Original Research Article

The Gasca-Maeztu conjecture for $n = 4$

Abstract

We consider planar GC_n node sets, i.e., n-poised sets whose all n-fundamental polynomials are products of n linear factors. Gasca and Maeztu conjectured in 1982 that every such set possesses a maximal line, i.e., a line passing through $n+1$ nodes of the set. Till now the conjecture is confirmed to be true for $n \leq 5$. The case $n = 5$ was proved recently by H. Hakopian, K. Jetter, and G. Zimmermann (Numer. Math. **127** (2014) 685–713). In this paper we bring a short and simple proof of the conjecture for $n = 4$.

Key words: Polynomial interpolation, Gasca-Maeztu conjecture, fundamental polynomial, maximal line, n-poised set, n-independent set.

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1 Introduction

Denote by Π_n the space of bivariate polynomials of total degree at most n :

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j : a_{ij} \in \mathbb{R} \right\}.$$

We have that

$$N := N_n := \dim \Pi_n = \binom{n+2}{2}.$$

Consider a set of distinct nodes

$$\mathcal{X}_s = \{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \qquad i = 1, 2, \ldots s, \tag{1.1}$$

is called interpolation problem.

Definition 1.1. The interpolation problem with the set of nodes $\mathcal{X}_s$ is called n-poised if for any data $\{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions (1.1).

A polynomial $p \in \Pi_n$ is called an n-fundamental polynomial for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik}, i = 1, \ldots, s,$$

where δ is the Kronecker symbol. We denote the n-fundamental polynomial of $A \in \mathcal{X}_s$ by $p_A^\star = p_{A,\mathcal{X}_s}^\star$.

A necessary condition of n-poisedness is: $s = N$. In this latter case the following holds:

Proposition 1.2. *The set of nodes $\mathcal{X}_N$ is n-poised if and only if for any polynomial $p \in \Pi_n$ we have*

$$p(x_i, y_i) = 0 \quad i = 1, \ldots, N \Rightarrow p = 0.$$

Definition 1.3. A set of nodes $\mathcal{X}$ is called n-independent if all its nodes have n-fundamental polynomials. Otherwise, $\mathcal{X}$ is called n-dependent.

Fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence is $\#\mathcal{X} \leq N$. Suppose a node set $\mathcal{X}_s$ is n-independent. Then we have following Lagrange formula for a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1):

$$p(x, y) = \sum_{A \in \mathcal{X}_s} c_A p_{A,\mathcal{X}_s}^\star. \tag{1.2}$$

In view of this formula we readily get that the node set $\mathcal{X}_s$ is n-independent if and only if the interpolating problem (1.1) is solvable, i.e., for any data $\{c_1, \ldots, c_s\}$ there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

We shall use the same letter, most often ℓ to denote the linear polynomial $\ell \in \Pi_1$ and the line defined by the equation $\ell(x, y) = 0$.

Definition 1.4. Given an n-poised set $\mathcal{X}$, we say, that a node $A \in \mathcal{X}$ uses a line ℓ, if ℓ is a factor of the fundamental polynomial $p_{A,\mathcal{X}}^\star$.

The following proposition is well-known (see e.g. [8] Proposition 1.3):

Proposition 1.5. *Suppose that ℓ is a line. Then for any polynomial $p \in \Pi_n$ vanishing at $n+1$ points of ℓ we have*

$$p = \ell r, \quad \textit{where} \quad r \in \Pi_{n-1}.$$

From here we readily get that at most $n+1$ nodes of an n-poised set $\mathcal{X}_N$ can be collinear and the line ℓ, containing $n+1$ nodes, is used by all the nodes in $\mathcal{X}_N \setminus \ell$. In view of this a line ℓ containing $n+1$ nodes of an n-poised set $\mathcal{X}$ is called a maximal line [3].

In the sequel we will use the particular case $n = 3$ of the following

Proposition 1.6. *Any set of at most $2n+1$ points in the plain is n-dependent if and only if $n+2$ of points are collinear.*

Now let us define the following set of nodes:

Definition 1.7. For the given line ℓ we define $\mathcal{N}_\ell$ to be the set of all nodes in $\mathcal{X}$, which do not lie in ℓ and do not use ℓ:

$$\mathcal{N}_\ell = \{A \in \mathcal{X} : A \notin \ell \quad \text{and} \quad A \quad \text{is not using} \quad \ell\}.$$

Theorem 1.8 ([5])**.** *Suppose, that we have a line ℓ and an n-poised set $\mathcal{X}$. Then the following hold:*

(i) *If the set $\mathcal{N}_\ell$ is nonempty, then it is $(n-1)$-dependent and for no node $A \in \mathcal{N}_\ell$, there exists a fundamental polynomial $p^\star_{A,\mathcal{N}_\ell}$ in Π_{n-1}.*

(ii) *$\mathcal{N}_\ell = \emptyset$ if and only if ℓ passes through $n+1$ nodes in $\mathcal{X}$.*

2 The Gasca-Maeztu conjecture and GC_n-sets

Now we are going to consider a special type of n-poised sets whose n-fundamental polynomials are products of n linear factors as it always takes place in the univariate case.

Definition 2.1 (Chung, Yao [6])**.** An n-poised set $\mathcal{X}$ is called GC_n-set, if each node $A \in \mathcal{X}$ has an n-fundamental polynomial which is a product of n linear factors.

Since the fundamental polynomial of an n-poised set is unique we get (see e.g. [9], Lemma 2.5)

Lemma 2.2 ([9])**.** *Suppose $\mathcal{X}$ is a poised set and a node $A \in \mathcal{X}$ uses a line $\ell:\ p^\star_A = \ell q, q \in \Pi_{n-1}$. Then ℓ passes through at least two nodes from $\mathcal{X}$, at which q does not vanish.*

Now we are in a position to present the Gasca-Maeztu conjecture.

Conjecture 2.3 (Gasca, Maeztu [7]). *Any GC_n-set $\mathcal{X}$ possesses a maximal line, i.e., a line passing through its $n+1$ nodes.*

The Gasca-Maeztu conjecture is proved to be true for $n \leq 5$. The case $n = 4$ was proved for the first time by J.R. Busch [4]. The case $n = 5$ was proved recently by H. Hakopian, K. Jetter, and G. Zimmermann in [**?**]. In this paper we bring a short and simple proof of the conjecture for $n = 4$.

2.1 The Gasca-Maeztu conjecture for $n = 4$

We start with the formulation of the Gasca-Maeztu conjecture for $n = 4$ as:

Theorem 2.4. *Any GC_4-set $\mathcal{X}$ of 15 nodes possesses a maximal line, i.e., a line passing through 5 nodes.*

To prove the theorem assume by way of contradiction the following.

Assumption 2.5. *The set $\mathcal{X}$ is a GC_4-set without any maximal line.*

We call a line k-node line if it passes through exactly k nodes of the set $\mathcal{X}$. In the next subsection we discuss the problem: Given a $2, 3$ or 4-node line. By how many nodes in $\mathcal{X}$ it can be used at most.

The following lemma is in ([9], Lemma 4.1). We bring it here for the sake of completeness.

Lemma 2.6. *Any 2 or 3-node line can be used by at most one node of $\mathcal{X}$.*

Proof. Assume by contradiction that ℓ is a 2 or 3-node line used by two points $A, B \in \mathcal{X}$. Consider the fundamental polynomial $p_A^\star$. The node A uses the line ℓ and three more lines, which contain the remaining ≥ 11 nodes of $\mathcal{X} \setminus (\ell \cup \{A\})$, including B. Since there is no 5-node line, we get

$$p_A^\star = \ell \ell_{=4} \ell'_{=4} \ell_{\geq 3}.$$

Here the subscript $= 4$ means that the corresponding line is a 4-node line, while the subscript ≥ 3 means that except the 3 nodes the corresponding line may also pass through some nodes belonging to the other lines. First suppose that B belongs to one of the 4-node lines, say to $\ell'_{=4}$. We have also

$$p_B^\star = \ell q, \text{ where } q \in \Pi_3.$$

Notice that q vanishes at 4 nodes of $\ell_{=4}$ and 3 nodes of $\ell'_{=4}$ (i.e., except B). Therefore by using Proposition 1.5 twice we get that $q = \ell_{=4} r,\ r \in \Pi_2$ and $r = \ell'_{=4} s,\ s \in \Pi_1$. Thus $p_B^\star = \ell \ell_{=4} \ell'_{=4} s$. Hence $p_B^\star$ vanishes at B ($B \in \ell'_{=4}$), which is a contradiction.

Now assume that B belongs to the line $\ell_{\geq 3}$. Then q vanishes at 4 nodes of $\ell_{=4}$, 4 (≥ 3) nodes of $\ell'_{=4}$ and at least 2 nodes of $\ell_{\geq 3}$. Therefore again, as above, by consecutive usage of Proposition 1.5 we get that $p^\star_B = \ell\ell_{=4}\ell'_{=4}\ell_{\geq 3}$. Hence again $p^\star_B$ vanishes at B ($B \in \ell_{\geq 3}$), which is a contradiction. □

The following lemma is in ([1], Lemma 2.6). Here we bring a very short proof of it.

Lemma 2.7. *Any* 4*-node line can be used by at most three nodes of* $\mathcal{X}$*.*

Proof. Assume by contradiction that ℓ is a 4-node line used by four points from $\mathcal{X}$. Therefore we have $\#\mathcal{N}_\ell \leq 15 - 4 - 4 = 7$. In view of Theorem 1.8 $\mathcal{N}_\ell \neq \emptyset$ is (essentially) 3-dependent. According to Theorem 1.6 a set of $\leq 2 \times 3 + 1 = 7$ nodes is 3-dependent if and only if there is a 5-node line, which contradicts Assumption 2.5. □

Now we are in a position to prove the Gasca-Maeztu conjecture for $n = 4$.

2.2 Proof of the Gasca-Maeztu conjecture for $n = 4$

Let us start with an observation from ([10], Section 3.2). Fix any node $A \in \mathcal{X}$, and consider all the lines through the node A and some other node(s) of $\mathcal{X}$. Denote this set of lines by $\mathcal{L}_A$. Let $n_m(A)$ be the number of m-node lines from $\mathcal{L}_A$. In view of Assumption 2.5 we have

$$1n_2(A) + 2n_3(A) + 3n_4(A) = \#\big(\mathcal{X} \setminus \{A\}\big) = 14. \tag{2.1}$$

Denote by $M(A)$ the total number of uses of the lines passing through A. By Lemma 2.2 each of 14 nodes of $\mathcal{X} \setminus \{A\}$ uses at least one line from $\mathcal{L}_A$. On the other hand, we get from Lemmas 2.6 and 2.7 that

$$14 \leq M(A) \leq 1n_2(A) + 1n_3(A) + 3n_4(A).$$

Comparing this with (2.1), we conclude that necessarily $M(A) = 14$ and $n_3(A) = 0$, i.e., there is no 3-node line in $\mathcal{L}_A$.

Thus we have

$$n_2(A) + 3n_4(A) = 14. \tag{2.2}$$

Therefore each 4-node line in $\mathcal{L}_A$ is used exactly three times and each 2-node line is used exactly once. From here we conclude easily that $n_2(A) \geq 2$. Next we show that actually $n_2(A) = 2$.

Consider two 2-node lines passing through A. Suppose except A they pass through B and C, respectively. Denote these two lines by ℓ_B and ℓ_C, respectively (see Fig 2.1).

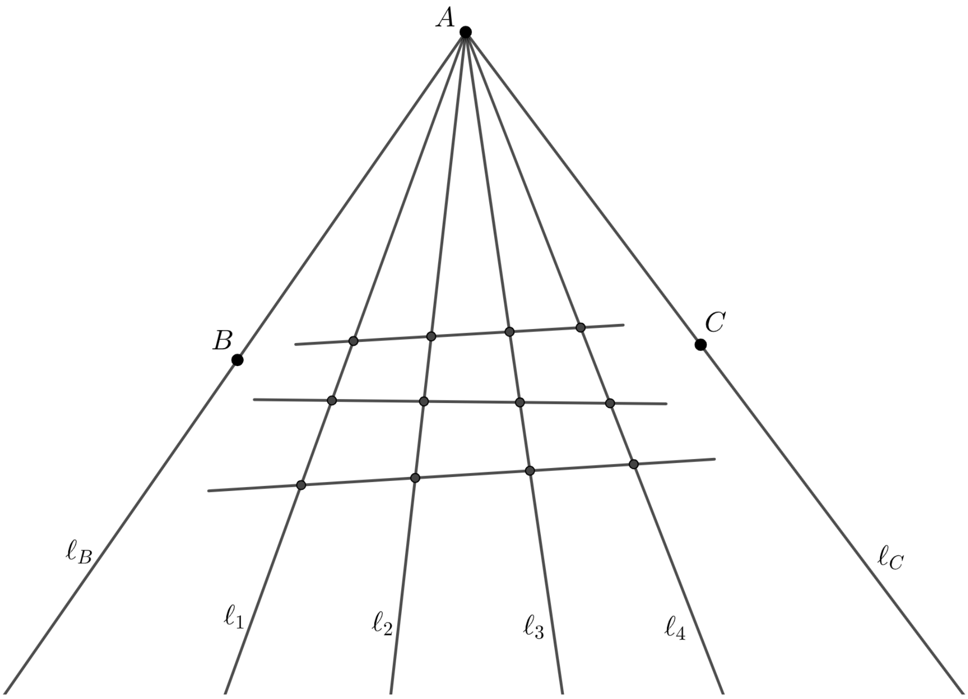

Figure 2.1: The lines of $\mathcal{L}_A$

Next, we will prove that B uses ℓ_C. Let us verify that in this case the node C uses ℓ_B. Indeed, if B uses ℓ_C we have $p_B^\star = \ell_C q$, where q is a product of three lines. Notice that the polynomial $\ell_B q$ is the fundamental polynomial of the node C, which means that C uses ℓ_B. Now, suppose by way of contradiction that B does not use ℓ_C. Therefore C does not use ℓ_B.

Thus, there are two nodes D and E in the 12 nodes of $\mathcal{X} \setminus \{A, B, C\}$ using the lines ℓ_B and ℓ_C respectively. In this case, we have $p_D^\star = \ell_B q_1$ and $p_E^\star = \ell_C q_2$, where q_1 and q_2 are polynomials of degree 3.

Since q_1 and q_2 have 10 common nodes we get from the Bezout theorem that they have common linear factor α, passing through at most 4 nodes. So we can write $q_1 = \alpha\beta_1$ and $q_2 = \alpha\beta_2$, where β_1 and β_2 have at least 6 common nodes. Therefore, β_1 and β_2 have common linear factor α_1, passing through at most 4 nodes.

Now, we have for the following presentations of the fundamental polynomials: $p_D^\star = \ell_B \alpha\alpha_1\alpha_2$ and $p_E^\star = \ell_C \alpha\alpha_1{\alpha_2}'$. Therefore α_2 and ${\alpha_2}'$ have at least two common nodes, which means that they coincide. We have that $E \in \alpha \cup \alpha_1 \cup \alpha_2$ and thus come to a contradiction, which proves that B uses ℓ_C.

Note that ℓ_C was an arbitrary 2-node line, which means that B uses all

2-node lines different from ℓ_B. It is easy to see that any node from $\mathcal{X}$ can use at most one 2-node line, since otherwise if some node uses two 2-node lines the remaining ≥ 10 nodes have to lie on two. Therefore, we conclude that there are no 2-node lines other than ℓ_B and ℓ_C, i.e., $n_2(A) = 2$. From here and the equality (2.2) we get $n_4(A) = 4$.

Thus, the 12 nodes of $\mathcal{X} \setminus \{A, B, C\}$ lie on four 4-node lines passing through A. We denote these lines by $\ell_1, ..., \ell_4$.

Finally, by taking $p(x, y) = \ell_1\ell_2\ell_3\ell_4$, in the Lagrange formula (1.2), we obtain

$$\ell_1\ell_2\ell_3\ell_4 = \lambda_1 p_B^\star + \lambda_2 p_C^\star, \tag{2.3}$$

since $\ell_1\ell_2\ell_3\ell_4$ vanishes in $\mathcal{X}\setminus\{B, C\}$. Now recall that $p_B^\star = \ell_C q$ and $p_C^\star = \ell_B q$, where q is a product of three 4-node lines passing through the 12 nodes of $\mathcal{X} \setminus \{A, B, C\}$. Thus we get

$$\ell_1\ell_2\ell_3\ell_4 = q(\lambda_1\ell_C + \lambda_2\ell_B).$$

Clearly none of the lines ℓ_i here is a factor of q. Hence this leads to a contradiction, which proves Theorem 2.4.

References

[1] V. Bayramyan, H. Hakopian and S. Toroyan, A simple proof of the Gasca-Maeztu conjecture for n = 4, Jaén J. Approx. **7** (2015) 137147.

[2] D. Eisenbud, M. Green and J. Harris (1996) Cayley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. (N.S.), 33(3), 295–324.

[3] C. de Boor, Multivariate polynomial interpolation: conjectures concerning GC-sets, Numer. Algorithms **45** (2007) 113–125.

[4] J. R. Busch, A note on Lagrange interpolation in $\mathbb{R}^2$, Rev. Un. Mat. Argentina **36** (1990) 33–38.

[5] J. M. Carnicer and M. Gasca, A conjecture on multivariate polynomial interpolation, Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.), Ser. A Mat. **95** (2001) 145–153.

[6] K. C. Chung and T. H. Yao, On lattices admitting unique Lagrange interpolations, SIAM J. Numer. Anal. **14** (1977) 735–743.

[7] M. Gasca and J. I. Maeztu, On Lagrange and Hermite interpolation in $\mathbb{R}^k$, Numer. Math. **39** (1982) 1–14.

[8] H. Hakopian, K. Jetter, and G. Zimmermann, Vandermonde matrices for intersection points of curves, Jaén J. Approx. **1** (2009) 67–81.

[9] H. Hakopian, K. Jetter, and G. Zimmermann, A new proof of the Gasca-Maeztu conjecture for $n = 4$, J. Approx. Theory **159** (2009) 224–242.

[10] H. Hakopian, K. Jetter, and G. Zimmermann, The Gasca-Maeztu conjecture for $n = 5$, Numer. Math. **127** (2014) 685–713.

[11] S. Toroyan, On a conjecture in bivariate interpolation, Proc. of YSU, Phys. and Math. Sci., **1** (2016) 30-34.