__Original Research Article__

Proposing a New Form of Fuzzy Sets and Its Properties

ABSTRACT

In this paper, a new form of fuzzy sets is defined, in which algebraic operations and some of their properties are different from algebraic operations on ordinary fuzzy sets. The definition provided enables us to examine the properties of fuzzy sets that are not possible using ordinary fuzzy sets.

Keywords: Fuzzy Sets, Fuzzy-2 Sets, Algebraic Operations, Ordinary Fuzzy Sets

1- INTRODUCTION

The aim of defining the fuzzy sets is to approximate the behavior and response of machine systems to the patterns that human beings treat based on them. Although intelligent systems may work powerfully, they follow the human pattern again. Zadeh's paper on fuzzy sets [1] was the beginning of a new approach in the science and engineering of systems and computers. Zadeh introduced the concept of fuzzy algorithms in 1968 [2] and posed the concept of using linguistic variables in memory and control systems in 1973 [3]. Fuzzy theory developed and practical applications appeared in the 1970s. The big event in this decade was the creation of fuzzy controllers for real systems. In 1975, Mamdani and Asilian identified the initial framework for a fuzzy controller and applied the fuzzy controller to a steam engine [4]. Subsequently, many applications of fuzzy sets and fuzzy logic have been used in industry and in different branches of science.

In the fuzzy theory, each object with a grade of membership can belong to a certain set. The more acceptable the membership of this object to the set, the grade of membership will be also higher [1]. Now consider a circle whose color is gray, so that its black and white proportions are equal to 50%. Thus the circle is 50% black and 50% white. Now if black and white colors are drawn to one side, e.g. black color to the north of the circle and white color to the south of the circle to completely black the north of the circle and completely white the south of the circle, the circle will be 50% black and 50% white in this case as well. Obviously, one cannot determine a part of the circle in a way that to be completely black or completely white in the first case, but it is possible in the second case.

Consider a piece of land that is owned by two people equally. The ownership of each of these people will be 50%. This is the fuzzy definition. Now, if we want to determine the share of each person in terms of location, this will not be easily possible. Now, suppose that the land is bounded in such a way that half of the land is owned by each person so that its location is completely clear; specifying the contribution of each individual in terms of location is something quite simple and doable in this case. Considering that there are many examples of such examples in our daily lives, introduction of a new type of fuzzy sets seems essential. In this paper, we introduce a new form of fuzzy sets that differ from ordinary fuzzy sets in terms of definition, but many algebraic properties of fuzzy sets are still maintained.

2- FUZZY SETS

Boundary of the set in the fuzzy set is not known precisely and has a state of ambiguity.

Definition 1: A fuzzy set *A* in set *X* is a function from *X* to [0,1] that its value in $x \in X$, i.e. $A(x)$, is grade of membership of *x* in *A*. The fuzzy set *A* is represented as follows:

$$A = \{(x, A(x)) | A : X \to [0,1]\}$$

Note 1: According to the above definition, set *X* is also a fuzzy set ($\forall x \in X, X(x) = 1$).

The main operations of fuzzy sets are defined as follows:

$A \subseteq B \Leftrightarrow A(x) \leq B(x), \quad \forall x \in X$ (*Containment*)

$(A \cup B)(x) = Max\{A(x), B(x)\}$ (*Union*)

$(A \cap B)(x) = Min\{A(x), B(x)\}$ (*Intersection*)

$\overline{A}(x) = 1 - A(x)$ (*Complement*)

Some properties of fuzzy sets are as follows:

$A \subseteq A$ (*Reflexive*)

$A \subseteq B, B \subseteq A \Rightarrow A = B$ (*Antisymmetric*)

$A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$ (*Transitive*)

$\overline{\overline{A}} = A$ (*Involution*)

$\left.\begin{array}{l} A \cup \overline{A} \neq X \quad , \ A \cap \overline{A} \neq \emptyset \\ \overline{\emptyset} = X \quad , \ \overline{X} = \emptyset \end{array}\right\}$ (*Failure of Complement*)

$A \cup A = A \quad , \ A \cap A = A$ (*Idempotent*)

$A \cup B = B \cup A \quad , \ A \cap B = B \cap A$ (*Commutative*)

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \;\; , \;\; \overline{(A \cap B)} = \overline{A} \cup \overline{B} \qquad (\textit{De Morgan})$$

$$\left.\begin{array}{l} A \cup X = X \quad , \; A \cap X = A \\ A \cup \varnothing = A \quad , \; A \cap \varnothing = \varnothing \end{array}\right\} \qquad (\textit{Identity})$$

$$\left.\begin{array}{l} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{array}\right\} \qquad (\textit{Associative})$$

$$\left.\begin{array}{l} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{array}\right\} \qquad (\textit{Distributive})$$

3- INTRODUCTION OF A NEW FORM OF FUZZY SETS

Given the issues raised, if we want to select a part of a single object (single existence) that differs from its other parts, it is natural that such a thing is not possible (completely gray circle), unless the object can be considered as a set of sets (a circle with a black northern half and a white southern half). Hence, introduction of the new fuzzy sets (in other words, fuzzy-2 sets) with above-mentioned properties should be a set of sets. With these interpretations, consider the following definitions:

Definition 2: Suppose that *X* is a set of mutually distinct and non-empty sets, $P(x), x \in X$ is the set of all subsets of *x* and μ is a function, we consider that:

$$\begin{cases} A = \{ x_A \,|\, x_A \subseteq x \;\; ; \;\; x \in X \} \\ \mu : \{ y \,|\, y \in P(x) ; \forall x \in X \} \rightarrow [0,1] \\ \mu(\varphi) = 0 \\ \mu(x) = 1 \;\; ; \;\; x \in X \\ \forall \alpha, \beta \in P(x) \;\; ; \;\; \alpha \subset \beta \Rightarrow \mu(\alpha) < \mu(\beta) \\ \forall \alpha, \beta \in P(x) \;\; ; \;\; \mu(\alpha \cup \beta) = \mu(\alpha) + \mu(\beta) - \mu(\alpha \cap \beta) \end{cases}$$

According to the above definitions, we represent and define the fuzzy-2 subset of the set *X* as follows:

$$\overset{\approx}{A} = \{ (x_A , \mu(x_A)) \,|\, x_A \in A \}$$

In this case, μ is the membership function.

4- Main operations on fuzzy-2 sets

Definition 3: Suppose that $\overset{\approx}{A}$ and $\overset{\approx}{B}$ are fuzzy-2 sets with the assumptions of definition (2), then we will show and define the union, intersection, and complement as follows:

$$\overset{\approx}{A} \bigcup \overset{\approx}{B} = \left\{ (y, \mu(y)) \middle| y = x_A \bigcup x_B ; x_A \in A, x_B \in B \right\}$$

$$\overset{\approx}{A} \bigcap \overset{\approx}{B} = \left\{ (y, \mu(y)) \middle| y = x_A \bigcap x_B ; x_A \in A, x_B \in B \right\}$$

$$\overline{\left(\overset{\approx}{A} \right)} = \left\{ (\overline{x_A}, \mu(\overline{x_A})) \middle| x_A \in A \right\}$$

Note 1: It should be noted $\overline{x_A} = x - x_A \quad ; \quad x_A \subseteq x \in X$.

Example 1: Suppose that sets X, $\overset{\approx}{A}$, $\overset{\approx}{B}$, and function μ are as follows:

$$X = \left\{ (n, n+1] \middle| n \in N \right\}$$

$$A = \left\{ x_A^n \middle| x_A^n = \left(\frac{n^2+1}{n}, \frac{2n^2+n+4}{2n} \right], n \in N, n > 10 \right\}$$

$$B = \left\{ x_B^n \middle| x_B^n = \left(\frac{2n^2+n-4}{2n}, \frac{n^2+n-3}{n} \right], n \in N, n > 10 \right\}$$

$$\mu((\alpha, \beta]) = \beta - \alpha$$

$$\overset{\approx}{A} = \left\{ (x_A^n, \mu(x_A^n)) \middle| x_A^n \in A \right\}$$

$$\overset{\approx}{B} = \left\{ (x_B^n, \mu(x_B^n)) \middle| x_B^n \in B \right\}$$

In this case, $\overset{\approx}{A}$ and $\overset{\approx}{B}$ are fuzzy-2 subsets of set X.

Example 2: Suppose that sets X, $\overset{\approx}{A}$, and $\overset{\approx}{B}$ are sets of example (1), in this case:

$$\overset{\approx}{A} \bigcup \overset{\approx}{B} = \left\{ (y^n, \mu(y^n)) \middle| y^n = x_A^n \bigcup x_B^n \right\}$$

$$\overset{\approx}{A} \bigcap \overset{\approx}{B} = \left\{ (y^n, \mu(y^n)) \middle| y^n = x_A^n \bigcap x_B^n \right\}$$

Example 3: Suppose that sets $\overset{\approx}{A}$ and $\overset{\approx}{B}$ are sets of example (1), in this case:

$$x_A^n = (\frac{n^2+1}{n}, \frac{2n^2+n+4}{2n}] \Rightarrow \mu(x_A^n) = \frac{n+2}{2n}$$

$$x_B^n = (\frac{2n^2+n-4}{2n}, \frac{n^2+n-3}{n}] \Rightarrow \mu(x_B^n) = \frac{n-2}{2n}$$

$$x_A^n \bigcup x_B^n = (\frac{n^2+1}{n}, \frac{n^2+n-3}{n}] \Rightarrow \mu(x_A^n \bigcup x_B^n) = \frac{n-4}{n}$$

$$x_A^n \bigcap x_B^n = (\frac{2n^2+n-4}{2n}, \frac{2n^2+n+4}{2n}] \Rightarrow \mu(x_A^n \bigcap x_B^n) = \frac{4}{n}$$

Example 4: The membership values of x^{20} in Example (1) are as follows:

$$\mu(x_A^{20}) = \frac{11}{20} = 0.55$$

$$\mu(x_B^{20}) = \frac{9}{20} = 0.45$$

$$\mu(x_A^{20} \bigcup x_A^{20}) = \frac{4}{5} = 0.8$$

$$\mu(x_A^{20} \bigcap x_A^{20}) = \frac{1}{5} = 0.2$$

5- Some Properties of fuzzy-2 sets

Theorem 1: Suppose that $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$ are fuzzy-2 sets with the assumptions of definition (2), in which case the following laws are established:

a) $\overline{\overline{\left(\tilde{\tilde{A}}\right)}} = \tilde{\tilde{A}}$

b) $\tilde{\tilde{A}} \bigcup \tilde{\tilde{A}} = \tilde{\tilde{A}}$, $\tilde{\tilde{A}} \bigcap \tilde{\tilde{A}} = \tilde{\tilde{A}}$

c) $\tilde{\tilde{A}} \bigcup \tilde{\tilde{B}} = \tilde{\tilde{B}} \bigcup \tilde{\tilde{A}}$, $\tilde{\tilde{A}} \bigcap \tilde{\tilde{B}} = \tilde{\tilde{B}} \bigcap \tilde{\tilde{A}}$

d) $\overline{(\tilde{\tilde{A}} \bigcup \tilde{\tilde{B}})} = \overline{\tilde{\tilde{A}}} \bigcap \overline{\tilde{\tilde{B}}}$, $\overline{(\tilde{\tilde{A}} \bigcap \tilde{\tilde{B}})} = \overline{\tilde{\tilde{A}}} \bigcup \overline{\tilde{\tilde{B}}}$

e) $\begin{cases} \tilde{\tilde{A}} \cup \overline{\left(\tilde{\tilde{A}}\right)} = X \quad , \quad \tilde{\tilde{A}} \cap \overline{\left(\tilde{\tilde{A}}\right)} = \varnothing \\ \bar{\varnothing} = X \quad , \quad \overline{X} = \varnothing \end{cases}$

f) $\tilde{A} \cup X = X \quad , \quad \tilde{A} \cap X = \tilde{A}$ و $\tilde{A} \cup \varnothing = \tilde{A} \quad , \quad \tilde{A} \cap \varnothing = \varnothing$

g) $\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C} \quad , \quad \tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C}$

h) $\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C}) \quad , \quad \tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$

***Proof*:**

a:

$$\overline{\overline{\left(\tilde{\tilde{A}}\right)}} = \left\{ (\overline{\overline{x_A}}, \mu(\overline{\overline{x_A}}) \middle| x_A \in A \right\} = \left\{ (x_A, \mu(x_A) \middle| x_A \in A \right\} = \tilde{\tilde{A}}$$

b:

$$\tilde{\tilde{A}} \cup \tilde{\tilde{A}} = \left\{ (y, \mu(y)) \middle| y = x_A \cup x_A ; x_A \in A \right\} = \left\{ (x_A, \mu(x_A)) \mid x_A \in A \right\} = \tilde{\tilde{A}}$$
$$\tilde{\tilde{A}} \cap \tilde{\tilde{A}} = \left\{ (y, \mu(y)) \middle| y = x_A \cap x_A ; x_A \in A \right\} = \left\{ (x_A, \mu(x_A)) \mid x_A \in A \right\} = \tilde{\tilde{A}}$$

c:

$$\begin{aligned} \tilde{A} \cup \tilde{B} &= \left\{ (y, \mu(y)) \middle| y = x_A \cup x_B ; x_A \in A, x_B \in B \right\} \\ &= \left\{ (y, \mu(y)) \middle| y = x_B \cup x_A ; x_A \in A, x_B \in B \right\} = \tilde{B} \cup \tilde{A} \\ \tilde{A} \cap \tilde{B} &= \left\{ (y, \mu(y)) \middle| y = x_A \cap x_B ; x_A \in A, x_B \in B \right\} \\ &= \left\{ (y, \mu(y)) \middle| y = x_B \cap x_A ; x_A \in A, x_B \in B \right\} = \tilde{B} \cap \tilde{A} \end{aligned}$$

d:

$$\overline{(A \cup B)} = \left\{ (\overline{y}, \mu(\overline{y})) \middle| y = x_A \cup x_B ; x_A \in A, x_B \in B \right\}$$

$$= \left\{ (\overline{y}, \mu(\overline{y})) \middle| \overline{y} = \overline{x_A} \cap \overline{x_B} ; x_A \in A, x_B \in B \right\} = \overline{A} \cap \overline{B}$$

$$\overline{(A \cap B)} = \left\{ (\overline{y}, \mu(\overline{y})) \middle| y = x_A \cap x_B ; x_A \in A, x_B \in B \right\}$$

$$= \left\{ (\overline{y}, \mu(\overline{y})) \middle| \overline{y} = \overline{x_A} \cup \overline{x_B} ; x_A \in A, x_B \in B \right\} = \overline{A} \cup \overline{B}$$

e:

$$\overset{\approx}{A} \bigcup \overline{\left(\overset{\approx}{A} \right)} = \left\{ (y, \mu(y)) \middle| y = x_A \cup \overline{x_A} = x ; x_A \in A \right\} = X$$

$$\overset{\approx}{A} \bigcap \overline{\left(\overset{\approx}{A} \right)} = \left\{ (y, \mu(y)) \middle| y = x_A \cap \overline{x_A} ; x_A \in A \right\} = \varnothing$$

$$\overline{\varnothing} = \overline{\overset{\approx}{A} \bigcap \overline{\left(\overset{\approx}{A} \right)}} = \overset{\approx}{A} \bigcup \overline{\left(\overset{\approx}{A} \right)} = X$$

$$\overline{X} = \overline{\overset{\approx}{A} \bigcup \overline{\left(\overset{\approx}{A} \right)}} = \overset{\approx}{A} \bigcap \overline{\left(\overset{\approx}{A} \right)} = \varnothing$$

f:

$$A \cup X = \left\{ (y, \mu(y)) \middle| y = x_A \cup x = x ; x_A \in A, x \in X \right\} = X$$

$$A \cap X = \left\{ (y, \mu(y)) \middle| y = x_A \cap x = x_A ; x_A \in A, x \in X \right\} = A$$

$$A \cup \varnothing = \left\{ (y, \mu(y)) \middle| y = x_A \cup x_\varnothing = x_A ; x_A \in A, x_\varnothing \in \varnothing \right\} = A$$

$$A \cap \varnothing = \left\{ (y, \mu(y)) \middle| y = x_A \cap x_\varnothing = \varnothing ; x_A \in A, x_\varnothing \in \varnothing \right\} = \varnothing$$

g:

$$
\begin{aligned}
\tilde{\tilde{A}} \cup (\tilde{\tilde{B}} \cup \tilde{\tilde{C}}) &= \left\{ (y, \mu(y)) \middle| y = x_A \cup (x_B \cup x_C); x_A \in A, x_B \in B, x_C \in C \right\} \\
&= \left\{ (y, \mu(y)) \middle| y = (x_A \cup x_B) \cup x_C; x_A \in A, x_B \in B, x_C \in C \right\} \\
&= (\tilde{\tilde{A}} \cup \tilde{\tilde{B}}) \cup \tilde{\tilde{C}}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\tilde{A}} \cap (\tilde{\tilde{B}} \cap \tilde{\tilde{C}}) &= \left\{ (y, \mu(y)) \middle| y = x_A \cap (x_B \cap x_C); x_A \in A, x_B \in B, x_C \in C \right\} \\
&= \left\{ (y, \mu(y)) \middle| y = (x_A \cap x_B) \cap x_C; x_A \in A, x_B \in B, x_C \in C \right\} \\
&= (\tilde{\tilde{A}} \cap \tilde{\tilde{B}}) \cap \tilde{\tilde{C}}
\end{aligned}
$$

h:

$$
\begin{aligned}
\tilde{\tilde{A}} \cup (\tilde{\tilde{B}} \cap \tilde{\tilde{C}}) &= \left\{ (y, \mu(y)) \middle| y = x_A \cup (x_B \cap x_C); x_A \in A, x_B \in B, x_C \in C \right\} \\
&= \left\{ (y, \mu(y)) \middle| y = (x_A \cup x_B) \cap (x_A \cup x_C); x_A \in A, x_B \in B, x_C \in C \right\} \\
&= (\tilde{\tilde{A}} \cup \tilde{\tilde{B}}) \cap (\tilde{\tilde{A}} \cup \tilde{\tilde{C}})
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\tilde{A}} \cap (\tilde{\tilde{B}} \cup \tilde{\tilde{C}}) &= \left\{ (y, \mu(y)) \middle| y = x_A \cap (x_B \cup x_C); x_A \in A, x_B \in B, x_C \in C \right\} \\
&= \left\{ (y, \mu(y)) \middle| y = (x_A \cap x_B) \cup (x_A \cap x_C); x_A \in A, x_B \in B, x_C \in C \right\} \\
&= (\tilde{\tilde{A}} \cap \tilde{\tilde{B}}) \cup (\tilde{\tilde{A}} \cap \tilde{\tilde{C}})
\end{aligned}
$$

Theorem 2: Suppose that $\tilde{\tilde{A}}$ is a fuzzy-2 set with the assumptions of definition (2), in this case:

$$\mu(\overline{x_A}) = 1 - \mu(x_A)$$

Proof:

$$
\begin{aligned}
1 = \mu(x) = \mu(\overline{x_A} \cup x_A) \\
&= \mu(\overline{A_x}) + \mu(A_x) - \mu(\overline{A_x} \cap A_x) \\
&= \mu(\overline{A_x}) + \mu(A_x) \\
&\Rightarrow \mu(\overline{A_x}) = 1 - \mu(A_x)
\end{aligned}
$$

6- Conclusion

In this paper, a new form of fuzzy sets (fuzzy-2 sets) is defined, so that the definition of union and intersection in it is different from the definition of union and intersection in ordinary fuzzy sets.

However, many algebraic properties of the fuzzy sets are maintained. The fuzzy-2 sets raised in some practical problems has more adaptability than the ordinary fuzzy sets.

References

1- Zadeh, L. A. "Fuzzy sets." Information and control 8, no. 3 (1965): 338-353.

2- Zadeh, L. A. "Communication-Fuzzy Algorithms." (1968).

3- Zadeh, L. A. "The Concept of a Linguistic Variable and Its Application to Approyimate Reasoninq." workinq paper, U. C. Berkeley (1973).

4- Mamdani, Ebrahim H., and Sedrak Assilian. "An experiment in linguistic synthesis with a fuzzy logic controller." International Journal of Man-Machine Studies 7, no. 1 (1975): 1-13.