Linear Maps Preserving Rank-additivity and Rank-sum-minimal on Tensor Products of Matrix Spaces

Abstract: The problems of characterizing maps that preserve certain invariant on given sets are called the preserving problems, which have become one of the core research areas in matrix theory. In this paper, linear maps that preserve rank-additivity and rank-sum-minimal on tensor products of matrix spaces $M_{n_1} \otimes \cdots \otimes M_{n_k}$ are characterized respectively.

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1. Introduction

Let F be any number field, F^* be the set of all non-zero elements in F. V is the matrix space given on the domain F. For any $A \in V$, $R(A)$ denotes the rank of the matrix A. If the matrix pairs $A, B \in V$ satisfy $R(A+B) = R(A) + R(B)$ or $R(A+B) = \left|R(A) - R(B)\right|$, then it is said to be rank-additivity or rank-sum-minimal. For the linear map ϕ on V and pairs A, B, if $R(A+B) = R(A) + R(B)$ are deduced from $R\phi(A+B) = R\phi(A) + R\phi(B)$, we say that ϕ preserves the rank-additivity. If $R(A+B) = \left|R(A) - R(B)\right|$ are deduced from $R\phi(A+B) = \left|R\phi(A) - R\phi(B)\right|$ we say that ϕ preserves the rank-sum-minimal.

Let $M_{m\times n}$ be the set of all $m \times n$ matrices. When $m = n$, $M_{m\times n}$ is abbreviated as M_n. E_{ij} denotes the matrix with 1 at the (i, j) entry and 0 elsewhere, the order being determined by context. The notation Γ denotes the set of all tensor product matrices of rank 1 in $M_{n_1} \otimes \cdots \otimes M_{n_k}$. Let Θ denotes the set of the all tensor product matrices satisfying the rank-additivity in $M_{n_1} \otimes \cdots \otimes M_{n_k}$, and

Θ_- denotes the set of the all tensor product matrices satisfying the rank-sum-minimal in $M_{n_1} \otimes \cdots \otimes M_{n_k}$. For any $A_i \in M_{n_i} (i = 1, \cdots, k)$, we call a linear map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ canonical, if $\pi(A_1 \otimes \cdots \otimes A_k) = \tau_1(A_1) \otimes \cdots \otimes \tau_k(A_k)$, where $\tau_i : M_{n_i} \to M_{n_i}$, is either the identity map or the transposition map, and we abbreviated as $\pi = \tau_1 \otimes \cdots \otimes \tau_k$.

Many scholars have characterized linear maps preserving rank-additivity and rank-sum-minimal on the general matrix space. For example, Alexander Guteran[1] and Beasley[2] respectively described the linear maps that preserving the rank-additivity and the rank-sum-minimal on the n order matrix space and the $m \times n$ matrix space. Zhang Xian[3] further generalized the results of [2]. Then, ZhangXian[4] discussed the linear map of the rank-additivity and the rank-sum-minimal on the symmetric matrix space. Tang Xiaomin[5] characterized the linear map of the rank-additivity on the Hermitian matrix space. At present, the research on the linear preserving problem of rank-additivity and the rank-sum-minimal on the general matrix space has been basically improved.

The tensor product matrix space, as a special matrix space, has played a certain role in promoting the development of quantum information science[6]. Therefore, it is particularly important to study the problem of preserving the tensor product matrix space. The research results about preserving the tensor product matrix space are not too many. Zheng BD[7] characterizes the linear maps preserving rank of the tensor products of matrices. Zejun Huang[8] further studied the linear rank preservers of tensor products of rank one. These laid the foundation for subsequent research. This paper characterizes the linear maps preserving rank-additivity and rank-sum-minimal on tensor products of matrix spaces, which enriches the results of the tensor product matrix space preserving problem.

2. Preliminaries

The following theorems and lemmas are required for the two theorems in this paper.

Theorem 1. (7, main theorem) For any $A_1 \otimes \cdots \otimes A_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, a linear map $\phi : M_{n_1} \otimes \cdots \otimes M_{n_k} \to M_m$, preserves rank of tensor products of matrices, i.e.

$$R\phi(A_1 \otimes \cdots \otimes A_k) = R(A_1 \otimes \cdots \otimes A_k)$$

if and only if there exist invertible matrices $P, Q \in M_m$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\begin{bmatrix} \pi(X) & 0 \\ 0 & 0 \end{bmatrix}Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

Lemma 1. (3, Lemma1) $(A, B) \in \Theta$ for any $A, B \in M_{m\times n}$ if and only if there exist invertible matrices $P \in M_m$ and $Q \in M_n$ such that

$$A = P(I_u \oplus O)Q \;, B = P(O \oplus I_v)Q \quad , u + v \le \min(m, n) .$$

Lemma 2. $(A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k) \in \Theta$ for any $A_1 \otimes \cdots \otimes A_k$, $B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ if and only if $(A_1 \otimes \cdots \otimes A_k, h(B_1 \otimes \cdots \otimes B_k)) \in \Theta$ for any $h \in F$.

Proof: The conclusion can be deduced immediately from the rank property of the matrix.

3. Main Results

Theorem 2. For any $A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, a linear map, $\phi : M_{n_1} \otimes \cdots \otimes M_{n_k} \to M_{n_1} \otimes \cdots \otimes M_{n_k}$, as

$$R(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k) = R(A_1 \otimes \cdots \otimes A_k) + R(B_1 \otimes \cdots \otimes B_k)$$

established, there is

$$R(\phi(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k)) = R(\phi(A_1 \otimes \cdots \otimes A_k)) + R(\phi(B_1 \otimes \cdots \otimes B_k))$$

then $\phi = 0$, or there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

Proof. We distinguish two cases:

(i) Suppose ϕ is not injective. It is easy to see that there is $A_1 \otimes \cdots \otimes A_k$ matrix with rank $s(s \geq 1)$ such that $\phi(A_1 \otimes \cdots \otimes A_k) = 0$. Without loss of generality, hypothesizing that $\phi(I_s \oplus O) = 0$. By the $(E_{11} \otimes \cdots \otimes E_{11}, 0 \oplus I_{s-1} \oplus O) \in \Theta$, we have

$$(\phi(E_{11} \otimes \cdots \otimes E_{11},), \phi(0 \oplus I_{s-1} \oplus O)) \in \Theta$$

or equivalently.

$$R((\phi(E_{11} \otimes \cdots \otimes E_{11}) + \phi(0 \oplus I_{s-1} \oplus O)) = R(\phi(E_{11} \otimes \cdots \otimes E_{11})) + R(\phi(0 \oplus I_{s-1} \oplus O))$$

but $\phi(E_{11} \otimes \cdots \otimes E_{11}) + \phi(0 \oplus I_{s-1} \oplus O) = \phi(I_s \oplus O)$, i.e. ,

$$R(\phi(I_s \oplus O)) = R(\phi(E_{11} \otimes \cdots \otimes E_{11})) + R(\phi(0 \oplus I_{s-1} \oplus O))$$

Then according to $\phi(I_s \oplus O) = 0$, we obtain

$$\phi(E_{11} \otimes \cdots \otimes E_{11}) = 0 \tag{1}$$

For $j_i = 1, \cdots, n_i$ ($i = 1, \cdots, k$,and $j_1, \cdots, j_k$ can't be 1 at the same time), by the

$$(E_{11} \otimes \cdots \otimes E_{11} - E_{1j_1} \otimes \cdots \otimes E_{1j_k}, E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}) \in \Theta,$$

$$(E_{11} \otimes \cdots \otimes E_{11} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}, E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k}) \in \Theta$$

we have

$$(\phi(E_{11} \otimes \cdots \otimes E_{11} - E_{1j_1} \otimes \cdots \otimes E_{1j_k}), \phi(E_{j_1 j_1} \oplus \cdots \otimes E_{j_k j_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k})) \in \Theta,$$

$$(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}), \phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k})) \in \Theta$$

or equivalently.

$$R(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k}))$$

$$= R(\phi(E_{11} \otimes \cdots \otimes E_{11} - E_{1j_1} \otimes \cdots \otimes E_{1j_k})) + R(\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}))$$

$$R(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}))$$

$$= R(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{1j_1} \otimes \cdots \otimes E_{1j_k})) + R(\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k}))$$

By calculation, this, together with (1), the above two formulas can be simplified to

$$R(\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k}) = R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k}) + R(\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}))$$

$$R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} + E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k})) = R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k})) + R(\phi(E_{j_1 j_1} \otimes \cdots \otimes E_{j_k j_k}))$$

Then, we have $R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k})) = 0$, i.e. ,

$$\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k}) = 0 \tag{2}$$

The same can be proved that for $p_i = 1,\cdots,n_i$ ($i = 1,\cdots,k$, and $p_1,\cdots,p_k$ can't be 1 at the same time), then we can get

$$\phi(E_{p_1 1} \otimes \cdots \otimes E_{p_k 1}) = 0 \tag{3}$$

For any $j_i = 1,\cdots,n_i; p_i = 1,\cdots n_i (i = 1,\cdots,k)$, by the

$$(E_{1j_1} \otimes \cdots \otimes E_{1j_k} - E_{p_1 j_1} \otimes \cdots \otimes E_{p_k j_k}, E_{p_1 1} \otimes \cdots \otimes E_{p_k 1} + E_{p_1 j_1} \otimes \cdots \otimes E_{p_k j_k}) \in \Theta$$

we can get

$$(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} - E_{p_1 j_1} \otimes \cdots \otimes E_{p_k j_k}),\ \phi(E_{p_1 1} \otimes \cdots \otimes E_{p_k 1} + E_{p_1 j_1} \otimes \cdots \otimes E_{p_k j_k})) \in \Theta$$

or equivalently.

$$R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} + E_{p_1 1} \otimes \cdots \otimes E_{p_k 1}) = R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} - E_{p_1 j_1} \otimes \cdots \otimes E_{p_k j_k})) \\ + R(\phi(E_{p_1 1} \otimes \cdots \otimes E_{p_k 1} + E_{p_1 j_1} \otimes \cdots \otimes E_{p_k j_k}))$$

By calculation, together with (2) and (3) we can get

$$R(\phi(E_{p_1 j_1} \otimes \cdots \otimes E_{p_k j_k})) = 0 \tag{4}$$

Therefore, together with(2)-(4) and the linearity of ϕ, we can obtain $\phi = 0$.

(ii) Suppose ϕ is injective.

For the convenience of discussion, let $n = n_1 \cdots n_k$. For any $A_{11} \otimes \cdots \otimes A_{1k} \in \Gamma$, It is easy to see that there are $A_{21} \otimes \cdots \otimes A_{2k}, \cdots, A_{n1} \otimes \cdots \otimes A_{nk} \in \Gamma$ such that

$$(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n-1} A_{t1} \otimes \cdots \otimes A_{tk},\ A_{n1} \otimes \cdots \otimes A_{nk}) \in \Theta$$

By ϕ preserving rank-additivity, we have

$$\begin{aligned} & R(\phi(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n} A_{t1} \otimes \cdots \otimes A_{nk}) = \\ & R(\phi(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n-1} A_{t1} \otimes \cdots \otimes A_{(n-1)\ k})) + R(\phi(A_{n1} \otimes \cdots \otimes A_{nk})) \\ & = R(\phi(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n-2} A_{t1} \otimes \cdots \otimes A_{(n-2)k})) + R(\phi(A_{(n-1)1} \otimes \cdots \otimes A_{(n-1)k})) + \\ & \quad R(\phi(A_{n1} \otimes \cdots \otimes A_{nk})) \\ & = \cdots \\ & = R(\phi(A_{11} \otimes \cdots \otimes A_{1k})) + \sum_{t=2}^{n} R(\phi(A_{t1} \otimes \cdots \otimes A_{tk})). \end{aligned}$$

By the $A_{t1} \otimes \cdots \otimes A_{tk} \in \Gamma(t = 1,2,\cdots,n)$, and ϕ is injective, We obtain $R(\phi(A_{11} \otimes \cdots \otimes A_{1k})) = 1$.

For $B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ with arbitrary rank $r(r \geq 2)$, it is obvious that there exist matrices

$$B_{11} \otimes \cdots \otimes B_{1k}, \cdots, B_{r1} \otimes \cdots \otimes B_{rk} \in \Gamma$$

such that

$$B_1 \otimes \cdots \otimes B_k = \sum_{t=1}^{r} B_{t1} \otimes \cdots \otimes B_{tk}$$

By the $(\sum_{t=2}^{r-1} B_{t1} \otimes \cdots \otimes B_{tk}, B_{r1} \otimes \cdots \otimes B_{rk}) \in \Theta$, we have

$$\begin{aligned} R(\phi(B_1 \otimes \cdots \otimes B_k)) &= R(\phi(\sum_{t=1}^{r} B_{t1} \otimes \cdots \otimes B_{tk})) \\ &= R(\phi(\sum_{t=1}^{r-1} B_{t1} \otimes \cdots \otimes B_{tk})) + R(\phi(B_{r1} \otimes \cdots \otimes B_{rk})) \\ &= \ldots \\ &= \sum_{t=1}^{r} R(\phi(B_{t1} \otimes \cdots \otimes B_{tk})) \end{aligned}$$

Thus, according to the $\phi(B_{t1} \otimes \cdots \otimes B_{tk}) \in \Gamma(t = 1, \cdots, r)$, we obtain $R(\phi(B_1 \otimes \cdots \otimes B_k)) = r$, therefore ϕ preserves the rank of the tensor product matrix. This, together with Theorem 1, we can know there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

Theorem 3. For any $A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, a linear map, $\phi : M_{n_1} \otimes \cdots \otimes M_{n_k} \to M_{n_1} \otimes \cdots \otimes M_{n_k}$, as

$$R(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k) = \left| R(A_1 \otimes \cdots \otimes A_k) - R(B_1 \otimes \cdots \otimes B_k) \right|$$

established, there is

$$R(\phi(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k)) = \left| R(\phi(A_1 \otimes \cdots \otimes A_k)) - R(\phi(B_1 \otimes \cdots \otimes B_k)) \right|$$

Then $\phi = 0$, or there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

Proof. For any $A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, if

$$(A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k) \in \Theta,$$

Then, by Lemma 2, it is known to any $h \in F$, and there is $(A_1 \otimes \cdots \otimes A_k, h(B_1 \otimes \cdots \otimes B_k)) \in \Theta$. Furthermore, we can get $(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k), -h(B_1 \otimes \cdots \otimes B_k)) \in \Theta_-$,

$$(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k), -(A_1 \otimes \cdots \otimes A_k)) \in \Theta_-.$$

This, ϕ preserves the rank-sum-minimal of the tensor product matrix, we get

$$(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)), \phi(-h(B_1 \otimes \cdots \otimes B_k)) \in \Theta_-,$$

$$(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)), \phi(-(A_1 \otimes \cdots \otimes A_k))) \in \Theta_-.$$

Further calculations, we can get

$$\begin{aligned} & R(\phi(A_1 \otimes \cdots \otimes A_k)) \\ & = \left| R(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)) - R(\phi(-h(B_1 \otimes \cdots \otimes B_k))) \right| \\ & = \left| R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k)) - R(\phi(h(B_1 \otimes \cdots \otimes B_k))) \right| \end{aligned} \tag{5}$$

$$\begin{aligned} & R(\phi(h(B_1 \otimes \cdots \otimes B_k)) \\ & = \left| R(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)) - R(\phi(-(A_1 \otimes \cdots \otimes A_k))) \right| \\ & = \left| R(\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k)) - R(\phi(A_1 \otimes \cdots \otimes A_k)) \right| \end{aligned} \tag{6}$$

Next, we will discuss it in three situations.

(i): Suppose that there is $h_0 \in F^*$, such that

$$R(\phi(A_1 \otimes \cdots \otimes A_k) + h_0\phi(B_1 \otimes \cdots \otimes B_k)) \geq R(\phi(h_0(B_1 \otimes \cdots \otimes B_k)))$$

Then, together with (5), yields that

$$R(\phi(A_1 \otimes \cdots \otimes A_k)) = R(\phi(A_1 \otimes \cdots \otimes A_k) + h_0\phi(B_1 \otimes \cdots \otimes B_k)) - R(\phi(h_0(B_1 \otimes \cdots \otimes B_k))$$

Therefore

$$R(\phi(A_1 \otimes \cdots \otimes A_k) + h_0\phi(B_1 \otimes \cdots \otimes B_k)) = R(\phi(A_1 \otimes \cdots \otimes A_k)) + R(\phi(h_0(B_1 \otimes \cdots \otimes B_k))$$

$(\phi(A_1 \otimes \cdots \otimes A_k), h_0\phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta$, this, together with lemma 2, we can obtain

$$(\phi(A_1 \otimes \cdots \otimes A_k), \phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta.$$

(ii)：Suppose that there is $h_0 \in F^*$, such that

$R((\phi(A_1 \otimes \cdots \otimes A_k + h_0\phi(B_1 \otimes \cdots \otimes B_k)) \geq R(\phi(A_1 \otimes \cdots \otimes A_k))$, by a similar argument to (i), and together with (6), we can gain $(\phi(A_1 \otimes \cdots \otimes A_k), \phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta$.

(iii)：For any $h \in F^*$, hypothesizing

$$R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k)) \leq R(\phi(h(B_1 \otimes \cdots \otimes B_k))),$$

$$R((\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k)) \leq R(\phi(A_1 \otimes \cdots \otimes A_k)).$$

This together with(5) and (6), it is easy to see that

$$R(\phi(A_1 \otimes \cdots \otimes A_k)) = R(\phi(h(B_1 \otimes \cdots \otimes B_k))) - R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k))$$
$$R(\phi(h(B_1 \otimes \cdots \otimes B_k))) = R(\phi(A_1 \otimes \cdots \otimes A_k)) - R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k))$$

consequently, $R(\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k)) = 0$, i.e. ,

$$\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k) = 0.$$

By the arbitrariness of h, we can know $\phi(A_1 \otimes \cdots \otimes A_k) = 0$ and $(\phi(A_1 \otimes \cdots \otimes A_k), \phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta$.

By combining the above three situations, we can obtain that ϕ preserves rank-additivity on tensor products of matrix spaces. Thus, by Theorem 2, we can get

$\phi = 0$, or there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

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