0}$:

$$\begin{aligned} 2 \cdot \mathbf{0}^7 \sin \mathbf{0} &= 2 \cdot \mathbf{0}^7 \left(\mathbf{0} - \frac{\mathbf{0}^3}{3!} + \frac{\mathbf{0}^5}{5!} - \cdots \right) \\ &= 2 \cdot \mathbf{0}^8 \left(1 - \frac{\mathbf{0}^2}{3!} + \frac{\mathbf{0}^4}{5!} - \cdots \right) \end{aligned}$$

which is absolute nothing. Next, we evaluate the denominator at $x = \mathbf{0}$:

$$\begin{aligned} 2 - \mathbf{0}^4 - 2\cos\mathbf{0}^2 &= 2 - \mathbf{0}^4 - 2\left(1 - \frac{\mathbf{0}^4}{2!} + \frac{\mathbf{0}^8}{4!} - \frac{\mathbf{0}^{12}}{6!} + \cdots\right) \\ &= 2 - \mathbf{0}^4 - 2 + \mathbf{0}^4 - \frac{\mathbf{0}^8}{12} + \frac{\mathbf{0}^{12}}{360} - \cdots \\ &= 2 - 2 - \mathbf{0}^4 + \mathbf{0}^4 - \frac{\mathbf{0}^8}{12} + \frac{\mathbf{0}^{12}}{360} - \cdots . \end{aligned}$$

We omit $2 - 2$ and $-\mathbf{0}^4 + \mathbf{0}^4$ since they are irreversibly equal to 0. Hence, we write

$$2 - \mathbf{0}^4 - 2\cos\mathbf{0}^2 = -\frac{\mathbf{0}^8}{12} + \frac{\mathbf{0}^{12}}{360} - \cdots = -\frac{\mathbf{0}^8}{12}\left(1 - \frac{\mathbf{0}^2}{30} + \cdots\right)$$

which is also absolute nothing. Thus the evaluation of

$$\frac{2x^7 \sin x}{2 - x^4 - 2\cos x^2}$$

at $x = \mathbf{0}$ is

$$\frac{2 \cdot \mathbf{0}^7 \sin\mathbf{0}}{2 - \mathbf{0}^4 - 2\cos\mathbf{0}^2} = \frac{2 \cdot \mathbf{0}^8\left(1 - \dfrac{\mathbf{0}^2}{3!} + \dfrac{\mathbf{0}^4}{5!} - \cdots\right)}{-\dfrac{0^8}{12}\left(1 - \dfrac{\mathbf{0}^2}{30} + \cdots\right)}$$

which becomes

$$\frac{2 \cdot \mathbf{0}^7 \sin\mathbf{0}}{2 - \mathbf{0}^4 - 2\cos\mathbf{0}^2} = \frac{2 \cdot \mathbf{0}^8}{-\dfrac{\mathbf{0}^8}{12}} = -24.$$

We wish to evaluate

$$\frac{b^x - a^x}{x}$$

at $x = \mathbf{0}$. This is done by first taking the series expansion of a^x:

$$a^x = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \cdots$$

Therefore the difference $b^x - a^x$ is expressed as:

$$b^x - a^x = 1 - 1 + (\ln b - \ln a)\, x + \frac{\left(\ln^2 b - \ln^2 a\right) x^2}{2!} + \cdots$$

which, omitting $1 - 1$ as it is irreversibly equal to 0, becomes

$$b^x - a^x = (\ln b - \ln a)\, x + \frac{\left(\ln^2 b - \ln^2 a\right) x^2}{2!} + \cdots .$$

Letting $x = \mathbf{0}$ gives

$$b^{\mathbf{0}} - a^{\mathbf{0}} = (\ln b - \ln a)\, \mathbf{0} + \frac{\left(\ln^2 b - \ln^2 a\right) \mathbf{0}^2}{2!} + \cdots .$$

Dividing both sides by $\mathbf{0}$ furnishes

$$\frac{b^{\mathbf{0}} - a^{\mathbf{0}}}{\mathbf{0}} = (\ln b - \ln a) + \frac{\left(\ln^2 b - \ln^2 a\right) \mathbf{0}}{2!} + \cdots$$

which becomes the final result

$$\frac{b^{\mathbf{0}} - a^{\mathbf{0}}}{\mathbf{0}} = \ln b - \ln a \tag{5.1}$$

since

$$\frac{\left(\ln^2 b - \ln^2 a\right)\mathbf{0}}{2!} + \cdots$$

is equal to naught.

5.2. **Zero Multiplied by Infinity.** Suppose we wish to evaluate

$$(x-1)\tan\frac{\pi}{2}x$$

at $x = 1$.Take the identity

$$\tan\frac{\pi}{2}x = -\cot\frac{\pi}{2}x - \frac{\pi}{2} = -\cot\frac{\pi}{2}(x-1).$$

and set $x = 1$. This furnishes

$$\tan\frac{\pi}{2}(1) = -\cot\frac{\pi}{2}(1-1)$$

which becomes

$$\tan\frac{\pi}{2}(1) = -\cot\left(\frac{\pi}{2}\cdot\mathbf{0}\right)$$

which becomes

$$\tan\frac{\pi}{2}(1) = -\left[\frac{1}{\left(\frac{\pi}{2}\cdot\mathbf{0}\right)} - \frac{\left(\frac{\pi}{2}\cdot\mathbf{0}\right)}{3} - \frac{\left(\frac{\pi}{2}\cdot\mathbf{0}\right)^3}{45} - \cdots\right].$$

The evaluation of $\tan\frac{\pi}{2}x$ at $x = 1$ is therefore

$$\tan\frac{\pi}{2}(1) = -\left[\frac{2}{\pi\cdot\mathbf{0}} - \frac{\pi\cdot\mathbf{0}}{6} - \frac{(\pi\cdot\mathbf{0})^3}{360} - \cdots\right] = -\frac{2}{\pi\cdot\mathbf{0}} = -\frac{2\boldsymbol{\infty}}{\pi}$$

which is an infinite number. Hence, we have the evaluation of

$$(x-1)\tan\frac{\pi}{2}x$$

at $x = 1$ as

$$\mathbf{0}\times\tan\frac{\pi}{2}(1) = -\mathbf{0}\left[\frac{2}{\pi\cdot\mathbf{0}} - \frac{\pi\cdot\mathbf{0}}{6} - \frac{(\pi\cdot\mathbf{0})^3}{360} - \cdots\right] = -\frac{2}{\pi}.$$

5.3. **Infinity Divided by Infinity.** Let us now evaluate

$$\frac{3x^2+5x-8}{7x^2-2x+1}$$

at $x = \boldsymbol{\infty}$. We do this as follows:

$$\begin{aligned}
\frac{3\cdot\boldsymbol{\infty}^2+5\cdot\boldsymbol{\infty}-8}{7\cdot\boldsymbol{\infty}^2-2\cdot\boldsymbol{\infty}+1} &= \frac{3(1/\mathbf{0})^2+5(1/\mathbf{0})-8}{7(1/\mathbf{0})^2-2(1/\mathbf{0})+1}\\
&= \frac{3/\mathbf{0}^2+5/\mathbf{0}-8}{7/\mathbf{0}^2-2/\mathbf{0}+1}\\
&= \frac{(3+5\cdot\mathbf{0}-8\cdot\mathbf{0}^2)/\mathbf{0}^2}{(7-2\cdot\mathbf{0}+\mathbf{0}^2)/\mathbf{0}^2}\\
&= \frac{3-5\cdot\mathbf{0}-8\cdot\mathbf{0}^2}{7-2\cdot\mathbf{0}+\mathbf{0}^2}\\
&= \frac{3}{7}
\end{aligned}$$

5.4. **Infinity Minus Infinity.** We next take up the evaluation of

$$\frac{1+x}{x-x^2}-\frac{1}{x+x^2}$$

at $x=\mathbf{0}$. We perform this as follows:

$$\begin{aligned}\frac{1+\mathbf{0}}{\mathbf{0}-\mathbf{0}^2}-\frac{1}{\mathbf{0}+0^2}&=\frac{(1+\mathbf{0})(\mathbf{0}+\mathbf{0}^2)-(\mathbf{0}-\mathbf{0}^2)}{(\mathbf{0}-\mathbf{0}^2)(\mathbf{0}+\mathbf{0}^2)}\\&=\frac{\mathbf{0}+\mathbf{0}^2+\mathbf{0}^2+\mathbf{0}^3-\mathbf{0}+\mathbf{0}^2}{\mathbf{0}^2-\mathbf{0}^4}\\&=\frac{\mathbf{0}-\mathbf{0}+\mathbf{0}^2+\mathbf{0}^2+\mathbf{0}^2+\mathbf{0}^3}{\mathbf{0}^2(1-\mathbf{0}^2)}\\&=\frac{3\cdot\mathbf{0}^2+\mathbf{0}^3}{\mathbf{0}^2(1-\mathbf{0}^2)}\\&=\frac{\mathbf{0}^2(3+\mathbf{0})}{\mathbf{0}^2(1-\mathbf{0}^2)}\\&=\frac{3+\mathbf{0}}{1-\mathbf{0}^2}\\&=3.\end{aligned}$$

We proceed further to evaluate

$$\frac{1}{x}-\frac{1}{e^x-1}$$

at $x=\mathbf{0}$. Setting $x=\mathbf{0}$ in the above expression gives the following:

$$\begin{aligned}\frac{1}{\mathbf{0}}-\frac{1}{e^{\mathbf{0}}-1}&=\frac{1}{\mathbf{0}}-\frac{1}{1+\mathbf{0}/1!+\mathbf{0}^2/2!+\mathbf{0}^3/3!+\cdots-1}\\&=\frac{1}{\mathbf{0}}-\frac{1}{1-1+\mathbf{0}/1!+\mathbf{0}^2/2!+\mathbf{0}^3/3!+\cdots}\\&=\frac{1}{\mathbf{0}}-\frac{1}{\mathbf{0}+\mathbf{0}^2/2+\mathbf{0}^3/6+\cdots}\\&=\frac{(\mathbf{0}+\mathbf{0}^2/2+\mathbf{0}^3/6+\cdots)-\mathbf{0}}{\mathbf{0}(\mathbf{0}+\mathbf{0}^2/2+\mathbf{0}^3/6+\cdots)}\\&=\frac{\mathbf{0}-\mathbf{0}+\mathbf{0}^2/2+\mathbf{0}^3/6+\cdots}{\mathbf{0}^2+\mathbf{0}^3/2+\mathbf{0}^4/6+\cdots}\\&=\frac{\mathbf{0}^2/2+\mathbf{0}^3/6+\cdots}{\mathbf{0}^2+\mathbf{0}^3/2+\mathbf{0}^4/6+\cdots}\\&=\frac{\mathbf{0}^2(1/2+\mathbf{0}/6+\cdots)}{\mathbf{0}^2(1+\mathbf{0}/2+\mathbf{0}^2/6+\cdots)}\\&=\frac{1/2+\mathbf{0}/6+\cdots}{1+\mathbf{0}/2+\mathbf{0}^2/6+\cdots}\\&=\frac{1}{2}.\end{aligned}$$

These examples suffice for the full comprehension of others.

6. Justifications of the Arithmetic of Zero and Infinity

6.1. **Euler Number e.** The operations $1+\mathbf{0}=1$ and $1-\mathbf{0}=1$ are rigidly valid in finite arithmetic. In infinite arithmetic, wherein infinite quantities are involved,

$$(1+\mathbf{0})^{\frac{1}{\mathbf{0}}} \neq 1 \quad \text{and} \quad (1-\mathbf{0})^{\frac{1}{\mathbf{0}}} \neq 1$$

but

$$(1+\mathbf{0})^{\frac{1}{\mathbf{0}}} = e \quad \text{and} \quad (1-\mathbf{0})^{\frac{1}{\mathbf{0}}} = \frac{1}{e}.$$

This is because of the further operations involving the $1/\mathbf{0}$ th power of the operations within the brackets.

We start with the binomial expansion

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{(1-1/n)}{2!}+\frac{(1-1/n)(1-2/n)}{3!}+\cdots.$$

Letting $n=1/\mathbf{0}$ gives

$$\left(1+\frac{1}{1/\mathbf{0}}\right)^{\frac{1}{\mathbf{0}}} = 1+1+\frac{\left(1-\frac{1}{1/\mathbf{0}}\right)}{2!}+\frac{\left(1-\frac{1}{1/\mathbf{0}}\right)\left(1-\frac{1}{1/\mathbf{0}}\right)}{3!}+\cdots$$

which becomes

$$(1+\mathbf{0})^{\frac{1}{\mathbf{0}}} = 1+1+\frac{(1-\mathbf{0})}{2!}+\frac{(1-\mathbf{0})(1-2\cdot\mathbf{0})}{3!}+\cdots = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots = e.$$

6.2. **Series Expansion of $\ln(1+x)$.** We start with the expansion

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots.$$

Letting $\alpha=0$ gives

$$(1+x)^{\mathbf{0}} = 1+\mathbf{0}x+\frac{\mathbf{0}(\mathbf{0}-1)}{2!}x^2+\frac{\mathbf{0}(\mathbf{0}-1)(\mathbf{0}-2)}{3!}x^3+\cdots \tag{6.1}$$

which actually becomes

$$(1+x)^{\mathbf{0}} = 1$$

since the rest terms are all absolute nothing. But because further operations impend, we must strictly uphold (6.1) and use it in these operations.

Now consider the series expansion

$$a^y = 1+y\ln a+\frac{y^2\ln^2 a}{2!}+\cdots.$$

Setting $a=1+x$ gives

$$(1+x)^y = 1+y\ln(1+x)+\frac{y^2\ln^2(1+x)}{2!}+\cdots$$

and letting $y=0$ gives

$$(1+x)^{\mathbf{0}} = 1+0\ln(1+x)+\frac{\mathbf{0}^2\ln^2(1+x)}{2!}+\cdots. \tag{6.2}$$

Equating (6.1) and (6.2) furnishes

$$1+\mathbf{0}x+\frac{\mathbf{0}(\mathbf{0}-1)}{2!}x^2+\frac{\mathbf{0}(\mathbf{0}-1)(\mathbf{0}-2)}{3!}x^3+\cdots = 1+\mathbf{0}\ln(1+x)+\frac{\mathbf{0}^2\ln^2(1+x)}{2!}+\cdots$$

which becomes

$$\mathbf{0}x+\frac{\mathbf{0}(\mathbf{0}-1)}{2!}x^2+\frac{\mathbf{0}(\mathbf{0}-1)(\mathbf{0}-2)}{3!}x^3+\cdots=1-1+\mathbf{0}\ln(1+x)+\frac{0^2\ln^2(1+x)}{2!}+\cdots.$$

The expression $1-1$ is omitted as it is irreversibly equal to 0. Thus we write

$$\mathbf{0}x+\frac{\mathbf{0}(\mathbf{0}-1)}{2!}x^2+\frac{\mathbf{0}(\mathbf{0}-1)(\mathbf{0}-2)}{3!}x^3+\cdots=\mathbf{0}\ln(1+x)+\frac{\mathbf{0}^2\ln^2(1+x)}{2!}+\cdots.$$

Let us now divide both sides of the above equations by $\mathbf{0}$. Accomplishing this, we have

$$x+\frac{(\mathbf{0}-1)}{2!}x^2+\frac{(\mathbf{0}-1)(\mathbf{0}-2)}{3!}x^3+\cdots=\ln(1+x)+\frac{\mathbf{0}\ln^2(1+x)}{2!}+\cdots.$$

Since this is the terminus of our calculation, we make $\mathbf{0}$ display the properties of 0. Hence, the equation above becomes

$$x+\frac{(-1)}{2!}x^2+\frac{(-1)(-2)}{3!}x^3+\cdots=\ln(1+x)$$

which, being simplified and rearranged, becomes

$$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots.$$

6.3. **Series Expansion of** $\ln(1+y/x)$**.** We start with the binomial theorem

$$(x+y)^n=x^n+\frac{n!}{1!(n-1)!}x^{n-1}y+\frac{n!}{2!(n-2)!}x^{n-2}y^2+\frac{n!}{3!(n-3)!}x^{n-3}y^3+\ldots$$

which can be rewritten as

$$(x+y)^n-x^n=\frac{n!}{1!(n-1)!}x^{n-1}y+\frac{n!}{2!(n-2)!}x^{n-2}y^2+\frac{n!}{3!(n-3)!}x^{n-3}y^3+\ldots$$

Letting $n=\mathbf{0}$ gives

$$(x+y)^{\mathbf{0}}-x^{\mathbf{0}}=\frac{\mathbf{0}!}{1!(\mathbf{0}-1)!}x^{\mathbf{0}-1}y+\frac{\mathbf{0}!}{2!(\mathbf{0}-2)!}x^{\mathbf{0}-2}y^2+\frac{\mathbf{0}!}{3!(\mathbf{0}-3)!}x^{\mathbf{0}-3}y^3+\ldots$$

which becomes

$$(x+y)^{\mathbf{0}}-x^{\mathbf{0}}=\frac{1}{1!(-1)!}x^{-1}y+\frac{1}{2!(-2)!}x^{-2}y^2+\frac{1}{3!(-3)!}x^{-3}y^3+\ldots.$$

Applying this we get

$$(x+y)^{\mathbf{0}}-x^{\mathbf{0}}=\frac{\mathbf{0}}{1!}x^{-1}y+\frac{-1!\cdot\mathbf{0}}{2!}x^{-2}y^2+\frac{2!\cdot 0}{3!}x^{-3}y^3+\ldots$$

which becomes

$$(x+y)^{\mathbf{0}}-x^{\mathbf{0}}=\frac{\mathbf{0}}{1}x^{-1}y-\frac{\mathbf{0}}{2}x^{-2}y^2+\frac{\mathbf{0}}{3}x^{-3}y^3-\ldots$$

which in turn, dividing by $\mathbf{0}$, becomes

$$\frac{(x+y)^{\mathbf{0}} - x^{\mathbf{0}}}{\mathbf{0}} = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \cdots.$$

Thus, applying (5.1), we obtain

$$\ln(x+y) - \ln x = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \ldots$$

which becomes

$$\ln\left(\frac{x+y}{x}\right) = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \ldots$$

which finally becomes

$$\ln\left(1+\frac{y}{x}\right) = \frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 - \cdots.$$

6.4. **Sum of Alternating Harmonic Series.** In one of his works, Euler showed that

$$1^n - 2^n + 3^n - 4^n + \cdots = (-1)^s \frac{2^{n+1}-1}{n+1} B_{n+1}$$

where $s = [(n+1)/2]$ is the integer part of $(n+1)/2$ and B_n is the nth Bernoulli number. Setting $n = -1$ gives

$$1^{-1} - 2^{-1} + 3^{-1} - 4^{-1} + \cdots = (-1)^{[(-1+1)/2]} \frac{2^{-1+1}-1}{-1+1} B_{-1+1}$$

which simplifies to

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = (-1)^{[(\mathbf{0})/2]} \frac{2^{\mathbf{0}} - 1^{\mathbf{0}}}{\mathbf{0}} B_{\mathbf{0}}.$$

With the understanding that $B_{\mathbf{0}} = 1$ and considering that

$$\frac{2^{\mathbf{0}} - 1^{\mathbf{0}}}{\mathbf{0}} = \ln 2 - \ln 1 = \ln 2,$$

we get

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

6.5. **Infinite Sum of the Harmonic Series.** We start with the Taylor series expansion of $\ln(x+1)$:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$

and let $x = -1$.

Accomplishing these, we obtain the following:

$$\ln \mathbf{0} = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$$

which becomes

$$-\ln \mathbf{0} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots.$$

Employing $-\ln \mathbf{0} = \ln\left(\frac{1}{\mathbf{0}}\right) = \ln(-1)!$, we arrived at the required result

$$\ln(-1)! = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Let us now turn to the derivation of a famous formula in analysis in order to give the reader an idea of the flavor of $\ln(-1)!$. There is a very interesting formula discovered by Euler in his 1776 paper, which presents a beautiful means of computing Euler's constant γ. This formula is

$$1-\gamma=\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}.$$

We now proceed to derive this formula and we begin from the Maclaurin series expansion for $\ln(x)!$ which reads

$$\ln x!=-\gamma x+\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)}{k}x^k.$$

If we let $x=-1$, we obtain the result

$$\ln(-1)!=\gamma+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}.$$

We have already established that the natural logarithm of $(-1)!$ is the sum of the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$. If we then replace $\ln(-1)!$ with the sum $\sum_{k=1}^{\infty}\frac{1}{k}$, we procure for ourselves

$$\sum_{k=1}^{\infty}\frac{1}{k}=\gamma+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}$$

which results in

$$1+\sum_{k=2}^{\infty}\frac{1}{k}-\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}=\gamma$$

which in turn furnishes our required formula

$$1+\sum_{k=2}^{\infty}\frac{1-\zeta(k)}{k}=\gamma$$

or

$$1-\gamma=\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}.$$

To be more convinced of our free use of the sum $\ln(-1)!$ of the harmonic series, we employ it again in the derivation of this same formula by taking another pathway. We begin with the familiar identity

$$H_x=\sum_{k=1}^{\infty}(-1)^{k+1}x^k\zeta(k+1)$$

and integrate both sides of it with respect to x, that is, we find

$$\int_{\mathbf{0}}^{n}H_x\,dx=\int_{\mathbf{0}}^{n}\sum_{k=1}^{\infty}(-1)^{k+1}x^k\zeta(k+1)\,dx$$

where H_x is the xth harmonic number. We apply the familiar relation

$$\int_{\mathbf{0}}^{n}H_x\,dx=n\gamma+\ln(n!) \tag{6.3}$$

and get

$$n\gamma + \ln(n!) = \int_{\mathbf{0}}^{n} \sum_{k=1}^{\infty} (-1)^{k+1} x^k \zeta(k+1)\, dx$$

which becomes

$$\begin{aligned} n\gamma + \ln(n!) &= \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(k+1) \int_{\mathbf{0}}^{n} x^k\, dx \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(k+1) \left[\frac{x^{k+1}}{k+1}\right]_{\mathbf{0}}^{n} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \zeta(k+1)\, n^{k+1}}{k+1}. \end{aligned}$$

Let us now set $n = -1$. We obtain

$$-\gamma + \ln(-1)! = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \zeta(k+1)\, (-1)^{k+1}}{k+1}$$

which furnishes

$$\gamma = \ln(-1)! - \sum_{k=1}^{\infty} \frac{\zeta(k+1)}{k+1}.$$

We set $k+1 = k$ and get

$$\gamma = \ln(-1)! - \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}.$$

Finally, setting

$$\ln(-1)! = \sum_{k=1}^{\infty} \frac{1}{k}$$

, we obtain

$$\gamma = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}$$

which, taking an easily construed step, becomes our proposed formula:

$$1 - \gamma = \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}.$$

6.6. **Riemann Zeta Constants in Relation to Bernoulli Numbers.** A well known functional equation is that which pertains to the famous Riemann zeta function ζ

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

where Γ is the familiar gamma function expressed in the factorial function as

$$\Gamma(s) = (s-1)!.$$

From this functional equation we can, by means zero and infinity, derive the famous formula

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

where B_{2n} is a Bernoulli number.

We start with the evaluation of

$$\sin\left(\frac{\pi s}{2}\right)$$

at $s = 2n$ where n is a natural number. Take the identity

$$\sin\left(\frac{\pi}{2}s - \frac{\pi}{2}c\right) = \sin\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{2}c\right) - \cos\left(\frac{\pi}{2}s\right)\sin\left(\frac{\pi}{2}c\right).$$

If we let $c = 2n$, we get

$$\sin\left(\frac{\pi}{2}s - \frac{\pi}{2}(2n)\right) = \sin\left[\frac{\pi}{2}s\right]\cos\left[\frac{\pi}{2}(2n)\right] - \cos\left[\frac{\pi}{2}sx\right]\sin\left[\frac{\pi}{2}(2n)\right]$$

which simplifies to

$$\sin\frac{\pi}{2}(s-2n) = \sin\left(\frac{\pi}{2}s\right)\cos(\pi n) - \cos\left(\frac{\pi}{2}s\right)\sin(\pi n)$$

which in turns becomes

$$\sin\frac{\pi}{2}(s-2n) = \sin\left(\frac{\pi}{2}s\right)\cos(\pi n).$$

This is rewritten as

$$\sin\left(\frac{\pi}{2}s\right) = (-1)^n \sin\frac{\pi}{2}(s-2n),$$

noting that $\cos(\pi n) = (-1)^n$ for $n = 1, 2, 3, \ldots$. Setting $s = 2n$ gives

$$\sin\left(\frac{\pi}{2}(2n)\right) = (-1)^n \sin\frac{\pi}{2}(2n-2n)$$

which turns into

$$\sin\left(\frac{\pi}{2}(2n)\right) = (-1)^n \sin\left(\frac{\pi}{2}\cdot\mathbf{0}\right)$$

which finally becomes

$$\sin\left(\frac{\pi}{2}(2n)\right) = (-1)^n\left[\frac{\pi}{2}\cdot\mathbf{0} - \frac{1}{3!}\left(\frac{\pi}{2}\cdot\mathbf{0}\right)^3 + \cdots\right]. \tag{6.4}$$

We now proceed find the expression for the function $\Gamma(1-2n)$. First of all, we note that $\Gamma(1-s) = (-s)!$. If p is any positive integer, then letting $s = p$, as already stated in section, furnishes

Letting $p = 2n$, we obtain

$$\Gamma(1-2n) = \frac{1}{(2n-1)!}\cdot\frac{1}{\mathbf{0}}. \tag{6.5}$$

It remains to evaluate the already mentioned Riemann functional equation at $s = 2n$. We thus have

$$\zeta(2n) = 2^{2n}\pi^{2n-1}\sin\left(\frac{\pi}{2}(2n)\right)\Gamma(1-2n)\zeta(1-2n)$$

which, applying (6.4) and (6.5), becomes

$$\zeta(2n) = 2^{2n}\pi^{2n-1}(-1)^n\left[\frac{\pi}{2}\cdot\mathbf{0} - \frac{1}{3!}\left(\frac{\pi}{2}\cdot\mathbf{0}\right)^3 + \cdots\right]\cdot\frac{1/\mathbf{0}}{(2n-1)!}\zeta(1-2n)$$

which becomes

$$\zeta(2n) = 2^{2n}\pi^{2n-1}(-1)^n\left[\frac{\pi}{2} - \frac{1}{3!}\left(\frac{\pi}{2}\right)^3\cdot\mathbf{0}^2 + \cdots\right]\cdot\frac{1}{(2n-1)!}\zeta(1-2n)$$

which simplifies to

$$\zeta(2n) = \frac{2^{2n}\pi^{2n-1}(-1)^n\pi}{2\cdot(2n-1)!}\zeta(1-2n)$$

or

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}(-1)^n}{(2n-1)!}\zeta(1-2n). \tag{6.6}$$

One familiar result in the analysis of the Riemann zeta function is

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

where n is a positive integer. If we let $n = 1-2n$, we get

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}. \tag{6.7}$$

Eliminating $\zeta(1-2n)$ from (6.6) and (6.7), we get

$$\zeta(2n) = -\frac{2^{2n}\pi^{2n}(-1)^n}{(2n-1)!} \cdot \frac{-B_{2n}}{2n}.$$

which becomes

$$\zeta(2n) = \frac{2^{2n}\pi^{2n}(-1)^{n+1}}{2n(2n-1)!}B_{2n}.$$

which in its turn becomes

$$\zeta(2n) = \frac{(-1)^{n+1}2^{2n-1}\pi^{2n}}{(2n)!}B_{2n}.$$

This identity becomes the final and required

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}.$$

.

6.7. **Fourier series.** Here, we are concerned with the use of trigonometric Fourier series in assigning sums to infinite series. But before we do this, it is necessary to first evaluate the function $\sin nx^\circ$ as $x = 180$, where $n = 1, 2, 3, \ldots$. We begin with the identity

$$\sin(nx - n\pi) = \sin nx \cos n\pi - \sin n\pi \cos nx.$$

For $n = 1, 2, 3, \ldots$, we have

$$\sin n(x-\pi) = (-1)^n \sin nx.$$

Letting $x = \pi$ gives

$$\sin n(\pi - \pi) = (-1)^n \sin n\pi$$

which becomes

$$\sin n \cdot \mathbf{0} = (-1)^n \sin n\pi$$

which in turns becomes

$$\sin n\pi = (-1)^n \sin n \cdot \mathbf{0}.$$

Applying the Taylor series expansion of $\sin n \cdot \mathbf{0}$, we get

$$\sin n\pi = (-1)^n n \cdot \mathbf{0}\left[1 - \frac{(n\cdot\mathbf{0})^2}{3!} + \cdots\right].$$

6.7.1. *Grandi's Series.* Let us now consider a problem which utilizes the result (**??**) in its solution that we may admire its utility. We wish to use the special Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}, \qquad \mathbf{0} < x < 2\pi \tag{6.8}$$

to determine the sum of the Grandi's series which has caused intense dispute and hot debates among mathematicians. Grandi was the first to treat this series, and he assigned the sum $1/2$ to it, but some disagreed, asserting that the series has no sum as it is divergent. However, Leibniz and Euler concurred with Grandi that the series in question has the sum of $1/2$ as it is the value which arises from the series expansion

$$1 - x + x^2 - \ldots = \frac{1}{1+x}$$

setting $x = 1$. Here, we shall show that the sum of the series is actually $1/2$.

Let us set the variable $x = \pi$ in (6.8). Performing this, we obtain

$$\sum_{n=1}^{\infty} \frac{\sin n\pi}{n} = \frac{\pi - \pi}{2}$$

which becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n n \cdot \mathbf{0}}{n} \left[1 - \frac{(n \cdot \mathbf{0})^2}{3!} + \cdots\right] = \frac{-\mathbf{0}}{2}$$

which, dividing both sides by $-\mathbf{0}$, turns into

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[1 - \frac{(n \cdot \mathbf{0})^2}{3!} + \cdots\right] = \frac{1}{2}.$$

which in its own turn becomes

$$\sum_{n=1}^{\infty} (-1)^{n-1} = \frac{1}{2}$$

or

$$1 - 1 + 1 - \cdots = \frac{1}{2}.$$

6.7.2. *Alternating Basel Series.* For more assurance of our technique, let us use the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n^3} = \frac{t(2\pi - t)(\pi - t)}{12}, \qquad \mathbf{0} < t < 2\pi$$

to show that the alternating Basel series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{12}.$$

Setting $t = \pi$ in the above Fourier series, we get

$$\sum_{n=1}^{\infty} \frac{\sin n\pi}{n^3} = \frac{\pi(2\pi - \pi)(\pi - \pi)}{12}$$

which becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n n \cdot \mathbf{0}}{n^3} \left[1 - \frac{(n \cdot \mathbf{0})^2}{3!} + \cdots\right] = \frac{\pi(\pi)(-\mathbf{0})}{12}$$

which, dividing both sides by $-\mathbf{0}$, becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \left[1 - \frac{(n \cdot \mathbf{0})^2}{3!} + \cdots\right] = \frac{\pi^2}{12}$$

This equation becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

or

$$1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{12}.$$

6.8. **Euler's constant and Zeta Constants.** One classical identify which relates γ to $\zeta(n)$ is exceedingly useful in deriving other fine formulas for γ, and this is the striking relation

$$\sum_{k=2}^{\infty} \frac{x^k \zeta(k)}{k} = \ln\left(\frac{\pi x}{\sin(\pi x)}\right) - \gamma x - \ln(x!). \tag{6.9}$$

We wish to use the analysis of zero and infinity together with this result to derive some known results.

Let us start with the evaluation of $\sin(x\pi)$ at $x = 1, 2, 3, \ldots$. Take the identity

$$\sin(\pi y - \pi x) = \sin \pi y \cos \pi x - \sin \pi x \cos \pi y.$$

For $x = 1, 2, 3, \ldots$, we have

$$\sin \pi(y - x) = (-1)^x \sin \pi y.$$

Letting $y = x$ gives

$$\sin \pi(x - x) = (-1)^x \sin \pi x.$$

which becomes

$$\sin \pi \cdot \mathbf{0} = (-1)^x \sin \pi x.$$

which in turns becomes

$$\sin \pi x = (-1)^x \sin \pi \cdot \mathbf{0}.$$

Applying the Taylor series expansion of $\sin \pi \cdot \mathbf{0}$, we get

$$\sin \pi x = (-1)^x \pi \cdot \mathbf{0} \left[1 - \frac{(\pi \cdot \mathbf{0})^2}{3!} + \cdots\right].$$

We apply the result above to derive the familiar identify

$$\Gamma(n)\,\Gamma(1 - n) = \frac{\pi}{\sin(n\pi)} \tag{6.10}$$

where $\Gamma(x)$ is the famous gamma function. A useful property of the gamma function is the recursive relation

$$\Gamma(x + 1) = x\Gamma(x), \qquad x > \mathbf{0}. \tag{6.11}$$

When x is a positive integer, say $x = n$, then the recursive relation (6.11) can be repeatedly applied to obtain

$$\Gamma(n) = n!. \tag{6.12}$$

We now apply the aforementioned formula (**??**):

$$(-n)! = \frac{(-1)!}{(-1)^{n-1}(n-1)!} \tag{6.13}$$

which can be used to express the factorial of any negative integer in terms of $(-1)!$. We rearrange this result as

$$(n-1)!\,(-n)! = \frac{(-1)!}{(-1)^{n-1}}$$

which, setting $(-1)! = 1/\mathbf{0}$, gives us

$$(n-1)!\,(-n)! = \frac{1}{(-1)^{n-1}\cdot\mathbf{0}}.$$

Multiplying the numerator and denominator of the fraction at the right hand side of the equation by π gives

$$(n-1)!\,(-n)! = \frac{\pi}{(-1)^{n-1}\,\pi\cdot\mathbf{0}}.$$

We now set

$$(-1)^{n-1}\pi\cdot\mathbf{0} = \frac{\sin(n\pi)}{\left[1-\dfrac{(\pi\cdot\mathbf{0})^2}{3!}+\cdots\right]}$$

and obtain

$$(n-1)!\,(-n)! = \frac{\pi\left[1-\dfrac{(\pi\cdot\mathbf{0})^2}{3!}+\cdots\right]}{\sin(n\pi)},$$

which, transforming the factorial functions on the left-hand side of the equation to the gamma functions, becomes the required identity

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}.$$

Armed with the ability to express $\sin(x\pi)$ in terms of zero for positive integers x, we turn to the derivation of notable formulas which connect γ to $\zeta(k)$ as we have aforementioned. In all the cases we shall always begin with (6.9).

First, we set $x=-1$ in (6.9) and obtain

$$\begin{aligned}
\sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)}{k} &= \ln\left(\frac{-\pi}{\sin(-\pi)}\right)+\gamma-\ln(-1)!\\
&= \ln\left(\frac{-\pi}{-\sin\pi}\right)+\gamma-\ln(-1)!\\
&= \ln\left(\frac{\pi}{\sin\pi}\right)+\gamma-\ln(-1)!\\
&= \ln\left[\frac{\pi}{\pi\cdot\mathbf{0}\left(1-\dfrac{(\pi\cdot\mathbf{0})^2}{3!}+\cdots\right)}\right]+\gamma-\ln(-1)!\\
&= \ln\left(\frac{1}{0}\right)-\ln\left(1-\frac{(\pi\cdot\mathbf{0})^2}{3!}+\cdots\right)+\gamma-\ln(-1)!\\
&= \ln(-1)!+\gamma-\ln(-1)!\\
&= \gamma.
\end{aligned}$$

Let now $x = 1$. We get

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{\zeta(k)}{k} &= \ln\left(\frac{\pi}{\sin(\pi)}\right) - \gamma - \ln 1! \\
&= \ln\left[\frac{\pi}{\pi \cdot \mathbf{0}\left(1 - \frac{(\pi \cdot \mathbf{0})^2}{3!} + \cdots\right)}\right] - \gamma - \ln 1! \\
&= \ln\left(\frac{1}{\mathbf{0}}\right) - \ln\left(1 - \frac{(\pi \cdot \mathbf{0})^2}{3!} + \cdots\right) - \gamma \\
&= \ln(-1)! - \gamma \\
&= \sum_{k=1}^{\infty} \frac{1}{k} - \gamma \\
&= 1 + \sum_{k=2}^{\infty} \frac{1}{k} - \gamma.
\end{aligned}$$

We conclude this section with the derivation of the formula which was first derived by Euler. Setting $x = -x$, we get

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{(-x)^k \zeta(k)}{k} &= \ln\left(\frac{-x\pi}{\sin(-x\pi)}\right) + \gamma x - \ln(-x)! \\
&= \ln\left(\frac{-x\pi}{-\sin(x\pi)}\right) + \gamma x - \ln(-x)! \\
&= \ln\left(\frac{x\pi}{\sin(x\pi)}\right) + \gamma x - \ln(-x)! \\
&= \ln\left[\frac{x\pi}{(-1)^{x-1}\pi \cdot \mathbf{0}\left(1 - \frac{(\pi \cdot \mathbf{0})^2}{3!} + \cdots\right)}\right] + \gamma x - \ln(-x)! \\
&= \ln\left(\frac{x}{(-1)^{x-1} \cdot \mathbf{0}}\right) - \ln\left(1 - \frac{(\pi \cdot \mathbf{0})^2}{3!} + \cdots\right) + \gamma x - \ln(-x)! \\
&= \ln\left(\frac{x(-1)!}{(-1)^{x-1}}\right) + \gamma x - \ln(-x)! \\
&= \ln\left(\frac{x(-1)!(x-1)!}{(-1)^{x-1}(x-1)!}\right) + \gamma x - \ln(-x)! \\
&= \ln\left(\frac{x!(-1)!}{(-1)^{x-1}(x-1)!}\right) + \gamma x - \ln(-x)! \\
&= \ln(x!(-x)!) + \gamma x - \ln(-x)! \\
&= \ln(x!) + \ln(-x)! + \gamma x - \ln(-x)! \\
&= \ln(x!) + \gamma x.
\end{aligned}$$

6.9. **Euler's Constant.** Here we use the way of zero and infinity to derive the famous identity

$$\gamma = H_\Omega - \ln \Omega$$

where Ω is the infinite integer for which $H_\Omega = 1 + 1/2 + 1/3 + \cdots + 1/\Omega$ is the harmonic series and $\gamma = 0.57721\ldots$ is the Euler's constant. We start with the sums of powers formula

$$\sum_{k=1}^{n} k^m = \frac{(B+n)^{m+1} - B^{m+1}}{m+1}$$

where B^m equals the mth Bernoulli number B_m. If we set $m = -1$, we get

$$\sum_{k=1}^{n} k^{-1} = \frac{(B+n)^{-1+1} - B^{-1+1}}{-1+1}$$

which becomes

$$\sum_{k=1}^{n} k^{-1} = \frac{(B+n)^{\mathbf{0}} - B^{\mathbf{0}}}{\mathbf{0}}.$$

If we consider (5.1), the above result becomes

$$\sum_{k=1}^{n} \frac{1}{k} = \ln(B+n) - \ln B = \ln\left(\frac{B+n}{B}\right) = \ln\left(1 + \frac{n}{B}\right).$$

This, setting $\sum_{k=1}^{n} \frac{1}{k} = H_n$ where H_n is the nth harmonic number, becomes

$$H_n = \ln\left(\frac{B+n}{B}\right) = \ln\left(1 + \frac{n}{B}\right). \tag{6.14}$$

The question now is, What is B? To answer this question we need to express B in terms of n and set $n = 1, 2, 3, \ldots$ to see what would happen to B. Now B expressed as the subject is

$$B = \frac{n}{e^{H_n} - 1}$$

where e is Euler's number. Computing B for the first few values of n, we observe that B varies with n. We conclude that B is a variable depending on n. Let now B be rewritten as the function $B(n)$. The above equation becomes

$$B(n) = \frac{n}{e^{H_n} - 1}.$$

It remains to compute the functional value of $B(n)$ when H_n becomes the harmonic series $H_\Omega = 1 + 1/2 + 1/3 + \cdots + 1/\Omega$. To perform this we construct a table of values of $B(n)$ as n becomes larger and larger. From this table, we see that as n becomes larger and larger, $B(n)$ becomes closer and closer to $0.561459\ldots = e^{-\gamma}$ where $e = 2.71828\ldots$. Let Ω be the value of n for which H_Ω is the harmonic series. It follows that

$$B(\Omega) = 0.561459\ldots = e^{-\gamma}.$$

We see immediately that, setting $n = \Omega$ in (6.14), the harmonic number becomes

$$H_\Omega = \ln\left(1 + \frac{\Omega}{e^{-\gamma}}\right)$$

which becomes

$$H_\Omega = \ln\left(1 + \Omega e^{\gamma}\right).$$

n	H_n	$B(n)$
10	$2.92896\ldots$	$0.62117\ldots$
100	$5.18737\ldots$	$0.56742\ldots$
1000	$7.48547\ldots$	$0.56205\ldots$
10000	$9.78760\ldots$	$0.56151\ldots$
100000	$12.0901\ldots$	$0.56146\ldots$
1000000	$14.3927\ldots$	$0.56146\ldots$
$\vdots$	$\vdots$	$\vdots$

TABLE 2. Values of $B(n)$

Since Ω is an infinite quantity, it is immutable in the presence of finite addends or minuends. Thus the above equation becomes

$$H_\Omega = \ln(\Omega e^\gamma)$$

which becomes the required identity

$$\gamma = H_\Omega - \ln\Omega.$$

6.10. **Generalizing a Family of Figurate Numbers.** A family of figurate numbers defined as

$$P_r(n) = \binom{n+r-1}{r}$$

is well worthy of our attention. One enchanting feature of these figurate numbers is that if the nth term of a sequence of any given r- figurate numbers be added to the $(n+1)$th term of the sequence of the preceding r- figurate numbers, the sum will be equal to the $(n+1)$th term of the sequence of the given r- figurate numbers. As an instance, let us take two sequences of the triangular numbers and the tetrahedral numbers:

$$1, \quad 3, \quad 6, \quad 10, \quad 15, \quad \ldots$$

$$1, \quad 4, \quad 10, \quad 20, \quad 35, \quad \ldots.$$

Here, if we add to any term in the upper sequence that term in the lower which stands one place to the left, the sum is the next term in the lower sequence. Starting with 6 sequences of 1's, all of the sequences of figurate numbers may be deduced in succession by the aid of this principle:

$r = \mathbf{0}$	:	1	1	1	1	1	1	1	1
$r = 1$	:	1	2	3	4	5	6	7	8
$r = 2$	:	1	3	6	10	15	21	28	36
$r = 3$	:	1	4	10	20	35	56	84	120
$r = 4$	:	1	5	15	35	70	126	210	330
$r = 5$	:	1	6	21	56	126	252	462	792.

By the just convention laid down that $(\mathbf{0}/\mathbf{0} = 1)$ we shall generalize these figurate numbers. Let us consider the cases where r is a negative integer. We begin by setting $r = -m$, that is

$$P_{-m}(n) = \binom{n-m-1}{-m}$$

which becomes

$$P_{-m}(n) = \binom{n-m-1}{-m}.$$

Before we proceed further let us first express the factorial of negative integers, namely $(-m)!$, in terms of $(-1)!$ since it will inevitably show up here. We start with the recurrence relation:

$$[-(m-1)]! = -(m-1)(-m)!$$

which turns into

$$(-m)! = -\frac{[-(m-1)]!}{m-1}.$$

Letting $m = 2$ gives

$$(-2)! = -\frac{(-1)!}{1!} = -\frac{1}{\mathbf{0}}.$$

and letting $m = 3$ gives

$$(-3)! = -\frac{(-2)!}{2!} = \frac{1}{2 \cdot \mathbf{0}}.$$

Similarly, we get

$$(-4)! = -\frac{1}{3! \cdot \mathbf{0}},$$
$$(-5)! = \frac{1}{4! \cdot \mathbf{0}},$$

and in general

$$(-m)! = (-1)^{m-1}\frac{1}{(m-1)! \cdot \mathbf{0}}.$$

We now express the binomial coefficient $P_{-m}(n)$ in terms of $\mathbf{0}$ as follows:

$$\begin{aligned} P_{-m}(n) &= \frac{(n-m-1)!}{(-m)!(n-1)!} \\ &= \frac{(n-m-1)!}{\left[\frac{(-1)^{m-1}}{(m-1)! \cdot \mathbf{0}}\right] \cdot (n-1)!} \\ &= \frac{(-1)^{m-1}(m-1)!(n-m-1)! \cdot \mathbf{0}}{(n-1)!}. \end{aligned}$$

Letting $m = 1$, we get

$$P_{-1}(n) = \frac{(-1)^{1-1}(1-1)!(n-1-1)! \cdot \mathbf{0}}{(n-1)!}$$

which becomes

$$\begin{aligned} P_{-1}(n) &= \frac{\mathbf{0}!(n-2)! \cdot \mathbf{0}}{(n-1)!} \\ &= \frac{\mathbf{0}}{n-1}. \end{aligned}$$

Setting $n = 1, 2, 3, \ldots$ gives

$$\begin{aligned}
P_{-1}(1) &= \frac{\mathbf{0}}{1-1} = \frac{\mathbf{0}}{\mathbf{0}} = 1 \\
P_{-1}(2) &= \frac{\mathbf{0}}{2-1} = \frac{\mathbf{0}}{1} = 0 \\
P_{-1}(3) &= \frac{\mathbf{0}}{3-1} = \frac{\mathbf{0}}{2} = 0 \\
P_{-1}(4) &= \frac{\mathbf{0}}{4-1} = \frac{\mathbf{0}}{3} = 0 \\
P_{-1}(5) &= \frac{\mathbf{0}}{5-1} = \frac{\mathbf{0}}{4} = 0
\end{aligned}$$

and so on. The case where $m = 1$ consists of only one finite figurate number, namely, 1. The rest numbers are all absolute nothing.

Letting $m = 2$, we get

$$P_{-2}(n) = \frac{(-1)^{2-1}(2-1)!(n-2-1)! \cdot \mathbf{0}}{(n-1)!}$$

which becomes

$$\begin{aligned}
P_{-2}(n) &= \frac{-1!(n-3)! \cdot \mathbf{0}}{(n-1)!} \\
&= \frac{-\mathbf{0}}{(n-1)(n-2)}.
\end{aligned}$$

Setting $n = 1, 2, 3, \ldots$ gives

$$\begin{aligned}
P_{-2}(1) &= \frac{-\mathbf{0}}{(1-1)(1-2)} = \frac{-\mathbf{0}}{(\mathbf{0})(-1)} = 1 \\
P_{-2}(2) &= \frac{-\mathbf{0}}{(2-1)(2-2)} = \frac{-\mathbf{0}}{(1)(\mathbf{0})} = -1 \\
P_{-2}(3) &= \frac{-\mathbf{0}}{(3-1)(3-2)} = \frac{-\mathbf{0}}{(2)(1)} = \frac{-\mathbf{0}}{2} = 0 \\
P_{-2}(4) &= \frac{-\mathbf{0}}{(4-1)(4-2)} = \frac{-\mathbf{0}}{(3)(2)} = \frac{-\mathbf{0}}{6} = 0 \\
P_{-2}(5) &= \frac{-\mathbf{0}}{(5-1)(5-2)} = \frac{-\mathbf{0}}{(4)(3)} = \frac{-\mathbf{0}}{12} = 0
\end{aligned}$$

and so on. For the case where $m = 2$ there exists only two finite figurate numbers, namely, 1 and -1. The other numbers are all absolute nothing.

Letting $m = 3$, we get

$$P_{-3}(n) = \frac{(-1)^{3-1}(3-1)!(n-3-1)! \cdot \mathbf{0}}{(n-1)!}$$

which becomes

$$\begin{aligned}
P_{-3}(n) &= \frac{2!(n-4)! \cdot \mathbf{0}}{(n-1)!} \\
&= \frac{2 \cdot \mathbf{0}}{(n-1)(n-2)(n-3)}.
\end{aligned}$$

Setting $n = 1, 2, 3, \ldots$ gives

$$P_{-3}(1) = \frac{2 \cdot \mathbf{0}}{(1-1)(1-2)(1-3)} = \frac{2 \cdot 0}{(\mathbf{0})(-1)(-2)} = 1$$
$$P_{-3}(2) = \frac{2 \cdot \mathbf{0}}{(2-1)(2-2)(2-3)} = \frac{2 \cdot \mathbf{0}}{(1)(\mathbf{0})(-1)} = -2$$
$$P_{-3}(3) = \frac{2 \cdot \mathbf{0}}{(3-1)(3-2)(3-3)} = \frac{2 \cdot \mathbf{0}}{(2)(1)(\mathbf{0})} = 1$$
$$P_{-3}(4) = \frac{2 \cdot \mathbf{0}}{(4-1)(4-2)(4-3)} = \frac{2 \cdot 0}{(3)(2)(1)} = \frac{\mathbf{0}}{3} = 0$$
$$P_{-3}(5) = \frac{2 \cdot \mathbf{0}}{(5-1)(5-2)(5-3)} = \frac{2 \cdot \mathbf{0}}{(4)(3)(2)} = \frac{\mathbf{0}}{12} = 0$$

and so on. The case where $m = 3$ comprises three finite figurate numbers which are 1, -2 and 1. The others are all naught.

The reader can apply the same approach for cases where $m = 4, 5, 6, \ldots$. If he gathers together all the finite figurate numbers for all the cases, he will discover that they form signed Pascal Triangle numbers. We thus generalize the figurate numbers by combining the old and new sequences of figurate numbers. The table below shows these sequences.

$r = -6$	:	1	-5	10	-10	5	-1		
$r = -5$	:	1	-4	6	-4	1			
$r = -4$	:	1	-3	3	-1				
$r = -3$	:	1	-2	1					
$r = -2$	:	1	-1						
$r = -1$	:	1							
$r = \mathbf{0}$	:	1	1	1	1	1	1	1	1
$r = 1$	:	1	2	3	4	5	6	7	8
$r = 2$	:	1	3	6	10	15	21	28	36
$r = 3$	:	1	4	10	20	35	56	84	120
$r = 4$	:	1	5	15	35	70	126	210	330
$r = 5$	:	1	6	21	56	126	252	462	792

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