Global Existence and Boundedness of a Two-Competing-Species Chemotaxis Model

Abstract: In this paper, we consider the following fully parabolic two-competing-species chemotaxis model

$$\begin{cases} u_{1t} = \Delta u_1 - \chi\nabla\cdot(u_1\nabla v_1) + \mu_1 u_1(1-u_1-e_1u_2), & x\in\Omega,\ t>0,\\ u_{2t} = \Delta u_2 - \xi\nabla\cdot(u_2\nabla v_2) + \mu_2 u_2(1-e_2u_1-u_2), & x\in\Omega,\ t>0,\\ v_{1t} = \Delta v_1 + u_1 - v_1, & x\in\Omega,\ t>0,\\ v_{2t} = \Delta v_2 + u_2 - v_2, & x\in\Omega,\ t>0 \end{cases}$$

under homogeneous Neumann boundary conditions, where $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ is a convex bounded domain with smooth boundary. Relying on a comparison principle, we show that the problem possesses a unique global bounded solution if μ_1 and μ_2 are large enough.

Keywords: Two-competing-species chemotaxis model; global existence; boundedness

MSC: 35K35; 35K45; 92C17.

1 Introduction and main result

In this paper, we consider the following two-competing-species chemotaxis model with two different chemicals

$$\begin{cases} u_{1t} = \Delta u_1 - \chi\nabla\cdot(u_1\nabla v_1) + \mu_1 u_1(1-u_1-e_1u_2), & x\in\Omega,\ t>0,\\ u_{2t} = \Delta u_2 - \xi\nabla\cdot(u_2\nabla v_2) + \mu_2 u_2(1-e_2u_1-u_2), & x\in\Omega,\ t>0,\\ v_{1t} = \Delta v_1 + u_1 - v_1, & x\in\Omega,\ t>0,\\ v_{2t} = \Delta v_2 + u_2 - v_2, & x\in\Omega,\ t>0,\\ \dfrac{\partial u_1}{\partial\nu} = \dfrac{\partial u_2}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_1(x,0)=u_{10}(x),\ u_2(x,0)=u_{20}(x),\ v(x,10)=v_{10}(x),\ v(x,20)=v_{20}(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a convex bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\nu}$ denotes the differentiation with respect to the outward normal derivative on $\partial\Omega$. $u_1(x,t)$ and $u_2(x,t)$ denote the densities of two competitive populations, whereas $v_1(x,t)$ and $v_2(x,t)$ represent the concentration of the chemicals produced by u_1 and u_2, respectively. The chemotactic sensitivities $\chi,\ \xi$, the growth rates of population $\mu_1,\ \mu_2$ and the competitive coefficients $e_1,\ e_2$ are all positive constants. The initial data $u_{10}, u_{20}, v_{10}, v_{20}$ are given functions satisfying

$$\begin{cases} u_{10} \in C^0(\overline{\Omega}) & \text{with } u_{10} \geq 0 \text{ in } \Omega, \\ u_{20} \in C^0(\overline{\Omega}) & \text{with } u_{20} \geq 0 \text{ in } \Omega, \\ v_{10} \in W^{1,q}(\overline{\Omega}) & \text{for some } q > n, \text{ with } v_{10} \geq 0 \text{ in } \Omega, \\ v_{20} \in W^{1,q}(\overline{\Omega}) & \text{for some } q > n, \text{ with } v_{20} \geq 0 \text{ in } \Omega. \end{cases} \tag{1.2}$$

For model (1.1), namely, multi-species and multi-stimuli chemotaxis model, only few results were studied. In the two-dimensional case, Black in [1] proved the corresponding Neumann problem possesses a unique global bounded solution for all positive parameters, it is also obtained that whenever $n \geq 1$, if $e_1,\ e_2 \in (0,1)$ any global bounded solution converges to the unique positive spatially homogeneous equilibrium in the large time for sufficiently large $\frac{\mu_1}{\chi^2}$ and $\frac{\mu_2}{\xi^2}$, and if $e_1 \geq 1 > e_2 > 0$ the solution $u_1(t) \to 0,\ u_2(t) \to 1,\ v_1(t) \to 1$ and $v_2(t) \to 0$ uniformly with respect to $x \in \Omega$ as $t \to \infty$ for large enough $\frac{\mu_2}{\xi^2}$. In the high-dimensional case, relying on the maximal Sobolev regularity and semigroup technique, Zheng et al. in [2] proved that the system has a unique globally bounded classical solution and there exists $\theta_0 > 0$ such that $\frac{\xi}{\mu_1} < \theta_0$ and $\frac{\chi}{\mu_2} < \theta_0$. However, to the best of our knowledge, in the high-dimensional convex domain, there are few papers concerned with the global existence of solution of (1.1).

Motivated on these studies, in this paper, we shall investigate the global existence and boundedness of solution to system (1.1) in a high-dimensional convex bounded domain.

Our main result in this paper is stated as follow.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a convex bounded domain with smooth boundary. Let $\chi,\ \xi,\ \mu_1,\ \mu_2,\ e_1$ and e_2 be some positive constants. Suppose that $\chi,\ \xi,\ \mu_1,\ \mu_2$ satisfy*

$$\frac{\mu_1}{\chi} > \frac{n}{4} \quad \text{and} \quad \frac{\mu_2}{\xi} > \frac{n}{4}. \tag{1.3}$$

Then for any choice of $(u_{10}, u_{20}, v_{10}, v_{20})$ fulfilling (1.2), problem (1.1) possesses a unique global classical solution (u_1, u_2, v_1, v_2) which is uniformly bounded in $\Omega \times (0,\infty)$.

Remark 1.1. We emphasize that Theorem 1.1 also holds for the spatial dimension $n \leq 2$.

Remark 1.2. As compared with the previous conditions Theorem 1.2 of [2], our conditions add the assumption that Ω is a convex region, but they are more natural, more symmetric, and only relate to the spatial dimension n.

This paper is organized as follows. Section 2 provides existence and uniqueness of local solutions. Section 3 is devoted to prove the global existence and boundedness of the classical solution of (1.1).

2 Existence and uniqueness of local solutions

To begin with, let us state a result on local existence and uniqueness of classical solutions.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary. Let χ, ξ, μ_1, μ_2, e_1 and e_2 be positive constants and let $q > \max\{2, n\}$. Then, for each nonnegative $u_{10} \in C^0(\overline{\Omega})$, $u_{20} \in C^0(\overline{\Omega})$, $v_{10} \in W^{1,q}(\Omega)$ and $v_{20} \in W^{1,q}(\Omega)$, there exists $T_{\max} \in (0, \infty]$ and a uniquely determined triple (u_1, u_2, v_1, v_2) of functions*

$$\begin{aligned}
&u_1 \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big),\\
&u_2 \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big),\\
&v_1 \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big) \cap L^\infty_{loc}\big([0, T_{\max}); W^{1,q}(\Omega)\big),\\
&v_2 \in C^0\big(\overline{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, T_{\max})\big) \cap L^\infty_{loc}\big([0, T_{\max}); W^{1,q}(\Omega)\big),
\end{aligned}$$

which solves (1.1) classically in $\Omega \times (0, T_{\max})$, and if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} (\|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} + \|v_1(\cdot, t)\|_{W^{1,q}(\Omega)} + \|v_2(\cdot, t)\|_{W^{1,q}(\Omega)}) = \infty.$$

Proof. This can be derived by standard arguments involving Banach's fixed point theorem and the parabolic regularity theory [3, Lemma 1.1]. □

3 Proof of Theorem 1.1

In this section we consider global bounded solution of (1.1) in a convex domain under suitable large assumption on the quotients $\frac{\mu_1}{\chi}$ and $\frac{\mu_2}{\xi}$. The proof is based on comparison argument similar to [3-5].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded convex domain. Then, under the same assumptions in Theorem 1.1, the solution of (1.1) satisfies*

$$\|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all} \;\; t \in (0, T_{\max}) \tag{3.1}$$

with some constant $C > 0$.

Proof. We consider the auxiliary function

$$J(x,t) := \frac{1}{\chi}u_1(x,t) + \frac{1}{\xi}u_2(x,t) + \frac{1}{2}|\nabla v_1(x,t)|^2 + \frac{1}{2}|\nabla v_2(x,t)|^2$$

for all $x \in \bar{\Omega}$ and $t \in (0, T_{\max})$. Using (1.1) and the pointwise identity $\nabla v \cdot \nabla \Delta v = \frac{1}{2}\Delta|\nabla v|^2 - |D^2 v|^2$, a straightforward calculation show the equation for J:

$$\begin{aligned} J_t - \Delta J + 2J =& \frac{2}{\chi}u_1 + \frac{2}{\xi}u_2 - u_1\Delta v_1 - u_2\Delta v_2 - |D^2 v_1|^2 - |D^2 v_2|^2 \\ &+ \frac{\mu_1}{\chi}u_1 - \frac{\mu_1}{\chi}u_1^2 - \frac{\mu_1 e_1}{\chi}u_1u_2 + \frac{\mu_2}{\xi}u_2 - \frac{\mu_2 e_2}{\xi}u_1u_2 - \frac{\mu_2}{\xi}u_2^2 \end{aligned} \tag{3.2}$$

in $\Omega \times (0, T_{\max})$. From Young's inequality and the inequality $|\Delta v|^2 \le n|D^2 v|^2$, we can estimate

$$-u_1\Delta v_1 \le \frac{|\Delta v_1|^2}{n} + \frac{n}{4}u_1^2 \le |D^2 v_1|^2 + \frac{n}{4}u_1^2 \quad \text{in } \ \Omega \times (0, T_{\max})$$

and

$$-u_2\Delta v_2 \le \frac{|\Delta v_2|^2}{n} + \frac{n}{4}u_2^2 \le |D^2 v_2|^2 + \frac{n}{4}u_2^2 \quad \text{in } \ \Omega \times (0, T_{\max}).$$

Combining this with (3.2), (1.3) and Young's inequality, we infer that

$$\begin{aligned} J_t - \Delta J + 2J \le& -(\frac{\mu_1}{\chi} - \frac{n}{4})u_1^2 + \frac{1}{\chi}(2+\mu_1)u_1 - (\frac{\mu_2}{\xi} - \frac{n}{4})u_2^2 + \frac{1}{\xi}(2+\mu_2)u_2 \\ \le& \frac{(2+\mu_1)^2}{(4\mu_1 - n\chi)\chi} + \frac{(2+\mu_2)^2}{(4\mu_2 - n\xi)\xi} \quad \text{in } \ \Omega \times (0, T_{\max}). \end{aligned}$$

Relying on the obvious property of functions fulfilling a homogenous Neumann boundary condition on convex domains Ω we note that $\frac{\partial|\nabla v|^2}{\partial\nu} \le 0$ on $\partial\Omega$ by [6, Lemma 3.2]. Then

$$\frac{\partial J}{\partial \nu} = \frac{1}{\chi}\frac{\partial u_1}{\partial \nu} + \frac{1}{\xi}\frac{\partial u_2}{\partial \nu} + \frac{1}{2}\frac{\partial|\nabla v_1|^2}{\partial \nu} + \frac{1}{2}\frac{\partial|\nabla v_2|^2}{\partial \nu} \le 0$$

for all $x \in \partial\Omega$ and $t \in (0, T_{\max})$. Let $y(t) \in C^1([0,\infty))$ denote the solution of

$$\begin{cases} y'(t) + y(t) = \dfrac{(2+\mu_1)^2}{(4\mu_1 - n\chi)\chi} + \dfrac{(2+\mu_2)^2}{(4\mu_2 - n\xi)\xi} \quad \text{for all } \ t \in (0,\infty), \\ y(0) = c_1, \end{cases}$$

where $c_1 := \frac{1}{\chi}\|u_{10}\|_{L^\infty(\Omega)} + \frac{1}{\xi}\|u_{20}\|_{L^\infty(\Omega)} + \frac{1}{2}\|\nabla v_{10}\|^2_{L^\infty(\Omega)} + \frac{1}{2}\|\nabla v_{20}\|^2_{L^\infty(\Omega)}$. Then, explicitly solving this initial-value problem, we conclude that

$$y(t) \to \frac{(2+\mu_1)^2}{(4\mu_1 - n\chi)\chi} + \frac{(2+\mu_2)^2}{(4\mu_2 - n\xi)\xi} \quad \text{as } \ t \to \infty.$$

On the other hand, by the comparison principle, we have

$$J(x,t) \le y(t) \quad \text{for all } \ x \in \Omega \ \text{ and } \ t \in (0, T_{\max}),$$

and this clearly proves the lemma. $\square$

We are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Because v_i solves the following parabolic equation

$$\begin{cases} v_{it} = \Delta v_i + u_i - v_i, & x \in \Omega,\ t > 0,\ i = 1, 2, \\ \dfrac{\partial v_i}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\ i = 1, 2, \\ v_i(x, 0) = v_{i0}(x), & x \in \Omega,\ i = 1, 2, \end{cases}$$

by (3.1) and the standard parabolic regularity theory [7, Lemma 4.1] or [8, Lemma 1], we see that

$$\|v_i(\cdot, t)\|_{W^{1,\infty}} \leq C \quad \text{for all} \ \ t \in (0, T_{\max}), \quad i = 1, 2.$$

This together with Lemma 3.1 proves Theorem 1.1 by Lemma 2.1. $\square$

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