Original Research Article

MODULES WHOSE ENDOMORPHISM RINGS ARE BAER

Abstract

In this paper, we study modules whose endomorphism rings are Baer, which we call endoBaer modules. We provide some characterizations of endoBaer modules and investigate their properties. Some classes of rings R are characterized in terms of endoBaer R-modules. It is shown that a direct summand of an endoBaer modules inherits the property, while a direct sum of endoBaer modules does not. Necessary and sufficient conditions for a finite direct sum of endoBaer modules to be an endoBaer module are provided.

Keywords: Baer module; endoBaer module; Rickart module; Rickart ring; Baer ring

1 Introduction

It is well known that Baer rings and Rickart rings (also known as p.p. rings) play an important role in providing a rich supply of idempotents and hence in the structure theory for rings. Rickart rings and Baer rings have their roots in functional analysis with close links to C^*-algebras and von Neumann algebras. Kaplansky [1] introduced the notion of Baer rings, which was extended to Rickart rings in ([2],[3]), and to quasi-Baer rings in [4], respectively. A number of research papers have been devoted to the study of Baer, quasi-Baer, and Rickart rings (see e.g [5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[1]). A ring R is said to be Baer if the right annihilator of any nonempty subset of R is generated by an idempotent as a right ideal of R. The notion of Baer rings was generalized to a module theoretic version and studied in recent years (see [15],[16]). An R-module M is called a Baer module if for each left ideal I of $S = \mathrm{End}_R(M)$, $r_M(I) = eM$ for $e^2 = e \in S$. A more general notion of a Baer ring is that of a right Rickart ring. A ring R is called a right Rickart ring if the right annihilator of any element in R is generated by an idempotent as a right ideal of R. A module M_R is called Rickart if the right annihilator of each left principal ideal of $\mathrm{End}_R(M)$ is generated by an idempotent, i.e, for each $\varphi \in S = \mathrm{End}_R(M)$, there exists $e = e^2$ in S such that $r_M(\varphi) = eM$. In this paper, we introduce the notion of endoBaer module, investigate some basic properties of these modules.

In section 2, we introduce the notion of endoBaer module, investigate some basic properties of these modules. The classes of a semiprimary rings, hereditary (Baer) rings, and von Neumann regular rings are characterized in terms of endoBaer R-modules.

It is shown that a direct summand of an endoBaer modules inherits the property, while a direct sum of endoBaer modules does not. Section 3 is devoted to investigating conditions that a direct sum of modules is endoBaer. In addition, we obtain necessary and sufficient conditions for a finite direct sum of copies of an endoBaer module to be endoBaer in terms of its endomorphism ring.

Throughout this paper, all rings are associative with unity. All modules are unital right R-modules unless otherwise indicated and $S = \mathrm{End}_R(M)$ is the ring of endomorphisms of M_R. Mod-R denotes the category of all right R-modules, and M_R a right R-module. By $N \subseteq M$, $N_R \leq M_R$ and $N_R \leq^{\oplus} M_R$ denote that N is a subset, submodule and direct summand of M, respectively. By $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ we denote the ring of real, integer and natural numbers, respectively. Z_n denotes Z/nZ, $M^{(n)}$ denotes the direct sum of n copies of M. The notations $r_R(.)$ (resp. $l_S(.)$) and $l_M(.)$ (resp. $r_M(.)$) denote the right (resp. left) annihilator of a subset of M with elements from R (resp. S) and the left (resp. right) annihilator of a subset of R (resp. S) with elements from M, respectively.

2 EndoBaer Modules

In this section, we introduce the notion of endoBaer modules, investigate some basic properties of these modules. It is shown that a direct summand of an endoBaer modules inherits the property, while a direct sum of endoBaer modules may not be endoBaer. The classes of a semiprimary rings, hereditary (Baer) rings, and von Neumann regular rings are characterized in terms of endoBaer R-modules.

Definition 2.1. A module M is called endoBaer if $\mathrm{End}_R(M)$ is a Baer ring.

Recall that a module M is extending if every closed submodule is a direct summand, or equivalently, every submodule is essential in a direct summand. Also recall that an R-module M is nonsingular if $mI = 0$ for an essential right ideal I of R implies that $m = 0$, equivalently, nonzero element of M has an essential right annihilator in R.

Remark 2.1. (1) Obviously, R_R is an endoBaer module if and only if R is a Baer ring.

(2) Every semisimple $\mathbb{Z}$-module and every nonsingular extending module are both endoBaer modules.

(3) Any Baer module is an endoBaer, since the endomorphism ring of a Baer module is a Baer. (see [17, Theorem 4.1]).

Next example shows that converse of Remark 2.1 (3) does not hold in general.

Example 2.1. *Let $M = \mathbb{Z}_{p^\infty}$, considered as a $\mathbb{Z}$-module. It is well-known that $\mathrm{End}_{\mathbb{Z}}(M)$ is the ring of p-adic integers ([18, Example 3, page 216]). Since the endomorphism ring of M is a Baer ring, $M = \mathbb{Z}_{p^\infty}$ is an endoBaer module. However M is not a Baer module.*

Remark 2.2. Let M be an R-module such that $l_S(N) = Se$ for all $N \leq M$, where $e = e^2 \in S$. Then M is an endoBaer module.

Proof. By [17, Lemma 1.9], we see that for all $I \leq S$, $r_M(l_S(r_M(I))) = r_M(I)$. But $l_S(N) = Se$, where $e = e^2 \in S$. Thus $r_M(I) = eM$, which implies that M is a Baer module, and then it is an endoBaer module by Remark 2.1. □

Recall that a right R-module M is retractable if $\mathrm{Hom}_R(M, N) \neq 0$ whenever N is a non-zero submodule of M.

Proposition 2.1. *Let M be a retractable module. Then the following conditions are equivalent:*

(i) *M is an endoBaer module.*

(ii) *M is a Baer module.*

Proof. (i) ⇒ (ii) Since M is an endoBaer module, $S = \mathrm{End}_R(M)$ is a Baer ring, Also M is a retractable, thus M is a Baer module by [17, Proposition 4.6].

(ii) ⇒ (i) follows from Remark 2.1. □

Recall that a module M is quasi-retractable if $\mathrm{Hom}_R(M, r_M(I)) \neq 0$ for every $I \leq S_S$ with $r_M(I) \neq 0$.

Proposition 2.2. *Let M be a quasi-retractable module. Then the following conditions are equivalent:*
(i) *M is an endoBaer module.*
(ii) *M is a Baer module.*

Proof. (i) $\Rightarrow$ (ii) Since M is an endoBaer module, $S = \mathrm{End}_R(M)$ is a Baer ring. Also M is a quasi-retractable, M is a Baer module by [16, Theorem 2.5].

(ii) $\Rightarrow$ (i) follows from Remark 2.1. □

We know that every retractable module is quasi-retractable, but the converse does not true in general. Each of the following examples exhibits an R-module M which is quasi-retractable endoBaer but not retractable endoBaer.

Recall that an element $m \in M$ is singular if $r_R(m) \leq^{ess} R_R$. We denote the set of all singular elements of M by $Z(M)$. Then we say a module M nonsingular if $Z(M) = 0$ and singular if $Z(M) = M$. A ring R is right nonsingular if R_R is nonsingular.

Example 2.2. *([19, Example 3.4]) Let K be a subfield of complex numbers $\mathbb{C}$. Let R be the ring $\begin{pmatrix} K & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$. Then R is a right nonsingular right extending ring. Consider the module $M = eR$ where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then M is projective, extending and nonsingular (as it is a direct summand of R). Hence M is endoBaer by [17, Theorem 1.12] and Remark 2.1 (3), and quasi-retractable by [16, Theorem 2.5]. But M is not retractable, since the endomorphism ring of M, which is isomorphic to K, consists of isomorphisms and the zero endomorphism.*

Example 2.3. *([20, Example 3.3]). Let*

$$R = \begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix},$$

and let $M = fR$, where

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then M is a nonsingular, projective extending right R-module. Hence M is endoBaer by [17, Theorem 1.12] and Remark 2.1(3), and quasi-retractable by [16, Theorem 2.5]. However, $End(M) = fRf$ is not a right extending ring, and since M is nonsingular, M is not retractable, because otherwise $End(M)$ would be right extending.

Recall that a module over a ring is torsion free if 0 is the only element annihilated by a regular element (nonzero divisor) of the ring.

Proposition 2.3. *Any finitely generated semisimple or torsion-free $\mathbb{Z}$-module M is endoBaer.*

Proof. If M is semisimple, then M is obviously endoBaer. If M is finitely generated and torsion-free, then $M \cong \mathbb{Z}^n$, where $n \in \mathbb{N}$. Note that $\mathbb{Z}^n$ is extending and nonsingular, hence it is endoBaer by Remark 2.1. □

Recall that a ring R is a principal ideal domain or PID if R is an integral domain in which every ideal is principal, i.e., can be generated by a single element.

Remark 2.3. The statement of Proposition 2.3 holds true for any finitely generated module over a Principal Ideal Domain.

Proposition 2.4. *Let M be an R-module and $S = \mathrm{End}_R(M)$. If for every $0 \neq \varphi \in S$, φ is a monomorphism, then M is an indecomposable endoBaer module.*

Proof. Assume that M is not indecomposable. Then $M = N_1 \oplus N_2$ with $N_1, N_2 \neq 0$. Take $\varphi = \pi_1$ the canonical projection of M onto N_1. Then $\mathrm{Ker}(\varphi) = N_2 \neq 0$, a contradiction (as φ is a monomorphism), and so M is indecomposable. It is obviously M is a Baer module, and hence an endoBaer module. □

Proposition 2.5. *If $\mathrm{End}(M)$ is a domain, then the module M is an indecomposable endoBaer.*

Proof. Note that every domain is trivially a Baer ring, so M is an endoBaer module. Since there are no idempotents other than 0 and 1 in a domain, M is also indecomposable. □

If M is an R-module, N a direct summand of M, and e the projection of M onto N, then it is easy to see that e is an idempotent of $S = \mathrm{Hom}_R(M, M)$ and $\mathrm{Hom}_R(N, N) = eSe$. This fact will be used in the next proposition.

Proposition 2.6. *Every direct summand of an endoBaer module is endoBaer.*

Proof. Let M be an endoBaer module, N a direct summand of M, $S = \mathrm{Hom}_R(M, M)$, and e the projection onto N. Then $\mathrm{Hom}_R(N, N) = eSe$. But for any Baer ring S and any idempotent $e \in S$, eSe is a Baer ring by [1, Theorem 2]. Thus N is endoBaer. □

Remark 2.4. If M is an endoBaer module then so are $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ for every regular $\varphi \in \mathrm{End}_R(M)$.

Proof. This follows from the fact that $\varphi \in \mathrm{End}_R(M)$ is regular if and only if $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of M by [21, Theorem 16]. □

Corollary 2.1. *If R is a Baer ring, then eR is an endoBaer R-module for every $e^2 = e \in R$.*

Corollary 2.1 also follows from the fact that if R is a Baer ring then so is eRe for every $e^2 = e \in R$ by [1, Theorem 2].

Proposition 2.7. *If M is an endoBaer module with only countably many direct summands, then M contains no infinite direct sums of disjoint summands.*

Proof. Since M is an endoBaer, S is Baer ring and M has countably many direct summands, thus S has only countably many idempotents. By [22, Theorem 2.3], S has no infinite sets of orthogonal idempotents. Hence there exists no infinite sets of mutually disjoint direct summands in M. □

Corollary 2.2. *If M is an endoBaer module with only countably many direct summands, then M is a finite direct sum of indecomposable summands.*

Proof. By Proposition 2.7, S has no infinite sets of orthogonal idempotents. Hence any direct sum decomposition of M must be finite. Thus M is a finite direct sum of indecomposable submodules. □

Recall that a ring is regular in the sense of commutative algebra if it is a commutative unit ring such that all its localizations at prime ideals are regular local rings.

Corollary 2.3. *Let M be an endoBaer module with only countably many direct summands and $S = \mathrm{End}_R(M)$ is a regular ring. Then M is a semisimple artinian module.*

Proof. It follows from Proposition 2.7 that S becomes a regular Baer ring with only countably many idempotents. Then S is a semisimple artinian ring by [22, Theorem 2.3]. It is easy to check that M is also a semisimple artinian module. □

Corollary 2.4. *Let M be $\mathbb{Z}$-module with only countably many direct summands and $S = \mathrm{End}_{\mathbb{Z}}(M)$ is a regular ring. Then M is an endoBaer module if and only if M is a semisimple artinian module.*

Proof. The proof follows directly from Remark 2.1 and Corollary 2.3. □

Recall that a module M is quasi-injective if every homomorphism of a submodule of M into M may be realized by an endomorphism of M.

Proposition 2.8. *Let M be a module and $S = \mathrm{End}_R(M)$ is a regular ring. Then M is endoBaer if any of the following conditions hold:*
(i) *M is a quasi-injective module.*
(ii) *M is extending module.*

Proof. Suppose (i) holds. Since M is a quasi-injective module, S is a Baer ring by [23, Proposition 5.1]. Thus M is endoBaer module.

Suppose (ii) holds. Since M is extending module and S is a regular ring, M is a Baer module by [17, Proposition 4.12]. Thus M is endoBaer module by Remark 2.1. □

Example 2.4. *A finite direct sum of endoBaer modules is not necessarily an endoBaer module. The $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ is not endoBaer while $\mathbb{Z}$ and $\mathbb{Z}_2$ are both endoBaer $\mathbb{Z}$-modules ($\mathbb{Z}$ and $\mathbb{Z}_2$ are both Baer modules). We note that the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ is a retractable module (Any direct sum of $\mathbb{Z}_{p^i}$ is retractable, where p is a prime number). For the endomorphism $f(x, \bar{y}) = \bar{x}$ where $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_2$, then $\mathrm{Ker} f = 2\mathbb{Z} \oplus \mathbb{Z}_2$ which is not a direct summand of $\mathbb{Z} \oplus \mathbb{Z}_2$, so $\mathbb{Z} \oplus \mathbb{Z}_2$ is not a Baer module (see [17, Example 2.24]). Thus using Proposition 2.1, $\mathbb{Z} \oplus \mathbb{Z}_2$ is not an endoBaer module.*

Recall that a module M have the summand intersection property (SIP) if the intersection of any two direct summands of M is a direct summand. A module is said to have the generalized summand intersection property $(GSIP)$ if the intersection of any family of direct summands of M is a direct summand.

Next, we provide a characterization of endoBaer modules in terms of $GSIP$.

Proposition 2.9. *Let M be a module has the generalized summand intersection property and $\mathrm{Ker}(\varphi) \leq^{\oplus} M$ for all $\varphi \in S$. Then M is endoBaer module.*

Proof. Let $N \subseteq S$ be a nonempty subset of S for each $\varphi \in N$. Then $\mathrm{Ker}(\varphi) \leq^{\oplus} M$. Also $r_M(N) = \bigcap_{\varphi \in N} \mathrm{Ker}(\varphi) \leq^{\oplus} M$ by the $GSIP$. Hence we get that M is a Baer, and then M is an endoBaer module. □

Proposition 2.10. *If M is a Rickart module with the $SSIP$, then M is an endoBaer module.*

Proof. This follows from Proposition 2.9. □

Remark 2.5. Note that every free module M of countable rank over a principal ideal domain (PID) R has the $SSIP$ (see [22, Exercise 51(c)]. Also by [15, Theorem 2.26], M is a Rickart R-module. An application of Proposition 2.10 yields that M is endoBaer.

The following example shows that the converse of Proposition 2.10 is not true in general.

Example 2.5. *As in Example 2.1, M is an endoBaer module. However, $\mathbb{Z}_{p^\infty}$ is not a Rickart module. Let $\varphi : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^\infty}$ be defined by $\varphi(a) = ap$. Since $0 \neq \mathrm{Ker}\varphi \lneq \mathbb{Z}_{p^\infty}$, $\mathbb{Z}_{p^\infty}$ is not a Rickart $\mathbb{Z}$-module, see[15, Example 2.17]. However $\mathbb{Z}_{p^\infty}$ has SIP, since that $\mathbb{Z}_{p^\infty}$ is indecomposable.*

Next, we characterize several classes of rings in terms of endoBaer modules.

Recall that R is hereditary ring if all submodules of projective modules over R are again projective. If this is required only for finitely generated submodules, it is called semihereditary. Also recall that a module M has the summand intersection property (SIP) if the intersection of any two direct summand is a direct summand of M. M is said to have the strong summand intersection property (SSIP) if the intersection of any family of direct summands is a direct summand of M.

Proposition 2.11. *The following conditions are equivalent for a ring R:*

(i) *Every free right R-module M is an endoBaer module and has the $SSIP$;*

(ii) *Every projective right R-module M is a Baer module and has the $SSIP$;*

(iii) *R is a semiprimary, hereditary (Baer) ring.*

Proof. (i) $\Rightarrow$ (ii) Since M is a free R-module, M is a retractable. Then M is a Baer module by Proposition 2.1, and since that every projective module is a direct summand of a free module. Hence every projective right R-module M is a Baer module by [17, Theorem 2.17]. Consequently, has the $SSIP$.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) Follows from [16, Theorem 3.3], [24, Corollary3.23] and Remark 2.1.

Proposition 2.12. *Let M be a free R-module. Then the following conditions are equivalent:*

(i) *M is an endoBaer module.*

(ii) *M is a Rickart module with the $SSIP$.*

Proof. (i) $\Rightarrow$ (ii) Since M is a free R-module, M is a retractable. Then M is a Baer module by Proposition 2.1. Hence it is a Rickart module with the $SSIP$ by [17, Proposition 2.22].

(ii) $\Rightarrow$ (i) Follows from Proposition 2.10. □

Corollary 2.5. *Let M be a free R-module. Then the following conditions are equivalent:*

(i) *M is an endoBaer module.*

(ii) *M is a Baer module.*

(iii) *M is a Rickart module with the $SSIP$.*

Proof. (i) $\Rightarrow$ (ii) Since M is a free R-module, M is a retractable. Hence M is a Baer module by Proposition 2.1.

(ii) $\Rightarrow$ (i) Follows from Remark 2.1.

(ii) $\Rightarrow$ (iii) Follows from [17, Proposition 2.22].

(iii) $\Rightarrow$ (i) Follows from Proposition 2.12. □

Recall that a ring R is a right (left) self injective ring if it is injective over itself as a right (left) module. If a von Neumann regular ring R is also right or left self injective, then R is Baer.

Corollary 2.6. *Let M be a projective module. Then the following statements hold:*

(i) *Every submodule of M over hereditary ring is an endoBaer module.*

(ii) *Every finitely generated submodule of M over a semiprimary ring is an endoBaer module.*

(iii) *Every finitely generated submodule of M over a right (left) self injective von Neumann regular ring is an endoBaer module.*

Proof. (i) and (ii) follows from the definition of a semiprimary, hereditary ring and Proposition 2.11.

(iii) Let L be a finitely generated submodule of M. It is well-known that a von Neumann regular ring is left and right semihereditary, and every finitely generated submodule of a projective module over a von Neumann regular ring R is isomorphic to a direct summand of a finitely generated free R-module by [25] . Hence $L \cong K \leq^{\oplus} R^{(n)}$. Therefore, L is an endoBaer module by Propositions 2.6 and 2.11(i). □

Recall that a ring R is a right (left) $\prod$-coherent if every finitely generated torsionless right (left) R-module is finitely presented. Clearly, a right (left) $\prod$-coherent ring is right (left) coherent, so $\prod$-coherent rings are also called strongly coherent rings.

Proposition 2.13. *Let R be a ring. The following statements are equivalent:*

(1) *Every finitely generated free right module over R is an endoBaer module;*

(2) *Every finitely generated free right module over R is a Baer module;*

(3) *Every finitely generated projective right module over R is a Baer module;*
(4) *Every finitely generated torsionless right R-module is projective;*
(5) *Every finitely generated torsionless left R-module is projective;*
(6) *R is left semihereditary and right Π-coherent;*
(7) *R is right semihereditary and left Π-coherent;*
(8) *$M_n(R)$ is Baer ring for every $n \in \mathbb{N}$.*

Proof. (1) ⇔ (2) Since M is a free R-module, M is retractable. Then M is an endoBaer module if and only if it is Baer by Proposition 2.1.

(1) ⇔ (8) Follows from [16, Theorem 2.5], Remark 2.1 and Proposition 2.1.

(1) ⇔ (4), (2) ⇔ (3),(4) ⇔ (5),(4) ⇔ (6) and (5) ⇔ (7) Follows from the fact that a free module is a retractable, and by [16, Theorem 3.5], Remark 2.1 and Proposition 2.1. □

3 Direct Sums Of EndoBaer Modules

It is interesting to investigate whether or not an algebraic property is inherited by direct summands and direct sums. In this section, we investigate when a direct sum of endoBaer modules is also endoBaer. In addition, we obtain necessary and sufficient conditions for a finite direct sum of copies of an endoBaer module to be endoBaer in terms of its endomorphism ring.

Recall that a module M is a quasi-continuous if every complement in M is a direct summand of M, and for any direct summands M_1 and M_2 of M such that $M_1 \cap M_2 = 0$, the submodule $M_1 \oplus M_2$ is also a direct summand of M.

Proposition 3.1. *Let M_i be a direct summand of a quasi-continuous endoBaer module M for all $i = 1, .., n$, such that $M_i \cap M_j = 0$ for $i \neq j$. Then M_i is an endoBaer module for all i and $\bigoplus_{i=1}^{n} M_i$ is an endoBaer module.*

Proof. Since M is a quasi-continuous module and $M_i \cap M_j = 0$ for all $i \neq j$, $\bigoplus_{i=1}^{n} M_i$ is a direct summand of M, Thus it is an endoBaer module by Proposition 2.6. □

Recall that the modules M and N relatively Rickart if for every $\varphi : M \to N$, $\mathrm{Ker}\varphi \leq^{\oplus} M$ and for every $\psi : N \to M$, $\mathrm{Ker}\psi \leq^{\oplus} N$.

Proposition 3.2. *Let $\{M_i\}_{1 \leq i \leq n}$ be a class of retractable endoBaer modules, where $n \in \mathbb{N}$. Assume that M_i and M_j are relative Rickart and relative injective for any $i \neq j$. Then $\bigoplus_{i=1}^{n} M_i$ is an endoBaer module.*

Proof. The result follows directly from [16, Lemma 2.8], [16, Theorem 3.19] and Proposition 2.1.

Proposition 3.3. *Let $\{M_i\}_{i \in I}$ (I an index set) be a class of retractable modules such that $\mathrm{Hom}_R(M_i, M_j) = 0$ for every $i \neq j \in I$. Then $M = \bigoplus_{i \in I} M_i$ is an endoBaer module if and only if M_i is an endoBaer module for every $i \in I$.*

Proof. The necessity is clear by Proposition 2.6.

To prove sufficiency, note that M is a Baer module by Proposition 2.1 and [16, Proposition 3.20]. Thus M is an endoBaer module by Remark 2.1. □

As a consequence of Proposition 2.13, we can obtain the following result for finite direct sums of copies of an arbitrary retractable endoBaer module M to be endoBaer(in this case, we do not require the modules to be finitely generated).

Theorem 3.1. *Let M be a finitely generated, retractable endoBaer module. Then an arbitrary direct sum of copies of M is an endoBaer module if and only if $S = \mathrm{End}(M)$ is semiprimary and (right) hereditary.*

Proof. We note that, for a finitely generated module M and $S = \mathrm{End}(M)$, we have that $\mathrm{End}(M^{(f)}) \cong \mathrm{End}(S^{(f)})$ as rings, where f is an arbitrary set. Hence, if an arbitrary direct sum of copies of M is endoBaer, its endomorphism ring $\mathrm{End}(M^{(f)})$ is a Baer ring. Hence $\mathrm{End}(S^{(f)})$ is also a Baer ring, and thus $S^{(f)}$ is an endoBaer module. Since $S^{(f)}$ is a free S-module, $S^{(f)}$ is a retractable S-module by Proposition 2.1 we get that $S^{(f)}$ is a Baer S-module, and then it has $SSIP$. Hence by Proposition 2.11, S is right semiprimary and right hereditary.

Conversely, let $S = \mathrm{End}(M)$ is semiprimary and (right) hereditary for an arbitrary set f. Since $S^{(f)}$ is a free S-module, we obtain that $S^{(f)}$ is an endoBaer S-module by Proposition 2.11. Hence $\mathrm{End}(S^{(f)})$ is a Baer ring. Thus $\mathrm{End}(M^{(f)})$ is a Baer ring, and $M^{(f)}$ is an endoBaer module.

The following result study finite direct sums of copies of an arbitrary retractable endoBaer module M.

Corollary 3.1. *Let M be a retractable endoBaer module. Then a finite direct sum of copies of M is an endoBaer module if and only if $S = \mathrm{End}(M)$ is left semihereditary and right Π-coherent.*

Proof. As M is retractable, using the same technique as used in the proof of Theorem 3.1, we can replace M with $S = \mathrm{End}(M)$. By Proposition 2.13, the endomorphism ring of a finite direct sum of copies of S is endoBaer if and only if S is left semihereditary and right Π-coherent. This gives us the desired result . □

Our next result illustrates an application to the case when the base ring R is commutative. Recall that a characterization for an $n \times n$ matrix ring over a commutative integral domain to be Baer is well-known ([1],[26]).

Recall that a ring R is a Prüfer domain if R is a commutative ring without zero divisors in which every non-zero finitely generated ideal is invertible.

Theorem 3.2. *([26, Corollary 15]). If R is a commutative integral domain, then $M_n(R)$ is a Baer ring (for some $n > 1$) if and only if every finitely generated ideal of R is invertible, i.e., if R is a Prüfer domain.*

We can show the following for a finite rank free module over a commutative domain.

Theorem 3.3. *Let R be a commutative integral domain and M a free R-module of finite $rank > 1$. Then M is endoBaer if and only if R is a Prüfer domain.*

Proof. Consider R is a Prüfer domain, then $M_n(R)$ is a Baer ring by Theorem 3.2. But $\mathrm{End}(M) \cong M_n(R)$. Thus $\mathrm{End}(M)$ is a Baer ring, so we obtain that M is an endoBaer module.

Conversely, if M is an endoBaer module, then $\mathrm{End}(M)$ is Baer. Hence $M_n(R)$ for $n > 1$ is a Baer ring, and thus R must be a Prüfer domain. □

We now characterize the semisimple artinian rings in terms of free endoBaer modules.

Theorem 3.4. *The following conditions are equivalent for a ring R :*

(i) *Every right R-module M is a Rickart module;*
(ii) *Every extending right R-module M is a Rickart module;*
(iii) *Every injective right R-module M is a Rickart module;*
(iv) *Every (injective) right R-module M is a Baer module;*
(v) *Every free right R-module M is an endoBaer module;*
(vi) *R is a semisimple artinian ring.*

Proof. (i) ⇒ (ii) ⇒ (iii) It is clear.

(iii) ⇔ (iv) It is easy to see because every injective Rickart module is Baer (see [15, Remark 2.13]).

(iv) ⇔ (v) Since a free module is retractable, the result follows from Remark 2.1 and Proposition 2.1.

(v) $\Rightarrow$ (vi) Since M is a Baer module by Proposition 2.1, R is a semisimple artinian ring by [27, Theorem 2.20].

(vi) $\Rightarrow$ (1) It is clear. □

In Proposition 3.4 a ring is semisimple artinian if and only if every free R-module is endoBaer. In the next proposition we obtain the same conclusion for commutative rings.

Proposition 3.4. *Let R be a commutative ring. Every free R-module is endoBaer if and only if R is semisimple artinian. In particular, every R-module is endoBaer if every free R-module is so.*

Proof. Since a free module is a retractable, the result follows from [16, Proposition 3.10] and Proposition 2.1 . □

If the endomorphism ring of a module is a PID, we obtain the following result, due to Wilson, which we have reformulated in our setting [28, Lemma 4].

Proposition 3.5. *Let M be a finite direct sum of copies of some finite rank, torsion-free module whose endomorphism ring is a PID. Then M is endoBaer module.*

Proof. By [28] M has $SSIP$ and the kernel of any endomorphism of M is a direct summand of M. Hence by our Proposition 2.10, M is endoBaer . □

For a fixed $n \in \mathbb{N}$, we obtain the following characterization for every n-generated free R-module to be endoBaer.

Proposition 3.6. *Let R be a ring and $n \in \mathbb{N}$. The following statements are equivalent:*

(1) *Every n-generated free right R-module is an endoBaer module;*
(2) *Every n-generated free right R-module is a Baer module;*
(3) *Every n-generated projective right R-module is a Baer module;*
(4) *Every n-generated torsionless right R-module is projective (therefore R is right n-hereditary).*

Proof. The proof follows the same outline as in Proposition 2.13, where we replace "finite" with "n elements." □

Recall that a ring R is a right n-fir if any right ideal that can be generated with $\leqslant n$ elements is free of unique rank (i.e., for every $I \leq R_R$, $I \cong R^k$ for some $k \leqslant n$, and if $I \cong R^l \Rightarrow k = l$) (for alternate definitions see Theorem 1.1, [29]).

The definition of (right) n-firs is left-right symmetric, thus we will call such rings simply n-firs.

Proposition 3.7. *Let M be a module with endomorphism ring S is n-fir. Then M is an endoBaer module and S^n is a Baer module.*

Proof. Since S is an n-fir, it is in particular an integral domain (see [29, page 45]), S is a Baer ring. Thus M is an endoBaer module. Also S^n is a Baer module by [16, Theorem 3.16.]. Consequently, $M_n(S)$ is a Baer ring. □

Remark 3.1. Let M be a module with endomorphism ring S is n-fir, then M is an endoBaer module and $M_n(S)$ is a Baer ring.

Proof. From Proposition 3.7 M is an endoBaer module and S^n is a Baer module. Consequently, $M_n(S)$ is a Baer ring. □

Proposition 3.8. *Let M be a finitely generated module with endomorphism ring S is n-fir, then M is an endoBaer module and a finite direct sum of copies of M is an endoBaer module.*

Proof. We note that, for a finitely generated module M and $S = \mathrm{End}(M)$, we have that $\mathrm{End}(M^n) \cong \mathrm{End}(S^n)$ as rings, where $n \in \mathbb{N}$. since S is n-fir, M is an endoBaer module and S^n is a Baer module by Proposition 3.7. Since S^n is a free S-module, we obtain that S^n is an endoBaer S-module by Proposition 3.6. Hence $\mathrm{End}(S^n)$ is a Baer ring. Thus $End(M^n)$ is a Baer ring, and M^n is an endoBaer. □

The example below proves the existence of a module M such that M^n is an endoBaer module, but M^{n+1} is not endoBaer.

Example 3.1. *(see[14]) Let n be any natural number and let R be the K-algebra (K is any commutative field) on the $2(n+1)$ generators X_i, $Y_i(i = 1, ..., n+1)$ with the defining relation*

$$\sum_{i=1}^{n+1} X_i Y_i$$

R is an n-fir. However not all $(n+1)$-generated ideals are flat (see [14, Theorem 2.3]).

In particular, R is not $(n+1)$-hereditary, since there exists an $(n+1)$-generated ideal which is not flat, hence not projective (see [16, Example 3.17]).

Thus, R^n is an endoBaer module (Baer module due to R being an n-fir); however, since R is not $(n+1)$-hereditary, R^{n+1} is not endoBaer by Proposition 3.6

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