Original Research Article

Existence of random attractors for the stochastic reaction-diffusion equation with distribution derivatives and multiplicative noise on $\mathbb{R}^n$

Abstract: In this paper, we prove the existence of random attractors for a stochastic reaction-diffusion equation with distribution derivatives and multiplicative noise defined on unbounded domains. In order to obtain the asymptotic compactness of the random dynamical system, we make use of a priori estimates for far-field values of solutions as well as the cut-off technique.

keywords : Random Attractors, Stochastic Reaction-Diffusion Equation, Distribution Derivatives, Asymptotic Compactness.

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1 Introduction

In this paper, we investigate the asymptotic behavior of solution to the following stochastic reaction-diffusion equation with distribution derivatives and multiplicative noise defined in the entire space $\mathbb{R}^n$:

$$du + (\alpha u - \Delta u)dt = (g(x,u) + f(x) + D_j f^j)dt + bu \circ dW(t), \tag{1.1}$$

with the initial value condition

$$u(x,0) = u_0(x) \quad , \qquad x \in \mathbb{R}^n, \tag{1.2}$$

where $-\Delta$ is the Laplacian operator with respect to the variable $x \in \mathbb{R}^n$, $u = u(x,t)$ is a real function of $x \in \mathbb{R}^n$ and $t \geq 0$; α, b are proper positive constants; $D_j = \frac{\partial}{\partial x_j}$ is distribution derivatives; $f^j, f \in L^2(\mathbb{R}^n)$ (j=1,2,....,n); g is a nonlinear function satisfying certain conditions; $W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ -algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$; $\circ$ denotes the Stratonovich sense in the stochastic term. We identify $\omega(t)$ with $W(t)$, i.e.,

$$W(t) = W(t, \omega) = \omega(t), \qquad t \in \mathbb{R}.$$

It is well known that the asymptotic behavior of a random dynamical system is presented by a random attractor. The existence of random attractors without distribution derivatives have been studied by many authors, see [2, 4, 5, 8, 9, 12, 17, 18] and the reference therein. Notice that the partial differential equations (PDEs) studied in these literatures are all defined on the bounded domains.

In the case of unbounded domains, the existence of random attractors without distribution derivatives was established for the stochastic reaction-diffusion equation with additive noise in [3], and with multiplicative noise in [16].

Recently, in our case of distribution derivatives on unbounded domains, the existence of global attractors was established for the deterministic reaction-diffusion equation with distribution derivatives in [14, 15], and for the stochastic reaction-diffusion equation with distribution derivatives and additive noise in [1].

However, there is no results on random attractors for stochastic reaction-diffusion equation with distribution derivatives and multiplicative noise on unbounded domain.

In this article, we will use the idea of uniform estimates on the tail of solutions to investigate the existence of a random attractor of the stochastic reaction-diffusion equation with distribution derivatives and multiplicative noise on unbounded domain. Since the equation (1.1) include the distribution derivatives, we can't use $-\Delta v$ as the test function to obtain a priori estimates of solution in a higher regular space. That is the essential different from [16]. Besides, we decrease the condition of the nonlinear function $g(x, u)$ comparing the condition of [16].

This paper is organized as follows. In section 2, we recall some basic concepts and properties for general random dynamics system. In section 3, we provide some basic settings about Eq. (1.1) and show that it generates a random dynamical system on $L^2(\mathbb{R}^n)$. In section 4, we prove the uniform estimates of solutions, which include the uniform estimates on the tails of solutions. In the last section, we first establish the asymptotic compactness of the solution operator by given uniform estimates on the tails of solutions, and then prove the existence of a random attractor.

In the sequel, we use $\|\cdot\|$ and $(\cdot, \cdot)$ to denote the norm and inner product of $L^2(\mathbb{R}^n)$, respectively.

2 Preliminaries

As mentioned in the introduction, our main purpose is to prove the existence of the random attractor. For that matter, first, we will recapitulate basic concepts related to random attractors for stochastic dynamical systems. The reader is referred to [2, 7] for more details.

Let $(X, \|\cdot\|_X)$ be separable Hilbert space with the Borel σ-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 2.1 $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\vartheta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, ϑ_0 is the identity on Ω, $\vartheta_{s+t} = \vartheta_t \circ \vartheta_s$ for all $s, t \in \mathbb{R}$ and $\vartheta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2 A continuous random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \longrightarrow X, \qquad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$- measurable and satisfies, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

(i) $\phi(0, \omega, \cdot)$ is the identity on X,
(ii) $\phi(t + s, \omega, \cdot) = \phi(t, \vartheta_s\omega, \cdot) \circ \phi(s, \omega, \cdot)$ for all $t, s \in \mathbb{R}^+$,
(iii) $\phi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.

Hereafter, we always assume that ϕ is continuous RDS on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$.

Definition 2.3 Let $\mathfrak{D}$ be a collection of random subset of X and $\{K(\omega)\} \in \mathfrak{D}$. Then $\{K(\omega)\}$ is called a random absorbing set for ϕ in $\mathfrak{D}$ for every $D \in \mathfrak{D}$ and $\mathbb{P}$-a.e, $\omega \in \Omega$, there exist $t_0(\omega)$ such that

$$\phi(t, \vartheta_{-t}\omega, D(\vartheta_{-t}\omega)) \subseteq K(\omega) \quad \text{ for all } t \geq t_0(\omega).$$

Definition 2.4 Let $\mathfrak{D}$ be the set of all random tempered sets in X. Then ϕ is said to be asymptotically compact in X if for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\{\phi(t_n, \vartheta_{-t_n}\omega, X_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $X_n \in B(\vartheta_{-t_n}\omega)$ with $\{B(\omega)\} \in \mathfrak{D}$.

Definition 2.5 A random compact set $\{\mathcal{A}(\omega)\}$ is said to be a random attractor if it is a random attracting set and $\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\vartheta_{-t}\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $t \geq 0$.

Theorem 2.6 Let ϕ be a continuous random dynamical system on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$. If there is a closed random tempered absorbing set $\{K(\omega)\}$ of ϕ and ϕ is asymptotically compact in X, then $\{\mathcal{A}(\omega)\}$ is a random attractor of ϕ, where

$$\mathcal{A}(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \phi(\tau, \vartheta_{-\tau}\omega, K(\vartheta_{-\tau}\omega))}, \quad \omega \in \Omega.$$

Moreover, $\{\mathcal{A}(\omega)\}$ is the unique attracor of ϕ.

3 The random reaction-diffusion equation on $\mathbb{R}^n$ with distribution derivatives and multiplicative noise

In this section, we show that there is a continuous random dynamical system generated by the stochastic reaction-diffusion equation defined on $\mathbb{R}^n$ with distribution derivatives and multiplicative noise:

$$du + (\alpha u - \Delta u)dt = (g(x,u) + f(x) + D_j f^j)dt + bu \circ dW(t), \tag{3.1}$$

with the initial value condition

$$u(x,0) = u_0(x) \quad , \qquad x \in \mathbb{R}^n, \tag{3.2}$$

where α, b are proper positive constants, $f^j, f \in L^2(\mathbb{R}^n)$, $D_j = \frac{\partial}{\partial x_j}$ is distribution derivatives, and $g(x,u)$ is a nonlinear function satisfying the same condition as [16], but except the condition $|\frac{\partial g}{\partial x}(x,u)| \le \tilde{g}(x)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$.

$$g \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}), \tag{3.3}$$

$$g(x,0) = 0, \quad g(x,u)u \le 0, \quad \text{for all } \; x \in \mathbb{R}^n, \; \text{ and } \; u \in \mathbb{R}, \tag{3.4}$$

$$\frac{\partial g}{\partial u}(x,u) \le \epsilon, \quad \text{for all } \; x \in \mathbb{R}^n, \; \text{ and } \; u \in \mathbb{R}, \tag{3.5}$$

$$\sup_{x\in\mathbb{R}^n} \sup_{|u|\le r} |\frac{\partial g}{\partial u}(x,u)| \le L(r), \; \text{ for all } \; x \in \mathbb{R}^n, \; u \in \mathbb{R} \; \text{ and } \; r \in \mathbb{R}^+, \tag{3.6}$$

where ϵ is a non-negative constant, $L(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$.

To model the random noise in Eq. (3.1), we need to define a shift operator $\{\vartheta_t\}_{t\in\mathbb{R}}$ on Ω (where Ω is defined in the introduction) by

$$\vartheta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$ is an ergodic metric dynamical system, see [2, 7].

For our purpose, it is convenient to convert the Eq.(3.1) into a deterministic system with a random parameter, and then show that it generates a random dynamical system.

We now introduce an Ornstein-Uhlenbeck process given by the Brownian motion. Put

$$z(\vartheta_t\omega) := -\int_{-\infty}^0 e^s(\vartheta_t\omega)(s)ds, \quad t \in \mathbb{R}, \tag{3.7}$$

which is called the Ornstein-Uhlenbeck process and solves the It$\hat{o}$ equation

$$dz + zdt = dW(t). \tag{3.8}$$

From [2, 3, 10, 11], it is known that the random variable $z(\omega)$ is tempered, and there is a ϑ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathbb{P}$ measure such that for every $\omega \in \tilde{\Omega},\ t \mapsto z(\vartheta_t\omega)$ is continuous in t; $\lim_{t\to\pm\infty} \frac{|z(\vartheta_t\omega)|}{|t|} = 0$; and $\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\vartheta_s\omega)ds = 0$.

To show that Eq. (3.1) generates a random dynamical system, like in [16], we let

$$v(t) = e^{-bz(\vartheta_t\omega)}u(t), \tag{3.9}$$

where u is a solution of Eq. (3.1). Then we can consider the following evolution equation with random coefficients but without white noise:

$$\frac{dv}{dt} + \alpha v - \Delta v = e^{-bz(\vartheta_t\omega)}(g(x, e^{bz(\vartheta_t\omega)}v) + f(x) + D_j f^j) + bz(\vartheta_t\omega)v, \tag{3.10}$$

with the initial value condition

$$v(x,0) = v_0(x) = e^{-bz(\vartheta_t\omega)}u_0(x), \quad x \in \mathbb{R}^n. \tag{3.11}$$

We will consider (3.10)-(3.11) for $\omega \in \tilde{\Omega}$ and write $\tilde{\Omega}$ as Ω from now on.

By using the standard Galerkin method following, see [6, 13], one may show that (3.10) has a unique solution $v(t, \omega, v_0)$ which is continuous with respect to v_0 in $L^2(\mathbb{R}^n)$ for all $t > 0$. Then (3.10) generates a continuous random dynamical system $\{\phi(t)\}_{t\geq 0}$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$, where

$$\phi(t, \omega, v_0) = v(t, \omega, v_0), \text{ for } v_0 \in L^2(\mathbb{R}^n),\ t \geq 0 \text{ and for all } \omega \in \Omega.$$

We define mapping $\varphi : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$\varphi(t, \omega, u_0) = u(t, \omega, u_0) = e^{bz(\vartheta_t\omega)}\phi(t, \omega, v_0), \text{ for } v_0 \in L^2(\mathbb{R}^n),\ t \geq 0 \text{ and for all } \omega \in \Omega.$$

Then φ is a continuous random dynamical system associated with the Eq. (3.1) on $L^2(\mathbb{R}^n)$.

Note that the two random dynamical system are equivalent. It is easy to check that φ has a random attractor provided ϕ possesses a random attractor. Then, we only need to consider the random dynamical system ϕ.

4 Uniform estimates of solutions

In this section, we derive uniform estimates on the solutions of (3.1)-(3.2) defined on $\mathbb{R}^n$ when $t \to \infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equation. In particular, we will show that the tails of the solutions for large space variable are uniformly small when time is sufficiently large.

From now on, we always assume that $\mathfrak{D}$ is the collection of all tempered random subsets of $L^2(\mathbb{R}^n)$ with respect to $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta_t\}_{t\in\mathbb{R}})$. The next lemma shows that ϕ has a random absorbing set in $\mathfrak{D}$.

Lemma 4.1 Assume that $f^j, f \in L^2(\mathbb{R}^n)$, and (3.3)-(3.6) hold. Then there exists a random ball $\{K(\omega)\} \in \mathfrak{D}$ centered at 0 with random radius $\rho(\omega) > 0$ such that $\{K(\omega)\}$ is a random absorbing set for ϕ in $\mathfrak{D}$, that is, for any $\{B(\omega)\} \in \mathfrak{D}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there is $T_B(\omega) > 0$ such that

$$\phi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega,)) \subseteq K(\omega) \text{ for all } t > T_B(\omega). \tag{4.1}$$

Proof Taking the inner product of Eq.(3.10) with v in $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + \alpha\|v\|^2 + \|\nabla v\|^2 = e^{-bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} g(x, e^{bz(\vartheta_t\omega)}v)vdx \\ + e^{-bz(\vartheta_t\omega)}((f, v) + (D_j f^j, v)) + bz(\vartheta_t\omega)\|v\|^2.\end{aligned} \tag{4.2}$$

In line with condition (3.4) and (3.6), we get

$$-\infty < -L(e^{bz(\vartheta_t\omega)}\|v\|)\|v\|^2 \leq e^{-2bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} g(x, u)udx \leq 0. \tag{4.3}$$

By the Hölder' inequality and the Young inequality, we conclude

$$e^{-bz(\vartheta_t\omega)}(f, v) \leq \frac{1}{2\alpha}e^{-2bz(\vartheta_t\omega)}\|f\|^2 + \frac{\alpha}{2}\|v\|^2, \tag{4.4}$$

$$e^{-bz(\vartheta_t\omega)}(D_j f^j, v) = e^{-bz(\vartheta_t\omega)}(\tilde{f}, \nabla v) \leq e^{-bz(\vartheta_t\omega)}\|\tilde{f}\| \cdot \|\nabla v\| \leq \frac{1}{2}e^{-2bz(\vartheta_t\omega)}\|\tilde{f}\|^2 + \frac{1}{2}\|\nabla v\|^2, \tag{4.5}$$

where $\tilde{f} = (f^1, ..., f^n)$ and $\|\tilde{f}\|^2 = \sum_{j=1}^n |f^j|^2$.

Then inserting (4.3)-(4.5) into (4.2), it yields

$$\frac{d}{dt}\|v\|^2 - (2bz(\vartheta_t\omega) - \alpha)\|v\|^2 + \|\nabla v\|^2 \leq \frac{1}{\alpha}e^{-2bz(\vartheta_t\omega)}(\|f\|^2 + \alpha\|\tilde{f}\|^2). \tag{4.6}$$

Hence, we can rewrite (4.6)as

$$\frac{d}{dt}\|v\|^2 - (2bz(\vartheta_t\omega) - \alpha)\|v\|^2 \leq \frac{1}{\alpha}e^{-2bz(\vartheta_t\omega)}(\|f\|^2 + \alpha\|\tilde{f}\|^2). \tag{4.7}$$

By applying the Gronwall's lemma to (4.7), we find that

$$\begin{aligned}\|v(t, \omega, v_0(\omega))\|^2 \leq e^{2\int_0^t bz(\vartheta_s\omega)ds - \alpha t}\|v_0(\omega)\|^2 \\ + \frac{\|f\|^2 + \alpha\|\tilde{f}\|^2}{\alpha}e^{2b\int_0^t z(\vartheta_s\omega)ds - \alpha t}\int_0^t e^{-2bz(\vartheta_s\omega) - 2b\int_0^s z(\vartheta_\tau\omega)d\tau + \alpha s}ds.\end{aligned} \tag{4.8}$$

By replacing ω by $\vartheta_{-t}\omega$ in (4.8), we get

$$\|v(t, \vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega))\|^2 \leq e^{2b\int_{-t}^0 z(\vartheta_s\omega)ds - \alpha t}\|v_0(\vartheta_{-t}\omega)\|^2$$

$$+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_{-\infty}^{0} e^{-2bz(\vartheta_s\omega)+2b\int_s^0 z(\vartheta_\tau\omega)d\tau+\alpha s}ds. \tag{4.9}$$

By the properties of Ornstein-Uhlenbeck process,

$$\int_{-\infty}^{0} e^{-2bz(\vartheta_s\omega)+2b\int_s^0 z(\vartheta_\tau\omega)d\tau+\alpha s}ds<+\infty. \tag{4.10}$$

Notice that $\{B(\omega)\}\in\mathfrak{D}$ is tempered, then for any $v_0(\vartheta_{-t}\omega)\in B(\vartheta_{-t}\omega)$,

$$\lim_{t\to+\infty} e^{2b\int_{-t}^0 z(\vartheta_s\omega)ds-\alpha t}\|v_0(\vartheta_{-t}\omega)\|^2=0. \tag{4.11}$$

We can choose

$$\rho(\omega)=1+\frac{\|f\|^2+\lambda\|\tilde{f}\|^2}{\lambda}\int_{-\infty}^{0} e^{-2bz(\vartheta_s\omega)+2b\int_s^0 z(\vartheta_\tau\omega)d\tau+\alpha s}ds. \tag{4.12}$$

And let

$$K(\omega)=\{u\in L^2(\mathbb{R}^n):\|u\|^2\le\rho(\omega)\}.$$

Then $\{K(\omega)\}\in\mathfrak{D}$, and $\{K(\omega)\}$ is a random absorbing set for ϕ in $\mathfrak{D}$, which completes the proof. $\square$

Lemma 4.2 Assume that $f^j,f\in L^2(\mathbb{R}^n)$, and (3.3)-(3.6) hold. Then there exists a tempered random variable $\tilde{R}_1(\omega)>0$ such that for any $\{B(\omega)\}\in\mathfrak{D}$ and $v_0(\omega)\in B(\omega)$, there exists a $T_B(\omega)>0$ such that the solution ϕ of (3.10) satisfies for $\mathbb{P}$-a.e. $\omega\in\Omega$, for all $t\ge T_B(\omega)$,

$$\int_t^{t+1}\|\nabla\phi(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\|^2ds\le\tilde{R}_1(\omega). \tag{4.13}$$

Proof By substituting t by $\hat{T}$ and ω by $\vartheta_{-t}\omega$ in (4.8) for any $\hat{T}\ge0$, we find that

$$\begin{aligned}\|v(\hat{T},\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2 &\le e^{2b\int_0^{\hat{T}} z(\vartheta_{s-t}\omega)ds-\alpha\hat{T}}\|v_0(\vartheta_{-t}\omega)\|^2\\ &+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}e^{2b\int_0^{\hat{T}} z(\vartheta_{s-t}\omega)ds-\alpha\hat{T}}\int_0^{\hat{T}} e^{-2bz(\vartheta_{s-t}\omega)-2b\int_0^s z(\vartheta_{\tau-t}\omega)d\tau+\alpha s}ds. \end{aligned}\tag{4.14}$$

Multiplying two sides of the Eq. (4.14) by $e^{2b\int_{\hat{T}}^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-\hat{T})}$, then simplifying it, we find that for all $t\ge\hat{T}$

$$\begin{aligned}e^{2b\int_{\hat{T}}^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-\hat{T})}\|v(\hat{T},\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2 &\le e^{2b\int_0^t z(\vartheta_{s-t}\omega)ds-\alpha t}\|v_0(\vartheta_{-t}\omega)\|^2\\ &+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_0^{\hat{T}} e^{-2bz(\vartheta_{s-t}\omega)+2b\int_s^t z(\vartheta_{s-t}\omega)ds-\alpha(t-s)}ds. \end{aligned}\tag{4.15}$$

By the Gronwall's lemma to (4.6), we get that for all $t \geq \hat{T}$,

$$\begin{aligned}\|v(t,\omega,v_0(\omega))\|^2 \leq{}& e^{2b\int_{\hat{T}}^t z(\vartheta_s\omega)ds-\alpha(t-\hat{T})}\|v(\hat{T},\omega,v_0(\omega))\|^2\\&+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_{\hat{T}}^t e^{-2bz(\vartheta_s\omega)+2b\int_s^t z(\vartheta_\tau\omega)d\tau+\alpha(s-t)}ds\\&-\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_\tau\omega)d\tau+\alpha(s-t)}\|\nabla v(s,\omega,v_0(\omega))\|^2 ds,\end{aligned}\tag{4.16}$$

which obviously gives

$$\begin{aligned}&\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_\tau\omega)d\tau+\alpha(s-t)}\|\nabla v(s,\omega,v_0(\omega))\|^2ds \leq e^{2b\int_{\hat{T}}^t z(\vartheta_s\omega)ds-\alpha(t-\hat{T})}\|v(\hat{T},\omega,v_0(\omega))\|^2\\&\quad+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_{\hat{T}}^t e^{-2bz(\vartheta_s\omega)+2b\int_s^t z(\vartheta_\tau\omega)d\tau+\alpha(s-t)}ds.\end{aligned}\tag{4.17}$$

By replacing ω by $\vartheta_{-t}\omega$ into (4.17), we get

$$\begin{aligned}&\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau+\alpha(s-t)}\|\nabla v(s,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2ds\\&\quad\leq e^{2b\int_{\hat{T}}^t z(\vartheta_{s-t}\omega)ds-\alpha(t-\hat{T})}\|v(\hat{T},\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2\\&\qquad+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_{\hat{T}}^t e^{-2bz(\vartheta_{s-t}\omega)+2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau+\alpha(s-t)}ds.\end{aligned}\tag{4.18}$$

Together with (4.15) and (4.18), we have

$$\begin{aligned}&\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau+\alpha(s-t)}\|\nabla v(s,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2ds\\&\quad\leq e^{2b\int_{-t}^0 z(\vartheta_s\omega)ds-\alpha t}\|v_0(\vartheta_{-t}\omega)\|^2+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_{-t}^0 e^{-2bz(\vartheta_s\omega)+2b\int_s^0 z(\vartheta_\tau\omega)d\tau+\alpha s}ds.\end{aligned}\tag{4.19}$$

Replacing $\hat{T}$ by t and t by $t+1$ in (4.19), we have

$$\begin{aligned}&\int_t^{t+1} e^{2b\int_s^{t+1} z(\vartheta_{\tau-t-1}\omega)d\tau+\alpha(s-t-1)}\|\nabla v(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\|^2ds\\&\leq e^{2b\int_{-t-1}^0 z(\vartheta_s\omega)ds-\alpha(t+1)}\|v_0(\vartheta_{-t-1}\omega)\|^2+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_{-t-1}^0 e^{-2bz(\vartheta_s\omega)+2b\int_s^0 z(\vartheta_\tau\omega)d\tau+\alpha s}ds.\end{aligned}\tag{4.20}$$

For $s \in [t, t+1]$, to yield that

$$\int_t^{t+1} e^{2b\int_s^{t+1} z(\vartheta_{\tau-t-1}\omega)d\tau+\alpha(s-t-1)}\|\nabla v(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\|^2ds$$

$$\geq \int_t^{t+1} e^{-2b\max_{0\leq\tau\leq 1}|z(\vartheta_\tau\omega)|-\alpha}\|\nabla v(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\|^2 ds. \tag{4.21}$$

By the property of $z(\omega)$ and temperedness of $\|v_0(\omega)\|$, there exists $T_B(\omega)>0$ such that for all $t\geq T_B(\omega)$, from (4.20) and (4.21) we find that

$$\begin{aligned}&\int_t^{t+1}\|\nabla v(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\|^2 ds\\ &\qquad\leq 1+\frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_{-\infty}^0 e^{-2bz(\vartheta_s\omega)+2be\max_{0\leq\tau\leq 1}|z(\vartheta_\tau\omega)|+2b\int_s^0 z(\vartheta_\tau\omega)d\tau+\alpha(s+1)}ds\\ &\qquad=\tilde{R_1}(\omega).\end{aligned} \tag{4.22}$$

It is easy to check that $\tilde{R_1}(\omega)$ is tempered.This completes the proof. $\square$

Lemma 4.3 Assume that $f^j, f\in L^2(\mathbb{R}^n)$,(3.3)-(3.6) hold. The random dynamical system $\{\phi(t)\}_{t\geq 0}$ has a $(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))$ and $(L^2(\mathbb{R}^n),H^1(\mathbb{R}^n))$-bounded absorbing set, that is, there exists a random radius $\tilde{\rho}(\omega)$ such that for any $\{B(\omega)\}\in\mathfrak{D}$ and $v_0(\omega)\in B(\omega)$, there exists a $T_B(\omega)>0$ such that the solution ϕ of (3.10) satisfies for $\mathbb{P}$-a.e. $\omega\in\Omega$, for all $t\geq T_B(\omega)$,

$$\|\phi(t,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2+\|\nabla\phi(t,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2\leq\tilde{\rho}(\omega). \tag{4.23}$$

Proof Taking the inner product of Eq.(3.10) with v in $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2+\alpha\|v\|^2+\|\nabla v\|^2 &= e^{-bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} g(x,e^{bz(\vartheta_t\omega)}v)vdx\\ &\quad+e^{-bz(\vartheta_t\omega)}((f,v)+(D_jf^j,v))+bz(\vartheta_t\omega)\|v\|^2.\end{aligned} \tag{4.24}$$

By (4.3) - (4.5) and Lemma 4.1, we conclude from (4.24) that

$$\frac{d}{dt}\|v\|^2+\alpha\|v\|^2+\|\nabla v\|^2\leq\frac{1}{\alpha}e^{-2bz(\vartheta_t\omega)}\|f\|^2+2b(z(\vartheta_t\omega))\rho(\omega)+e^{-2bz(\vartheta_t\omega)}\|\tilde{f}\|^2 \tag{4.25}$$

Noticing that

$$\|\nabla v+\tilde{f}\|^2\leq 2\|\nabla v\|^2+2\|\tilde{f}\|^2, \tag{4.26}$$

by (4.26), we conclude from (4.25) that

$$\frac{d}{dt}\|v\|^2+\mathcal{C}(\|\nabla v+\tilde{f}\|^2+\|v\|^2)\leq\frac{1}{\alpha}e^{-2bz(\vartheta_t\omega)}\|f\|^2+2b(z(\vartheta_t\omega))\rho(\omega)+(1+e^{-2bz(\vartheta_t\omega)})\|\tilde{f}\|^2\leq\tilde{\rho}(\omega), \tag{4.27}$$

where $\mathcal{C}=\min\{\alpha,\frac{1}{2}\}$. Integrating the Eq. (4.27) from t to $t+1$, and using Lemma 4.1, we can find a $T_B(\omega)>0$, such that for all $t\geq T_B(\omega)$,

$$\int_t^{t+1}(\|\nabla v+\tilde{f}\|^2+\|v\|^2)\leq\tilde{\rho}(\omega) \quad . \tag{4.28}$$

On the other hand , multiplying Eq. (3.10) with v_t, and integrating over $\mathbb{R}^n$ we find that

$$\|v_t\|^2+\frac{1}{2}\frac{d}{dt}(\|\nabla v\|^2+\alpha\|v\|^2)$$
$$=(e^{-bz(\vartheta_t\omega)}g(x,e^{bz(\vartheta_t\omega)}v),v_t)+e^{-bz(\vartheta_t\omega)}((f,v_t)-\frac{d}{dt}(\tilde{f},\nabla v))+\frac{1}{2}b^2|z(\vartheta_t\omega)|^2\|v\|^2+\frac{1}{2}\|v_t\|^2. \tag{4.29}$$

By the Hölder inequality and the Young inequality, we conclude

$$e^{-bz(\vartheta_t\omega)}(f,v_t)\leq e^{-2bz(\vartheta_t\omega)}\|f\|^2+\frac{1}{4}\|v_t\|^2, \tag{4.30}$$

and

$$(e^{-bz(\vartheta_t\omega)}g(x,e^{bz(\vartheta_t\omega)}v),v_t)\leq e^{-2bz(\vartheta_t\omega)}\|g(x,u)\|^2+\frac{1}{4}\|v_t\|^2. \tag{4.31}$$

Then inserting (4.30)-(4.31) into (4.29), it yields

$$\frac{d}{dt}(\|\nabla v\|^2+2(\tilde{f},\nabla v)+\|\tilde{f}\|^2+\alpha\|v\|^2)$$
$$\leq 2e^{-2bz(\vartheta_t\omega)}\|f\|^2+2e^{-2bz(\vartheta_t\omega)}\|g(x,u)\|^2+b^2|z(\vartheta_t\omega)|^2\|v\|^2. \tag{4.32}$$

By using condition 3.5, we conclude that

$$\|g(x,u)\|^2\leq\int_{\mathbb{R}^n}|\frac{\partial g}{\partial u}(x,\theta u)|^2|u|^2dx\leq\epsilon^2\|u\|^2, \tag{4.33}$$

where $0<\theta<1$.

By (4.33) and Lemma 4.1, we can rewrite (4.32) as

$$\frac{d}{dt}(\|\nabla v+\tilde{f}\|^2+\alpha\|v\|^2)\leq 2e^{-2bz(\vartheta_t\omega)}\|f\|^2+(2\epsilon^2+b^2|z(\vartheta_t\omega)|^2)\rho(\omega)\leq\tilde{\rho}(\omega). \tag{4.34}$$

Combining with (4.28) and (4.34), by the uniform Gronwall lemma, we deduce that

$$\|\nabla v+\tilde{f}\|^2+\alpha\|v\|^2\leq\tilde{\rho}(\omega). \tag{4.35}$$

Thus, thanks to $\|\nabla v\|^2\leq 2\|\nabla v+\tilde{f}\|^2+2\|\tilde{f}\|^2$ and Eq. (4.35), we achieve that for $t\geq T_B(\omega)+1$,

$$\|\nabla v\|^2+\|v\|^2\leq\tilde{\rho}(\omega). \tag{4.36}$$

This proof is completed. $\square$

Now we will prove the solution is enough small in a large space using the method and skill of [22 – 24].

Lemma 4.4 Assume that $f^j, f \in L^2(\mathbb{R}^n)$, and (3.3)-(3.6) hold. Let $\{B(\omega)\} \in \mathfrak{D}$ and $v_0(\omega) \in B(\omega)$. Then, for any $\zeta > 0$, there exist $\tilde{T} = \tilde{T}(\zeta, \omega, B) > 0$ and $\tilde{K} = \tilde{K}(\zeta, \omega) > 0$, such that the solution ϕ of Eq. (3.10) satisfies for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\forall t \geq \tilde{T}$,

$$\int_{|x|\geq\tilde{R}} |\phi(t, \vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega))|^2 dx \leq \zeta. \tag{4.37}$$

Proof We first need to define a smooth function $\sigma(\cdot)$ from $\mathbb{R}^+$ into $[0,1]$ such that $\sigma(\cdot) = 0$ on $[0,1]$ and $\sigma(\cdot) = 1$ on $[2, +\infty)$, which evidently implies that there is a positive constant c such that the $|\sigma'(s)| \leq c$ for all $s \geq 0$. For convenience, we write $\sigma_\kappa = \sigma(\frac{|x|^2}{\kappa^2})$.

Multiplying Eq. (3.10) with $\sigma_\kappa v$ and integrating over $\mathbb{R}^n$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \sigma_\kappa |v|^2 dx + \alpha \int_{\mathbb{R}^n} \sigma_\kappa |v|^2 dx &= \int_{\mathbb{R}^n} (\Delta v)\sigma_\kappa v dx + bz(\vartheta_t\omega)\int_{\mathbb{R}^n} \sigma_\kappa |v|^2 dx \\ &+ e^{-bz(\vartheta_t\omega)}\left(\int_{\mathbb{R}^n} \sigma_\kappa g(x,u) v dx + \int_{\mathbb{R}^n} \sigma_\kappa f v dx + \int_{\mathbb{R}^n} D_j f^j \sigma_\kappa v dx\right),\end{aligned} \tag{4.38}$$

where

$$\begin{aligned}\int_{\mathbb{R}^n} (\Delta v)\sigma_\kappa v dx &= -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa dx - \int_{\mathbb{R}^n} v \sigma'_\kappa \frac{2x}{\kappa^2}(\nabla v) dx \\ &\leq -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa dx + \frac{C_0}{\kappa}(\|v\|^2 + \|\nabla v\|^2),\end{aligned} \tag{4.39}$$

where C_0 is a non-negative constant.

By condition (3.4) and (3.6), we get

$$-\infty < e^{-bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} \sigma_\kappa g(x,u) v dx = e^{-2bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} \sigma_\kappa g(x,u) u dx \leq 0. \tag{4.40}$$

For the fourth term on the right-hand side of (4.38), we have that

$$e^{-bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} \sigma_\kappa f v dx \leq \frac{\alpha}{2}\int_{\mathbb{R}^n} \sigma_\kappa |v|^2 dx + \frac{1}{2\alpha} e^{-2bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} \sigma_\kappa |f|^2 dx. \tag{4.41}$$

Next, we estimate the last term on the right-hand side of (4.38), we get that

$$\begin{aligned}e^{-bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} D_j f^j \sigma_\kappa v dx &= -e^{-bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} \tilde{f}\, \frac{2x}{\kappa^2}\, \sigma'_\kappa\, v\, dx - e^{-bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} \sigma_\kappa\, \tilde{f}\, (\nabla v) dx \\ &\leq \frac{\mathcal{C}_1}{\kappa}(\|\tilde{f}\|^2 + \|v\|^2) + \frac{1}{2} e^{-2bz(\vartheta_t\omega)} \int_{\mathbb{R}^n} \sigma_\kappa |\tilde{f}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} \sigma_\kappa |\nabla v|^2 dx,\end{aligned} \tag{4.42}$$

where $\mathcal{C}_1$ is a non-negative constant. Then inserting (4.39) - (4.42) into (4.38) to see that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\sigma_\kappa|v|^2dx-(2bz(\vartheta_t\omega)-\alpha)\int_{\mathbb{R}^n}\sigma_\kappa|v|^2dx+\int_{\mathbb{R}^n}|\nabla v|^2\sigma_\kappa dx\\&\leq\frac{1}{\alpha}e^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}\sigma_\kappa(|f|^2+\alpha|\tilde{f}|^2)dx+\frac{\mathcal{C}_2}{\kappa}\|\tilde{f}\|^2+\frac{\mathcal{C}_3}{\kappa}\|v\|^2+\frac{\mathcal{C}_4}{\kappa}\|\nabla v\|^2,\end{aligned}\tag{4.43}$$

where $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are non-negative constants. Hence, we can rewrite (4.43) as

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\sigma_\kappa|v|^2dx-(2bz(\vartheta_t\omega)-\alpha)\int_{\mathbb{R}^n}\sigma_\kappa|v|^2dx\\&\leq\frac{1}{\alpha}e^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}\sigma_\kappa(|f|^2+\alpha|\tilde{f}|^2)dx+\frac{\mathcal{C}_2}{\kappa}\|\tilde{f}\|^2+\frac{\mathcal{C}_3}{\kappa}\|v\|^2+\frac{\mathcal{C}_4}{\kappa}\|\nabla v\|^2.\end{aligned}\tag{4.44}$$

By applying the Gronwall's lemma to (4.44), for every $t\geq\hat{T}$, we find that

$$\begin{aligned}&\int_{\mathbb{R}^n}\sigma_\kappa|v(t,\omega,v_0(\omega))|^2dx\leq e^{2b\int_{\hat{T}}^t z(\vartheta_\tau\omega)d\tau-\alpha(t-\hat{T})}\int_{\mathbb{R}^n}\sigma_\kappa|v(\hat{T},\omega,v_0(\omega))|^2dx\\&+\frac{1}{\alpha}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_\tau\omega)d\tau-\alpha(t-s)-2bz(\vartheta_s\omega)}\int_{\mathbb{R}^n}\sigma_\kappa(|f|^2+\alpha|\tilde{f}|^2)dx\\&+\frac{\mathcal{C}_3}{\kappa}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_\tau\omega)d\tau-\alpha(t-s)}\|v(s,\omega,v_0(\omega))\|^2ds\\&+\frac{\mathcal{C}_4}{\kappa}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_\tau\omega)d\tau-\alpha(t-s)}\|\nabla v(s,\omega,v_0(\omega))\|^2ds\\&+\frac{\mathcal{C}_2}{\kappa}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_\tau\omega)d\tau-\alpha(t-s)}\|\tilde{f}\|^2ds.\end{aligned}\tag{4.45}$$

Then, substitutingω by $\vartheta_{-t}\omega$ into (4.45), we have that

$$\begin{aligned}&\int_{\mathbb{R}^n}\sigma_\kappa|v(t,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))|^2dx\leq e^{2b\int_{\hat{T}}^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-\hat{T})}\int_{\mathbb{R}^n}\sigma_\kappa|v(\hat{T},\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))|^2dx\\&+\frac{1}{\alpha}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)-2bz(\vartheta_{s-t}\omega)}\int_{\mathbb{R}^n}\sigma_\kappa(|f|^2+\alpha|\tilde{f}|^2)dxds\\&+\frac{\mathcal{C}_3}{\kappa}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)}\|v(s,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2ds\\&+\frac{\mathcal{C}_4}{\kappa}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)}\|\nabla v(s,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2ds\\&+\frac{\mathcal{C}_2}{\kappa}\int_{\hat{T}}^t e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)}\|\tilde{f}\|^2ds.\end{aligned}\tag{4.46}$$

Then,we estimate every term on the right-hand side of (4.46). Firstly by Eq. (4.8) replacing t by $\hat{T}$ and ω by $\vartheta_{-t}\omega$, then we get

$$\begin{aligned} & e^{2b\int_{\hat{T}}^{t} z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-\hat{T})} \int_{\mathbb{R}^n} \sigma_\kappa |v(\hat{T},\vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega))|^2 dx \\ & \le e^{2b\int_0^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha t}\|v_0(\vartheta_{-t}\omega)\|^2 + \frac{\|f\|^2+\alpha\|\tilde{f}\|^2}{\alpha}\int_0^{\hat{T}} e^{-2bz(\vartheta_{s-t}\omega)+2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)}ds. \end{aligned} \tag{4.47}$$

It easy to see that there exists $\tilde{T}_1=\tilde{T}_1(B,\zeta,\omega)>\hat{T}$, such that for all $t>\tilde{T}_1$, then

$$e^{2b\int_{\hat{T}}^{t} z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-\hat{T})} \int_{\mathbb{R}^n} \sigma_\kappa |v(\hat{T},\vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega))|^2 dx \le \zeta. \tag{4.48}$$

For the second term on the right-hand side of (4.46), Since $f,\tilde{f}\in L^2(\mathbb{R}^n)$, there are $\tilde{T}_2=\tilde{T}_2(\zeta,\omega)>\hat{T}$ and $\tilde{K}_1=\tilde{K}_1(\zeta,\omega)>0$, such that for all $t>\tilde{T}_2$ and $\kappa>\tilde{K}_1$, then

$$\begin{aligned} & \frac{1}{\alpha}\int_{\hat{T}}^{t} e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)-2bz(\vartheta_{s-t}\omega)}\int_{\mathbb{R}^n}\sigma_\kappa(|f|^2+\alpha|\tilde{f}|^2)dxds \\ & \le \frac{1}{\alpha}\int_{\hat{T}}^{t} e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)-2bz(\vartheta_{s-t}\omega)}\int_{|x|\ge\kappa}|f|^2dxds \\ & \quad + \int_{\hat{T}}^{t} e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)-2bz(\vartheta_{s-t}\omega)}\int_{|x|\ge\kappa}|\tilde{f}|^2dxds \\ & \le \ \zeta. \end{aligned} \tag{4.49}$$

For the third term on the right-hand side of (4.46). By replacing t by s and ω by $\vartheta_{-t}\omega$ in (4.8),we get

$$\begin{aligned} & \frac{\mathcal{C}_3}{\kappa}\int_{\hat{T}}^{t} e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)}\|v(s,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2ds \\ & \quad \le \frac{\mathcal{C}_3}{\kappa}(t-\hat{T})e^{2b\int_0^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha t}\|v_0(\vartheta_{-t}\omega)\|^2 \\ & \qquad + \frac{\mathcal{C}_3(\|f\|^2+\alpha\|\tilde{f}\|^2)}{\kappa\alpha}\int_{\hat{T}}^{t}\int_0^s e^{2b\int_{\tilde{s}}^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-\tilde{s})-2bz(\vartheta_{\tilde{s}-t}\omega)}d\tilde{s}ds. \end{aligned} \tag{4.50}$$

Then, by $f,\tilde{f}\in L^2(\mathbb{R}^n)$, there exist $\tilde{T}_3=\tilde{T}_3(B,\zeta,\omega)>\hat{T}$ and $\tilde{K}_2=\tilde{K}_2(\zeta,\omega)>0$, such that for all $t>\tilde{T}_3$ and $\kappa>\tilde{K}_2$, we find that

$$\frac{\mathcal{C}_3}{\kappa}\int_{\hat{T}}^{t} e^{2b\int_s^t z(\vartheta_{\tau-t}\omega)d\tau-\alpha(t-s)}\|v(s,\vartheta_{-t}\omega,v_0(\vartheta_{-t}\omega))\|^2ds \le \zeta. \tag{4.51}$$

Next, we estimate the fourth term on the right-hand side of (4.46). Since $f, \tilde{f} \in L^2(\mathbb{R}^n)$, by using(4.19), there exist $\tilde{T}_4 = \tilde{T}_4(B, \zeta, \omega) > \hat{T}$ and $\tilde{K}_3 = \tilde{K}_3(\zeta.\omega) > 0$, such that for all $t > \tilde{T}_4$ and $\kappa > \tilde{K}_3$, we get that

$$\frac{\mathcal{C}_4}{\kappa} \int_{\hat{T}}^{t} e^{2b \int_s^t z(\vartheta_{\tau-t}\omega)d\tau - \alpha(t-s)} \|\nabla v(s, \vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega))\|^2 ds \leq \zeta. \tag{4.52}$$

Finally, we estimate the last term on the right-hand side of (4.46). Since $\tilde{f} \in L^2(\mathbb{R}^n)$, there exist $\tilde{T}_5 = \tilde{T}_5(\zeta, \omega) > \hat{T}$ and $\tilde{K}_4 = \tilde{K}_4(\zeta.\omega) > 0$, such that for all $t > \tilde{T}_5$ and $\kappa > \tilde{K}_4$, we have that

$$\frac{\mathcal{C}_2}{\kappa} \int_{\hat{T}}^{t} e^{2b \int_s^t z(\vartheta_{\tau-t}\omega)d\tau - \alpha(t-s)} \|\tilde{f}\|^2 ds \leq \zeta. \tag{4.53}$$

By letting

$$\tilde{T} = \max\{\tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4, \tilde{T}_5\}, \text{ and } \quad \tilde{K} = \max\{\tilde{K}_1, \tilde{K}_2, \tilde{K}_3, \tilde{K}_4\}.$$

Then, inserting $(4.48) - (4.49), (4.51) - (4.53)$ into (4.46), for all $t > \tilde{T}$ and $\kappa > \tilde{K}$, we obtain that

$$\int_{\mathbb{R}^n} \sigma_\kappa |v(t, \vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega))|^2 dx \leq 5\zeta, \tag{4.54}$$

which shows that

$$\int_{|x| \geq \tilde{K}} |\phi(t, \vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega))|^2 dx \leq 5\zeta. \tag{4.55}$$

This proof is completed. □

5 Random attractors

In this section, we prove the existence of a global random attractor for the random dynamical system ϕ associated with the stochastic reaction-diffusion equation (3.1)-(3.2) on $\mathbb{R}^n$. The main result of this section can now be stated as follows.

Lemma 5.1 Assume that $f^j, f \in L^2(\mathbb{R}^n)$, and (3.3)-(3.6) hold. Then the random dynamical system ϕ generated by (3.10) is asymptotically compact in $L^2(\mathbb{R}^n)$, that is, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the sequence $\{\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega))\}$ has a convergent subsequence in $L^2(\mathbb{R}^n)$ provided $t_n \to +\infty$, $\{B(\omega)\} \in \mathfrak{D}$ and $v_{0,n}(\vartheta_{-t_n}\omega) \in B(\vartheta_{-t_n}\omega)$.

Proof Let $t_n \to +\infty$, $\{B(\omega)\} \in \mathfrak{D}$ and $v_{0,n}(\vartheta_{-t_n}\omega) \in B(\vartheta_{-t_n}\omega)$. Then by Lemma 4.1, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have that

$$\{\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega))\}_{n=1}^{\infty} \text{ is bounded in } L^2(\mathbb{R}^n).$$

Hence, there exist $\xi \in L^2(\mathbb{R}^n)$ such that, up to a subsequence,

$$\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega)) \to \xi \quad \text{ weakly in } L^2(\mathbb{R}^n). \tag{5.1}$$

Next, we prove the weak convergence of (5.1) is actually strong convergence. Given $\zeta > 0$, by Lemma 4.4, there exist $\hat{T}_1 = \hat{T}_1(B, \zeta, \omega) > 0$, $\hat{\kappa}_1 = \hat{\kappa}_1(\zeta, \omega) > 0$ and $\hat{N}_1 = \hat{N}_1(B, \zeta, \omega) > 0$, such that $t_n \geq \hat{T}_1$ for every $n \geq \hat{N}_1$

$$\int_{|x| \geq \hat{\kappa}_1} |\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega))|^2 dx \leq \zeta. \tag{5.2}$$

On the other hand, by Lemma 4.1 and 4.3, there exist $\hat{T}_2 = \hat{T}_2(B, \omega) > 0$, such that for all $t \geq \hat{T}_2$,

$$\|\phi(t, \vartheta_{-t}\omega, v_0(\vartheta_{-t}\omega)) - \xi\|^2_{H^1(\mathbb{R}^n)} \leq R_1(\omega). \tag{5.3}$$

Let $\hat{N}_2 = \hat{N}_2(B, \omega)$ be large enough such that $t_n \geq \hat{T}_2$ for $n \geq \hat{N}_2$. Then by (5.3) we find that, for all $n \geq \hat{N}_2$,

$$\|\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega)) - \xi\|^2_{H^1(\mathbb{R}^n)} \leq R_1(\omega). \tag{5.4}$$

Denote $Q_{\hat{\kappa}_1} = \{x \in \mathbb{R}^n : |x| \leq \hat{\kappa}_1\}$ be a ball. By the asymptotic a priori estimates of the random dynamical system ϕ with respect to L^2-norm, which play a crucial role in the proof of the $L^2(\mathbb{R}^n)$-asymptotic compactness $H^1(Q_{\hat{\kappa}_1}) \hookrightarrow L^2(Q_{\hat{\kappa}_1})$. It follows from (5.4) that, up to a subsequence depending on $\hat{\kappa}_1$

$$\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega)) \to \xi \quad \text{ strongly in } L^2(Q_{\hat{\kappa}_1}), \tag{5.5}$$

which shows that for the given $\zeta > 0$, there exist $\hat{N}_3 = \hat{N}_3(B, \omega)(B, \zeta, \omega) > 0$, such that for all $n \geq \hat{N}_3$,

$$\|\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega)) - \xi\|^2_{L^2(Q_{\hat{\kappa}_1})} \leq \zeta. \tag{5.6}$$

Note that $\xi \in L^2(\mathbb{R}^n)$. Therefore, there exist $\hat{\kappa}_2 = \hat{\kappa}_2(\zeta) > 0$, such that

$$\int_{|x| \geq \hat{\kappa}_2} |\xi(x)|^2 dx \leq \zeta. \tag{5.7}$$

By letting $\quad \hat{N} = \max\{\hat{N}_1, \hat{N}_2, \hat{N}_3\}$, and $\hat{\kappa} = \max\{\hat{\kappa}_1, \hat{\kappa}_2\}$.

Then, by (5.2),(5.6) and (5.7), we find that for all $n \geq \hat{N}$,

$$\|\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega)) - \xi\|^2_{L^2(\mathbb{R}^n)} \leq \int_{|x| \leq \hat{\kappa}} |\phi(t_n, \vartheta_{-t_n}\omega, v_{0,n}(\vartheta_{-t_n}\omega)) - \xi|^2 dx$$

$$+\int_{|x|\geq\hat{\kappa}}|\phi(t_n,\vartheta_{-t_n}\omega,v_{0,n}(\vartheta_{-t_n}\omega))-\xi|^2dx$$
$$\leq 6\zeta. \tag{5.8}$$

which shows that

$$\phi(t_n,\vartheta_{-t_n}\omega,v_{0,n}(\vartheta_{-t_n}\omega))\to\xi \quad \text{strongly in} \quad L^2(\mathbb{R}^n). \tag{5.9}$$

This as desired. □

We are now in a position to present our main result, the existence of a global random attractor for ϕ in $L^2(\mathbb{R}^n)$.

Lemma 5.2 Assume that $f^j, f\in L^2(\mathbb{R}^n)$, and (3.3)-(3.6) hold. Then the random dynamical system ϕ generated by (3.10) has a unique global random attractor in $L^2(\mathbb{R}^n)$.

Proof Notice that the random dynamical system ϕ has a random absorbing set $\{K(\omega)\}$ in $\mathfrak{D}$ by Lemma 4.1. On the other hand, by Lemma 5.1, the random dynamical system ϕ is asymptotically compact in $L^2(\mathbb{R}^n)$. Then by Theorem 2.6, the random dynamical system ϕ generated by (3.10) has a unique global random attractor in $L^2(\mathbb{R}^n)$. □

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