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# Initial value problems for second order neutral impulsive integro-differential equations with advanced argument

***Original research papers***

## Abstract

Aims/ This paper discusses initial value problems for second order neutral impulsive integro-differential equations with advanced argument. By using the fixed point theorem of either Leray-Schaude or Banach, some existence results are obtained.

*Keywords: Neutral impulsive integro-differential equation; Second order; Initial value; Fixed point*

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# 1 Introduction

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc. About initial value problems for impulsive differential equations, many authors have obtained very good existence results (for example, see [1-7]). Now consider the following equation

$$\begin{cases} (u(\phi(t)))'' = f(t, u(t), u'(t), Ku(t), Hu(t)),\ t \in J = [0, a],\ t \neq \xi_k, \\ \Delta u(t_k) = I_{0k}(u(t_k)),\ \Delta u'(t_k) = I_{1k}(u'(t_k)),\ k = 1, \cdots, p, \\ u(0) = u_0,\ u'(0) = u_0', \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = a$, $\phi \in C^2(J, \mathbb{R})$, $\phi$ is monotone increasing with $t \leq \phi(t) \leq a$ $(t \in J)$, $\phi(0) = 0$, $\phi(a) = a$, $\phi'(t) > 0$ with $\phi^{-1} \in C^2(J, \mathbb{R})$, and let $\phi(\xi_k) = t_k$ $(k = 1, \cdots, p)$, $J^* = J \backslash \{t_1, \cdots, t_p\}$, $\bar{J} = J \backslash \{\xi_1, \cdots, \xi_p\}$, $f : J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous everywhere except at $\{\xi_k\} \times \mathbb{R}^4$, $f(\xi_k^+, x, x', y_1, y_2)$ and $f(\xi_k^-, x, x', y_1, y_2)$ exist, $f(\xi_k^-, x, x', y_1, y_2) = f(\xi_k, x, x', y_1, y_2)$, and $Ku(t) = \int_0^t k(t, s)u(s)ds$, $Hu(t) = \int_0^T h(t, s)u(s)ds$, $k(t, s) \in C(D, \mathbb{R}^+)$, $h(t, s) \in C(J \times J, \mathbb{R}^+)$, $D = \{(t, s) \in \mathbb{R}^2, 0 \leq s \leq t \leq a\}$, $k_0 = \max\{k(t, s) : (t, s) \in D\}$, $h_0 = \max\{h(t, s) : (t, s) \in J \times J\}$, further and $I_{0k}, I_{1k} \in C(\mathbb{R}, \mathbb{R})$, $\Delta u(t_k) = u(t_k^+) - u(t_k), \Delta u'(t_k) = u'(t_k^+) - u'(t_k)$. Denote by $PC(X, Y)$, where $X \subset \mathbb{R}, Y \subset \mathbb{R}$, the set of all functions $u : X \to Y$ which are piecewise continuous in $X$ with points of discontinuity of the first kind at the points $t_k \in X$, i.e., there exist the limits $u(t_k^+) < \infty$ and $u(t_k^-) = u(t_k) < \infty$.

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## 2 Preliminaries

According to the properties of $\phi$, there exist positive constants $m_1$ and $m_2$ such that $m_1 \leq \phi'(t) \leq m_2$ for all $t \in J$.

Let $E_0 = \{u|u, u' \in PC(J, \mathbb{R})\} \bigcap C^2(J^*, \mathbb{R})$. Evidently, $E_0$ is a real Banach space with norm $\|u(t)\|_{E_0} = \max\{\|u(t)\|_{PC}, \|u'(t)\|_{PC}\}$, where $\|u(t)\|_{PC} = \sup_{t\in J} |u(t)|$, $\|u'(t)\|_{PC} = \sup_{t\in J} |u'(t)|$. Further, let $E = \{u(\phi(t))|u(t) \in E_0\}$. We can check that $E$ is also a real Banach space with norm $\|u(\phi(t))\| = \max\{\|u(\phi(t))\|_{PC}, \|(u(\phi(t)))'\|_{PC^*}\}$, where $\|u(\phi(t))\|_{PC} = \sup_{t\in J} |u(\phi(t))| = \|u(t)\|_{PC}$, $\|(u(\phi(t)))'\|_{PC^*} = \sup\limits_{\phi(t)\in J} \left|\frac{du(\phi(t))}{d\phi(t)}\right| \cdot \sup\limits_{t\in J} \frac{d\phi}{dt} = \sup\limits_{t\in J} |u'(t)| \cdot m_2 = m_2\|u'(t)\|_{PC}$.

Define operator $B : u(t) \longmapsto u(\phi(t))$, where $u(t) \in E_0$ and $u(\phi(t)) \in E$. It is evident that $B$ is topological linear isomorphic, which implies that $E$ is a real Banach space.

Since $\frac{\phi(a) - \phi(0)}{a - 0} = \phi'(\bar{t})$ $(0 < \bar{t} < a)$, i.e., $\phi'(\bar{t}) = 1$, we get $m_2 \geq 1$, next $\|(u(\phi(t)))'\|_{PC^*} = m_2\|u'(t)\|_{PC} \geq \|u'(t)\|_{PC}$, so

$$\|u(t)\|_{E_0} \leq \|u(\phi(t))\|. \tag{2.1}$$

**Lemma 2.1.** *$u(t) \in E_0$ is a solution of (1.1) if and only if $u(t) \in E_0$ is a solution of the following integral equation*

$$\begin{aligned} u(\phi(t)) = u_0 + u_0't + \int_0^t (t-s)f(s, u(s), u'(s), Ku(s), Hu(s))ds + \\ \textstyle\sum_{0<\xi_k<t}[I_{0k}(u(t_k)) + (t - \xi_k)I_{1k}(u'(t_k))],\ t \in J. \end{aligned} \tag{2.2}$$

*Proof.* (i) Necessity

For $\xi_k < t \leq \xi_{k+1}$ $(k = 0, 1, \cdots, p)$, by (1.1), we get

$$u(t_1) - u(0) = u(\phi(\xi_1)) - u(\xi(0)) = \int_0^{\xi_1} (u(\phi(s)))'ds,$$

$$u(t_2) - u(t_1^+) = u(\phi(\xi_2)) - u(\phi(\xi_1^+)) = \int_{\xi_1}^{\xi_2} (u(\phi(s)))'ds,$$

$$\cdots\cdots$$

$$u(t_k) - u(t_{k-1}^+) = u(\phi(\xi_k)) - u(\phi(\xi_{k-1}^+)) = \int_{\xi_{k-1}}^{\xi_k} (u(\phi(s)))'ds,$$

$$u(\phi(t)) - u(t_k^+) = u(\phi(t) - u(\phi(\xi_k^+)) = \int_{\xi_k}^t (u(\phi(s)))'ds.$$

Adding these together, we obtain

$$u(\phi(t)) = u(0) + \int_0^t (u(\phi(s)))'ds + \sum_{i=1}^k [x(t_i^+) - x(t_i)],$$

$$u(\phi(t)) = u_0 + \int_0^t (u(\phi(s)))'ds + \sum_{0<\xi_k<t} I_{0k}(u(t_k)),\ t \in J. \tag{2.3}$$

Similarly, we obtain

$$(u(\phi(t)))' = u_0' + \int_0^t (u(\phi(s)))''ds + \sum_{0<\xi_k<t} I_{1k}(u'(t_k)),\ t \in J. \tag{2.4}$$

Substituting (2.4) into (2.3), it is easy to get (2.2)

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(ii) Sufficiency
According to (2.2), it is clear that

$$u(0) = u_0,\ \Delta u(t_k) = I_{0k}(u(t_k)). \tag{2.5}$$

Differentiating both sides of (2.2), we have

$$(u(\phi(t)))' = u_0' + \int_0^t f(s, u(s), u'(s), Ku(s), Hu(s))ds + \sum_{0<\xi_k<t} I_{1k}(u'(t_k)),\ t \in J. \tag{2.6}$$

Similarly, we also have

$$(u(\phi(t)))'' = f(t, u(t), u'(t), Ku(t), Hu(t)),\ t \in \bar{J}. \tag{2.7}$$

By(2.6), it is evident that

$$u'(0) = u_0',\ \Delta u'(t_k) = I_{1k}(u'(t_k)). \tag{2.8}$$

From (2.5),(2.7) and (2.8), we get that $u(t)$ is a solution of (1.1).

□

**Lemma 2.2.** *(Leray-Schauder [6]) Let the operator $A : X \to X$ be completely continuous, where $X$ is a real Banach space. If the set $G = \{\|x\| \,|\, x \in X, x = \lambda Ax, 0 < \lambda < 1\}$ is bounded, then the operator $A$ has at least one fixed point in the closed ball $T = \{x | x \in X, \|x\| \le R\}$, where $R = \sup G$.*

**Lemma 2.3.** *(Compactness criterion [7]) $H \subset PC(J, \mathbb{R})$ is a relatively compact set if and only if $H \subset PC(J, \mathbb{R})$ is uniformly bounded and equicontinuous on every $J_k$ $(k = 0, \cdots, p)$, where $J_0 = [t_0, t_1], J_k = (t_k, t_{k+1}]$ $(k = 1, \cdots, p)$.*

# 3 Main Result

Let us introduce the following conditions for later use:

(H1) There exist nonnegative constants $b, c, d_i$ $(i = 1, 2)$, $b_k, c_k$ $(k = 1, \cdots, p)$, and $g \in L(J, \mathbb{R}^+)$ such that $|f(t, x_2, y_2, z_{12}, z_{22}) - f(t, x_1, y_1, z_{11}, z_{21})|$
$\le g(t)\big(b\|x_2 - x_1\|_{PC} + c\|y_2 - y_1\|_{PC} + \sum_{i=1}^{2} d_i\|z_{i2} - z_{i1}\|_{PC}\big),\ t \in J,$
$|I_{0k}(x_2(t_k)) - I_{0k}(x_1(t_k))| \le b_k|x_2(t_k) - x_1(t_k)|,\ I_{0k}(0) = 0,\ k = 1, \cdots, p,$
$|I_{1k}(y_2(t_k)) - I_{1k}(y_1(t_k))| \le c_k|y_2(t_k) - y_1(t_k)|,\ I_{1k}(0) = 0,\ k = 1, \cdots, p,$
where $x_1, x_2 \in E_0,\ y_i(t) = \bar{y}_i'(t),\ \bar{y}_i(t)\ (i = 1, 2) \in E_0,\ z_{1i} = K\bar{z}_{1i},\ z_{2i} = H\bar{z}_{2i},\ \bar{z}_{1i},\ \bar{z}_{2i}$ $(i = 1, 2) \in E_0,\ a_0 = \int_0^a g(t)dt.$

(H2) There exist positive constant $M$ such that $|f(t, u(t), u'(t), Ku(t), Hu(t))| \le M(1 + \|u(t)\|_{E_0})$.

(H3) $l = \max\{l_1, l_2\} < 1$, where $l_1 = a^2M + \sum_{k=1}^{p}(b_k + ac_k),\ l_2 = \frac{m_2}{m_1}\big(aM + \sum_{k=1}^{p} c_k\big)$.

(H4) $r = \max\{r_1, r_2\} < 1$, where $r_1 = aa_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p}(b_k + ac_k)$,
$r_2 = \frac{m_2}{m_1}\big[a_0\big(b + c + ad_1k_0 + ad_2h_0\big) + \sum_{k=1}^{p} c_k\big].$

**Theorem 3.1.** *If conditions (H1),(H2) and (H3) are satisfied, then (1.1) has at least one solution in the closed ball $\overline{B} = \{u(\phi(t)) | u(\phi(t)) \in E,\ \|u(\phi(t))\| \le R\}$, where $R = \sup G,\ G = \{\|u(\phi(t))\| \,|\, u(\phi(t)) \in E,\ u(\phi(t)) = \lambda Au(\phi(t)),\ 0 < \lambda < 1\}$.*

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*Proof.* (i) For any $u(\phi(t)) \in E$ define the operator $A$ by

$$Au(\phi(t)) = u_0 + u_0't + \int_0^t (t-s)f(s,u(s),u'(s),Ku(s),Hu(s))ds + \sum_{0<\xi_k<t}[I_{0k}(u(t_k)) + (t-\xi_k)I_{1k}(u'(t_k))],\ t \in J. \tag{3.1}$$

It is easy to see that $Au(\phi(t)) \in E_0$. According to the properties of $\phi$, for any $v(t) \in E_0$, we have $v(t) = v(\phi^{-1}(\phi(t))) = v\phi^{-1}(\phi(t))$. Let $u = v\phi^{-1}$. Next, it is clear that $v(t) = u(\phi(t)) \in E$. It follows that $A$ maps $E$ into $E$. Thus $Au(\phi(t)) \in E$ with

$$(Au(\phi(t)))' = u_0' + \int_0^t f(s,u(s),u'(s),Ku(s),Hu(s))ds + \sum_{0<\xi_k<t} I_{1k}(u'(t_k)),\ t \in J. \tag{3.2}$$

$A$ is a completely continuous operator will be verified by the following tree steps.

Step1. $A$ is continuous.

Let any $u_n(\phi(t))\ (n = 1,2,\cdots),\ u(\phi(t)) \in E$ with $\|u_n(\phi(t)) - u(\phi(t))\| \to 0$ as $n \to \infty$. By (3.1) and (H1), we have

$$\begin{aligned}|Au_n(\phi(t)) - Au(\phi(t))| \le &\int_0^t (t-s)g(s)\big[b\|u_n(s)-u(s)\|_{PC} + \\ &c\|u_n'(s)-u'(s)\|_{PC} + d_1\|Ku_n(s)-Ku(s)\|_{PC} + d_2\|Hu_n(s)-Hu(s)\|_{PC}\big]ds + \\ &\sum_{0<\xi_k<t}[b_k|u_n(t_k)-u(t_k)| + (t-\xi_k)c_k|u_n'(t_k)-u'(t_k)|] \\ \le &\big(b+c+ad_1k_0+ad_2h_0\big)\|u_n(t)-u(t)\|_{E_0}\int_0^t (t-s)g(s)ds + \\ &\|u_n(t)-u(t)\|_{E_0}\sum_{0<\xi_k<t}[b_k + (t-\xi_k)c_k],\end{aligned}$$

$$|Au_n(\phi(t)) - Au(\phi(t))| \le \Big[aa_0\big(b+c+ad_1k_0+ad_2h_0\big) + \sum_{k=1}^p (b_k + ac_k)\Big]\|u_n(t)-u(t)\|_{E_0},\ t \in J. \tag{3.3}$$

Then from (3.3) and (2.1), we have

$$\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \le \Big[aa_0\big(b+c+ad_1k_0+ad_2h_0\big) + \sum_{k=1}^p (b_k + ac_k)\Big]\|u_n(t)-u(t)\|_{E_0},$$

$$\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \le \Big[aa_0\big(b+c+ad_1k_0+ad_2h_0\big) + \sum_{k=1}^p (b_k + ac_k)\Big]\|u_n(\phi(t))-u(\phi(t))\|. \tag{3.4}$$

Thus

$$\|Au_n(\phi(t)) - Au(\phi(t))\|_{PC} \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Similarly, from (3.2) and (2.1), we get

$$\begin{aligned}\left|\frac{d[Au_n(\phi(t)) - Au(\phi(t))]}{d\phi(t)}\right|\frac{d\phi}{dt} &= |(Au_n(\phi(t)) - Au(\phi(t)))'| = |(Au_n(\phi(t)))' - (Au(\phi(t)))'| \\ &\le \Big[a_0\big(b+c+ad_1k_0+ad_2h_0\big) + \sum_{k=1}^p c_k\Big]\|u_n(\phi(t))-u(\phi(t))\|,\end{aligned}$$

$$\left|\frac{d[Au_n(\phi(t)) - Au(\phi(t))}{d\phi(t)}\right| \le \frac{1}{m_1}\Big[a_0\big(b+c+ad_1k_0+ad_2h_0\big) + \sum_{k=1}^p c_k\Big]\|u_n(\phi(t))-u(\phi(t))\|,\ t \in J,$$

$$\|(Au_n(\phi(t)) - Au(\phi(t)))'\|_{PC^*} \le \frac{m_2}{m_1}\Big[a_0\big(b+c+ad_1k_0+ad_2h_0\big) + \sum_{k=1}^p c_k\Big]\|u_n(\phi(t))-u(\phi(t))\|. \tag{3.6}$$

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Thus
$$\|(Au_n(\phi(t)) - Au(\phi(t)))'\|_{PC^*} \to 0 \text{ as } n \to \infty. \tag{3.7}$$
By (3.5) and (3.7), it is easy to see that $\|Au_n(\phi(t)) - Au(\phi(t))\| \to 0$ as $n \to \infty$, that is to say, $A$ is continuous.

Step 2. $A$ maps any bounded subset of $E$ into one bounded subset of $E$.

Let $T$ be any bounded subset of $E$. Then there exist $h > 0$ such that $\|u(\phi(t))\| \le h$ for all $u(\phi(t)) \in J$.
By (3.1),(H1),(H2) and (2.1), we have

$$\begin{aligned}|Au(\phi(t))| &\le |u_0| + |u_0'|t + \int_0^t (t-s)M(1+\|u(s)\|_{E_0})ds + \sum_{0<\xi_k<t}[b_k|u(t_k)| + (t-\xi_k)c_k|u'(t_k)|] \\ &\le |u_0| + a|u_0'| + M(1+\|u(t)\|_{E_0})\int_0^a ads + \|u(t)\|_{E_0}\sum_{0<\xi_k<t}(b_k + ac_k) \\ &\le |u_0| + a|u_0'| + M(1+\|u(\theta(t))\|)\int_0^a ads + \|u(\phi(t))\|\sum_{k=1}^p(b_k + ac_k) \\ &\le |u_0| + a|u_0'| + a^2M(1+h) + h\sum_{k=1}^p(b_k + ac_k),\ t \in J,\end{aligned}$$

so
$$\|Au(\phi(t))\|_{PC} \le |u_0| + a|u_0'| + a^2M(1+h) + h\sum_{k=1}^p(b_k + ac_k). \tag{3.8}$$
Similarly, from(3.2),(H1),(H2) and (2.1), we get
$$\left|\frac{dAu(\phi(t))}{d\phi(t)}\right| \cdot \frac{d\phi}{dt} = |(Au(\phi(t)))'| \le |u_0'| + aM(1+h) + h\sum_{k=1}^p c_k,\ t \in J,$$
$$\left|\frac{dAu(\phi(t))}{d\phi(t)}\right| \le \frac{1}{m_1}\big[|u_0'| + aM(1+h) + h\sum_{k=1}^p c_k\big],\ t \in J,$$
so
$$\|(Au(\phi(t)))'\|_{PC^*} \le \frac{m_2}{m_1}\big[|u_0'| + aM(1+h) + h\sum_{k=1}^p c_k\big]. \tag{3.9}$$
According to (3.8) and (3.9), we obtain
$$\|Au(\phi(t))\| \le \max\Big\{|u_0| + a|u_0'| + a^2M(1+h) + h\sum_{k=1}^p(b_k + ac_k), \frac{m_2}{m_1}\big[|u_0'| + aM(1+h) + h\sum_{k=1}^p c_k\big]\Big\}.$$
Therefore $A(T)$ is uniformly bounded.

Step 3. $A(T)$ is equicontinuous on every $J_k\ (k = 0, \cdots, p)$, where $J_0 = [0, \xi_1]$, $J_k = (\xi_k, \xi_{k+1}]\ (k = 1, \cdots, p)$.

For any $Au(\phi(t)) \in A(T)$ and any $\varepsilon > 0$, take $\delta = \big[|u_0'| + aM(1+h) + h\sum\limits_{k=1}^p c_k\big]^{-1}\varepsilon$. Then if $t_1, t_2 \in J_k$ and $|t_1 - t_2| < \delta$ with $t_1 < t_2$, from (3.1),(H1),(H2) and (2.1), we have

$$\begin{aligned}&|Au(\phi(t_2)) - Au(\phi(t_1))| \le |u_0'|(t_2 - t_1) + \int_{t_1}^{t_2}(t-s)M(1+\|u(s)\|_{E_0})ds + \sum_{i=1}^k (t_2 - t_1)c_i|u'(t_i)| \\ &\le \big[|u_0'| + aM(1+\|u(t)\|_{E_0}) + \|u(t)\|_{E_0}\sum_{i=1}^k c_i\big](t_2 - t_1) \\ &\le \big[|u_0'| + aM(1+\|u(\phi(t))\|) + \|u(\phi(t))\|\sum_{k=1}^p c_k\big]|t_2 - t_1| \le \big[|u_0'| + aM(1+h) + h\sum_{k=1}^p c_k\big]|t_2 - t_1| < \varepsilon.\end{aligned}$$
Thus, $A(T)$ is equicontinuous on every $J_k\ (k = 0, \cdots, p)$.

As a consequence of Step 1-3, $A$ is completely continuous.

(ii) For any $\|u(\phi(t))\| \in G$, similar with getting (3.8) and (3.9), we have respectively

$$\|Au(\phi(t))\|_{PC} \le |u_0| + a|u_0'| + a^2M + \big[a^2M + \sum_{k=1}^{p}(b_k + ac_k)\big]\|u(\phi(t))\|$$
$$= |u_0| + a|u_0'| + a^2M + l_1\|u(\phi(t))\|,$$
$$\|(Au(\phi(t)))'\|_{PC^*} \le \frac{m_2}{m_1}(|u_0'| + aM) + \frac{m_2}{m_1}\big(aM + \sum_{k=1}^{p} c_k\big)\|u(\phi(t))\| = \frac{m_2}{m_1}(|u_0'| + aM) + l_2\|u(\phi(t))\|.$$

Then $\|u(\phi(t))\| = \lambda\|Au(\phi(t))\| \le \|Au(\phi(t))\| \le L + l\|u(\phi(t))\|$, where $L = \max\big\{|u_0| + a|u_0'| + a^2M,\ \frac{m_2}{m_1}(|u_0'| + aM)\big\}$. It follows that $\|u(\phi(t))\| \le \frac{L}{1-l}$, i.e., $G$ is bounded.

From (i) and (ii), now all conditions of Lemma 2.2 are satisfied and therefore the proof is complete. □

**Theorem 3.2.** *If conditions (H1) ($I_{0k}(0) = 0,\ I_{1k}(0) = 0$ are not needed) and (H4) are satisfied, then (1.1) has a unique solution.*

The proof of Theorem 2 is similar to that of Theorem 1, and is omitted here.

# 4 An Example

**Example 4.1.** *Consider the equation*

$$\begin{cases} \big(u(t + \frac{1}{2}t(1-t))\big)'' = \frac{t}{66}\Big[11\sin(u(t) + e^t) - 2u'(t) + 6\int_0^t (ts)u(s)ds + \\ \qquad 3\int_0^1 (ts^2)u(s)ds\Big],\ t \in J = [0,1],\ t \neq \xi_1 = \frac{1}{2}, \\ \Delta u(t_1) = \frac{1}{12}u(t_1),\ \Delta u'(t_1) = \frac{1}{12}u'(t_1),\ t_1 = \frac{5}{8}, \\ u(0) = u_0,\ u'(0) = u_0', \end{cases} \tag{4.1}$$

*Firstly, it is easy to verify that* $\phi(t) = t + \frac{1}{2}t(1-t),\ k(t,s) = ts,\ k_0 = 1,\ h(t,s) = ts^2,\ h_0 = 1$ *all satisfy the requisitions of (1.1). From* $\phi'(t) = \frac{3}{2} - t$*, we get* $m_1 = 1/2,\ m_2 = 3/2$*. Next, since* $f(t,x,y,z_1,z_2) = \frac{t}{66}\big[11\sin(x + e^t) - 2y + 6z_1 + 3z_2\big]$*, and* $|\sin(x_2(t) + e^t) - \sin(x_1(t) + e^t)| = |(x_2(t)+e^t) - (x_1(t)+e^t)|\cdot|\cos(\bar{x}(t)+e^t)| \le |x_2(t) - x_1(t)|$ *($\bar{x}(t)$ is located between $x_1(t)$ and $x_2(t)$), we have*

$$|f(t,x_2,y_2,z_{12},z_{22}) - f(t,x_1,y_1,z_{11},z_{21})|$$
$$\le \frac{t}{66}\big[11|\sin(x_2 + e^t) - \sin(x_1 + e^t)| + 2|y_2 - y_1| + 6|z_{12} - z_{11}| + 3|z_{22} - z_{21}|\big]$$
$$\le t\big[\frac{1}{6}|x_2 - x_1| + \frac{1}{33}|y_2 - y_1| + \frac{1}{11}|z_{12} - z_{11}| + \frac{1}{22}|z_{22} - z_{21}|\big]$$
$$\le t\big[\frac{1}{6}\|x_2 - x_1\|_{PC} + \frac{1}{33}\|y_2 - y_1\|_{PC} + \frac{1}{11}\|z_{12} - z_{11}\|_{PC} + \frac{1}{22}\|z_{22} - z_{21}\|_{PC}\big],\ t \in J,$$

*where* $b = \frac{1}{6},\ c = \frac{1}{33},\ d_1 = \frac{1}{11},\ d_2 = \frac{1}{22},\ a = 1,\ a_0 = \int_0^1 tdt = \frac{1}{2}$*. From* $I_{01}(x) = \frac{1}{12}x,\ I_{11}(y) = \frac{1}{12}y$*, we have*

$$|I_{01}(x_2(t_1)) - I_{01}(x_1(t_1))| \le \frac{1}{12}|x_2(t_1) - x_1(t_1)|,\ I_{01}(0) = 0,$$
$$|I_{11}(y_2(t_1)) - I_{11}(y_1(t_1))| \le \frac{1}{12}|y_2(t_1) - y_1(t_1)|,\ I_{11}(0) = 0,$$

*where* $b_1 = c_1 = \frac{1}{12}$*.Further, we have*

$$\begin{aligned}
&|f(t, u(t), u'(t), Ku(t), Hu(t))| \\
&\quad \le \frac{1}{66}\Big[11|\sin(u(t)+e^t)| + 2|u'(t)| + 6\int_0^t k(t,s)|u(s)|ds + 3\int_0^1 h(t,s)|u(s)|ds\Big] \\
&\quad \le \frac{1}{66}[11 + 2\|u(t)\|_{E_0} + 6\|u(t)\|_{E_0} + 3\|u(t)\|_{E_0}] = \frac{1}{6}(1 + \|u(t)\|_{E_0}),
\end{aligned}$$

*where* $M = \frac{1}{6}$*. Finally, since* $l_1 = a^2M + (b_1 + ac_1) = \frac{1}{3}$, $l_2 = \frac{m_2}{m_1}(aM + c_1) = \frac{3}{4}$*, we get* $l = \max\{l_1, l_2\} = \frac{3}{4} < 1$.

*Thus (4.1) satisfies all conditions of Theorem 3.1. It follows that (4.1) has at least one solution in the closed ball* $\overline{B}$.

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