Random attractor for non-autonomous stochastic extensible plate equation on unbounded domains

Abstract: We study the asymptotic behavior of solutions to the non-autonomous stochastic extensible plate equation driven by additive noise defined on unbounded domains. We first prove the uniform estimates of solutions, and then establish the existence of a random attractor.

Keywords: pullback attractors, extensible plate equation, unbounded domains, the splitting technique, additive noise.

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1 Introduction

Consider the following non-autonomous stochastic extensible plate equations with additive noise and nonlinear damping defined in the entire space $\mathbb{R}^n$:

$$u_{tt} + h(u_t) + \Delta^2 u + (p - \varrho\|\nabla u\|^2)\Delta u + \lambda u + f(x,u) = g(x,t) + \phi(x)\frac{dW}{dt}, \tag{1.1}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \quad u_t(x,\tau) = u_1(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, λ, ϱ is positive constant, p is a negative constant, f is a nonlinearity satisfying certain growth and dissipative conditions, $g(x,\cdot)$ and ϕ are given functions in $L^2_{loc}(\mathbb{R}, H^1(\mathbb{R}^n))$ and $H^2(\mathbb{R}^n) \cap H^3(\mathbb{R}^n)$, respectively, $W(t)$ is a two-sided real-valued Wiener process on a probability space.

Plate equations have been investigated for many years due to their importance in some physical areas such as vibration and elasticity theories of solid mechanics. The study of the long-time dynamics of plate equations has become an outstanding area in the field of the infinite-dimensional dynamical system. While the attractors is regarded as a proper notation to describe the long-time dynamics of solutions. Equations of type (1.1)-(1.2) model transversal vibrations of thin extensible elastic plates, which was established based on the theory of elastic vibration in [3, 18].

When $h \equiv 0$, and $p = \varrho \equiv 0$, (1.1)-(1.2) reduces to a standard deterministic plate equation, which has been extended studied by some authors. For instance, Yang and Zhong [34, 35] investigated the existence of the global attractors for the autonomous plate equation with nonlinear damping on the bounded domain as well as the non-autonomous plate equation with a localized damping. In [14, 15], Khanmamedov scrutinized the existence of global attractors for the plate equation with critical exponent under the case of the different damping on an unbounded domain; similar problems were surveyed by Xiao in [32,33]. Yue and Zhong considered the global attractors for the plate equation with critical exponent in a locally uniform space [38]. A global attractor of the plate equation with displacement-dependent damping was achieved by Khanmamedov in [16]. Carbone et.al. investigated the pullback attractors of a singularly non-autonomous plate equation, see [5].

As $h \equiv 0$, p and ϱ are not zero, the equation is so called a deterministic Kirchhoff type problem. In [17], Kirchhoff first paid attention to the oscillations of stretched strings and plates. Later, the analogous problems were considered by several authors such as Giorgi and Pata et.al. [11, 12], Bochicchio and Vuk [4]. Barbosa and Ma [1] investigated the long-time behavior of an extensible plate equation with thermal memory. Yao and Ma [36] proved the existence of a global attractor for the plate equations of Kirchhoff type with nonlinear damping and memory using the contraction function method.

In the case when $h \neq 0$, (1.1)-(1.2) is just the stochastic plate equation that we are concerned with in this paper. As $p = \varrho \equiv 0$, in [20, 23], the authors proved the existence of random attractors on a bounded domain and unbounded domain; Yao and Ma et.al. [37] obtained the asymptotic behavior of a class of stochastic plate equations with rotational inertia and Kelvin-Voigt dissipative term. Ma and Xu [19] studied the random attractors of the extensible suspension bridge equation with white noise. In recent years, the existence of random attractors for stochastic dynamical system on unbounded domains have been investigated by several authors, such as Reaction-diffusion equations with additive noise [2], Reaction-diffusion equations with multiplicative noise [31], FitzHugh- Nagumo equations with additive noise [27], Navier-Stokes equations with additive noise [13], wave equations with additive noise [26, 27, 30], wave equations with multiplicative noise [29].

Motivated by above literatures, the goal of the present paper is to study random attractors of non-autonomous stochastic extensible equation (1.1)-(1.2) on unbounded domain. By applying the abstract results in [25], we will prove the stochastic strongly damped plate equation (1.1)-(1.2) has tempered random attractors in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

In general, the existence of global random attractor depends on some kind compactness (see, e.g., [6–8, 10]). Involving to our problem (1.1)-(1.2), two main difficulties needed to be overcome. One difficulty is to prove the existence of random attractors for (1.1)-(1.2) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, we must establish the pullback asymptotic compactness of solutions. Since Sobolev embeddings are not compact on unbounded domain, we cannot get the desired asymptotic compactness directly from the regularity of solutions. We here overcome the difficulty by using the uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$ and the splitting technique, see [26,29] for details; another difficulty is brought by the term $-\|\nabla u\|^2_{L^2(\mathbb{R}^n)}\Delta u$, they make the estimates more complex than those in [7,8]. Besides, in fact, four-order derivative term $\triangle^2 u$ can also lead to some obstacles in deducing the regularity of the solution.

The framework of this paper is as follows. In the next Section, we recall some definitions and already known results concerning random attractors. In Section 3, we define a continuous cocycle for Eq.(1.1) in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then we derive all necessary uniform estimates of solutions in Section 4. Finally, in Section 5, we prove the existence and uniqueness of tempered random attractor for the non-autonomous stochastic extensible plate equation.

Throughout the paper, the letters c and c_i $(i = 1, 2, \ldots)$ are generic positive constants which may change their values from line to line or even in the same line.

2 Preliminaries

In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems.

Let X be a separable Banach space and $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$ which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega,\ t \in \mathbb{R}. \tag{2.1}$$

We often say that $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

The following four definitions and one proposition are from [25].

Definition 2.1. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)-(4) are satisfied:

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;

(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;

(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Hereafter, we assume Φ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and $\mathcal{D}$ is the collection of all tempered families of nonempty bounded subsets of X parameterized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\}\}.$$

$\mathcal{D}$ is said to be tempered if there exists $x_0 \in X$ such that for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the following holds:

$$\lim_{t\to-\infty} e^{ct} d(D(\tau + t, \theta_t\omega), x_0) = 0. \tag{2.2}$$

Given $D \in \mathcal{D}$, the family $\Omega(D) = \{\Omega(D, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Ω-limit set of D where

$$\Omega(D, \tau, \omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega))}. \tag{2.3}$$

The cocycle Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X \tag{2.4}$$

whenever $t_n \to \infty$, and $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$ with $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.2. A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T = T(D, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T. \tag{2.5}$$

If, in addition, $K(\tau, \omega)$ is closed in X and is measurable in ω with respect to $\mathcal{F}$, then K is called a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.3. A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (1)-(3) are fulfilled: for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(1) $\mathcal{A}(\tau, \omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$.

(2) $\mathcal{A}$ is invariant, that is,

$$\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t\omega). \tag{2.6}$$

(3) For every $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t \to \infty} d_H(\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0, \tag{2.7}$$

where d_H is the Hausdorff semi-distance given by $d_H(F, G) = \sup_{u \in F} \inf_{v \in G} ||u - v||_X$, for any $F,\ G \subset X$.

As in the deterministic case, random complete solutions can be used to characterized the structure of a $\mathcal{D}$-pullback attractor. The definition of such solutions are given below.

Definition 2.4. A mapping $\Psi : \mathbb{R} \times \mathbb{R} \times \Omega \to X$ is called a random complete solution of Φ if for every $\tau \in \mathbb{R}^+$, $s, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau + s, \theta_s\omega, \Psi(s, \tau, \omega)) = \Psi(t + s, \tau, \omega). \tag{2.8}$$

If, in addition, there exists a tempered family $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\Psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_t\omega)$ for every $t \in \mathbb{R}, \tau \in \mathbb{R}$ and $\omega \in \Omega$, then Ψ is called a tempered random complete solution of Φ.

Proposition 2.1. *Suppose Φ is $\mathcal{D}$-pullback asymptotically compact in X and has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. Then Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{D \in \mathcal{D}} \Omega(D, \tau, \omega) \tag{2.9}$$

$$= \{\Psi(0, \tau, \omega) : \Psi \textit{ is a tempered random complete solution of } \Phi\}. \tag{2.10}$$

3 Cocycles for stochastic plate equation

In this section, we outline some basic settings about (1.1)-(1.2) and show that it generates a continuous cocycle in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $D(A)=H^4$. We can define the powers A^ν of A for $\nu\in\mathbb{R}$. The space $V_\nu=D(A^{\frac{\nu}{4}})$ is a Hilbert space with the following inner product and norm

$$(u,v)_\nu=(A^{\frac{\nu}{4}}u,A^{\frac{\nu}{4}}v),\qquad\qquad \|\cdot\|_\nu=\|A^{\frac{\nu}{4}}\cdot\|.$$

For brevity, the notation $(\cdot,\cdot)$ for L^2-inner product will also be used for the notation of duality pairing between dual spaces.

Let $E=H^2\times L^2$, with the Sobolev norm

$$\|y\|_{H^2\times L^2}=(\|v\|^2+\|u\|^2+\|\Delta u\|^2)^{\frac{1}{2}},\quad \text{for}\quad y=(u,v)^\top\in E. \tag{3.1}$$

For simplicity, let $\varrho\equiv 1$ and $\xi=u_t+\delta u$, where δ is a small positive constant whose value will be determined later, then (1.1)-(1.2) can be rewritten as the equivalent system

$$\begin{cases} \frac{du}{dt}+\delta u=\xi,\\ \frac{d\xi}{dt}-\delta\xi+(\lambda+\delta^2+A)u+h(\xi-\delta u)+(p-\|\nabla u\|^2)\Delta u\\ \qquad\qquad\qquad\qquad +f(x,u)=g(x,t)+\phi(x)\frac{dW}{dt}, \end{cases} \tag{3.2}$$

with the initial value conditions

$$u(x,\tau)=u_0(x),\qquad\qquad \xi(x,\tau)=\xi_0(x), \tag{3.3}$$

where $\xi_0(x)=u_1(x)+\delta u_0(x),\ x\in\mathbb{R}^n$.

Assumption I. Assume that the functions $h\in C^1(\mathbb{R})$ and $f\in C^1(\mathbb{R})$ satisfy the following conditions:

(1) Let $F(x,u)=\int_0^u f(x,s)ds$ for $x\in\mathbb{R}^n$ and $u\in\mathbb{R}$, there exist positive constants $c_i(i=1,2,3,4)$, such that

$$|f(x,u)|\le c_1|u|^p+\eta_1(x),\qquad\qquad \eta_1\in L^2(\mathbb{R}^n), \tag{3.4}$$

$$f(x,u)u-c_2F(x,u)\ge\eta_2(x),\qquad\qquad \eta_2\in L^1(\mathbb{R}^n), \tag{3.5}$$

$$F(x,u)\ge c_3|u|^{p+1}-\eta_3(x),\qquad\qquad \eta_3\in L^1(\mathbb{R}^n), \tag{3.6}$$

$$|\frac{\partial f}{\partial u}(x,u)|\le\beta,\quad |\frac{\partial f}{\partial x}(x,u)|\le\eta_4(x),\quad \eta_4\in L^2(\mathbb{R}^n), \tag{3.7}$$

where $\beta>0,\ 1\le p\le\frac{n+4}{n-4}$. Note that (3.4) and (3.5) imply

$$F(x,u)\le c(|u|^2+|u|^{p+1}+\eta_1^2+\eta_2). \tag{3.8}$$

(2) There exist two constants β_1,β_2 such that

$$h(0)=0,\qquad 0<\beta_1\le h'(v)\le\beta_2<\infty. \tag{3.9}$$

For our purpose, it is convenient to convert the problem (1.1)-(1.2) (or (3.2)-(3.3)) into a deterministic system with a random parameter, and then show that it generates a cocycle over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

We identify $\omega(t)$ with $W(t)$, i.e., $\omega(t) = W(t) = W(t,x),\ t \in \mathbb{R}$. Set $v(t) = \xi(t) - \phi\omega(t)$, we obtain the equivalent system of (3.2)-(3.3),

$$\begin{cases} \frac{du}{dt} + \delta u = v + \phi\omega(t), \\ \frac{dv}{dt} - \delta v + (\lambda + \delta^2 + A)u + (p - \|\nabla u\|^2)\Delta u + f(x,u) = g(x,t) \\ \qquad\qquad -h(v + \phi\omega(t) - \delta u) + \delta\phi\omega(t), \end{cases} \tag{3.10}$$

with the initial value conditions

$$u(x,\tau,\tau) = u_0(x), \qquad v(x,\tau,\tau) = v_0(x), \tag{3.11}$$

where $v_0(x) = \xi_0(x) - \phi\omega(t),\ x \in \mathbb{R}^n$.

The well-posedness of the deterministic problem (3.10)-(3.11) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ can be established by standard methods as in [21,24], more precisely, if Assumption I is fulfilled, then we can prove the following Lemma.

Lemma 3.2 *Put* $\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0) = (u(t+\tau,\tau,\theta_{-\tau}\omega,u_0), v(t+\tau,\tau,\theta_{-\tau}\omega,v_0))^\top$, *where* $\varphi_0 = (u_0,v_0)^\top$, *and let Assumption I and Assumption II below hold. Then for every* $\omega\in\Omega,\ \tau\in\mathbb{R}$ *and* $\varphi_0 \in E(\mathbb{R}^n)$, *problem (3.10)-(3.11) has a unique* $(\mathcal{F},\ \mathcal{B}(H^2(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$*-measurable solution* $\varphi(\cdot,\tau,\omega,\varphi_0) \in C([\tau,\infty), E(\mathbb{R}^n))$ *with* $\varphi(\tau,\tau,\omega,\varphi_0) = \varphi_0,\ \varphi(t,\tau,\omega,\varphi_0) \in E(\mathbb{R}^n)$ *being continuous in* φ_0 *with respect to the usual norm of* $E(\mathbb{R}^n)$ *for each* $t > \tau$. *Moreover, for every* $(t,\tau,\omega,\varphi_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$, *the mapping*

$$\Phi(t,\tau,\omega,\varphi_0) = \varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0) \tag{3.12}$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$ *to* $E(\mathbb{R}^n)$ *over* $\mathbb{R}$ *and* $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Introducing the homeomorphism $P(\theta_t\omega)(u,v)^\top = (u, v + z(\theta_t\omega))^\top,\ (u,v)^\top \in E(\mathbb{R}^n)$ with an inverse homeomorphism $P^{-1}(\theta_t\omega)(u,v)^\top = (u, v - z(\theta_t\omega))^\top$. Then, the transformation

$$\widetilde{\Phi}(t,\tau,\omega,(u_0,\xi_0)) = P(\theta_t\omega)\Phi(t,\tau,\omega,(u_0,v_0))P^{-1}(\theta_t\omega) \tag{3.13}$$

generates a continuous cocycle with (3.2)-(3.3) over over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Note that these two continuous cocycles are equivalent. By (3.13), it is easy to check that $\widetilde{\Phi}$ has a random attractor provided Φ possesses a random attractor. Then, we only need to consider the continuous cocycle Φ.

Next we make another assumption:

Assumption II. We assume that σ, δ and $g(x,t)$ satisfy the following conditions:

$$\sigma = \min\{\delta, \frac{\delta c_2}{2}\}, \quad \lambda + \delta^2 - \beta_2\delta > 0, \quad \beta_1 > 4\delta + \frac{3\beta^2}{\delta(\lambda+\delta^2-\beta_2\delta)}. \tag{3.14}$$

Moreover,

$$\int_{-\infty}^0 e^{\sigma s}\|g(\cdot,\tau+s)\|_1^2 ds < \infty, \quad \forall\ \tau \in \mathbb{R}, \tag{3.15}$$

and

$$\lim_{k\to\infty}\int_{-\infty}^{0} e^{\sigma s}\int_{|x|\geq k}|g(x,\tau+s)|^2dxds=0,\ \forall\ \tau\in\mathbb{R}, \tag{3.16}$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

Given a bounded nonempty subset B of E, we write $\|B\|=\sup_{\phi\in B}\|\phi\|_E$. Let $D=\{D(\tau,\omega):\tau\in\mathbb{R},\ \omega\in\Omega\}$ be a family of bounded nonempty subsets of E such that for every $\tau\in\mathbb{R},\omega\in\Omega$,

$$\lim_{s\to-\infty} e^{\sigma s}\|D(\tau+s,\theta_s\omega)\|_E^2=0. \tag{3.17}$$

Let $\mathcal{D}$ be the collection of all such families, that is,

$$\mathcal{D}=\{D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}:\ D \text{ satisfies } (3.17)\}. \tag{3.18}$$

4 Uniform estimates of solutions

In this section, we conduct uniform estimates on the weak solutions of the stochastic plate equations (3.2)-(3.3) defined on $\mathbb{R}^n$, through the converted random equation (3.10)-(3.11), for the purposes of showing the existence of a pullback absorbing sets and the pullback asymptotic compactness of the random dynamical system.

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E=(\|v\|^2+(\lambda+\delta^2-\beta_2\delta)\|u\|^2+\|\Delta u\|^2)^{\frac{1}{2}},\quad \text{for}\ \ Y=(u,v)\in E. \tag{4.1}$$

It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^2\times L^2}$ in (3.1).

First we show that the cocycle Φ has a pullback $\mathcal{D}$-absorbing set in $\mathcal{D}$.

Lemma 4.1 *Under Assumptions I and II, for every $\tau\in\mathbb{R},\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T=T(\tau,\omega,D)>0$ such that for all $t\geq T$ the solution of problem (3.10)-(3.11) satisfies*

$$\|Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|_E^2+\frac{1}{2}(\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2-p)^2\leq R_1(\tau,\omega),$$

and $R_1(\tau,\omega)$ is given by

$$R_1(\tau,\omega)=M+M\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|^2ds+c\int_{-\infty}^{0} e^{\sigma s}(|\omega(s)|^2+|\omega(s)|^4+|\omega(s)|^{p+1})ds, \tag{4.2}$$

where M is a positive constant independent of τ,ω,D.

Proof. Taking the inner product of the second equation of (3.10) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2-\delta\|v\|^2+(\lambda+\delta^2)(u,v)+(Au,v)+(p-\|\nabla u\|^2)(\Delta u,v)+(f(x,u),v)\\&=(g(x,t),v)-(h(v+\phi\omega(t)-\delta u),v)+\delta(\phi,v)\omega(t).\end{aligned} \tag{4.3}$$

By the first equation of (3.10), we have

$$v=u_t-\phi\omega(t)+\delta u. \tag{4.4}$$

By (3.9) and Lagranges mean value theorem, we have

$$\begin{aligned}
&-(h(v+\phi\omega(t)-\delta u),v)\\
=&-(h(v+\phi\omega(t)-\delta u)-h(0),v)\\
=&-(h'(\vartheta)(v+\phi\omega(t)-\delta u),v)\\
\le&-\beta_1\|v\|^2-(h'(\vartheta)(\phi\omega(t)-\delta u),v)\\
\le&-\beta_1\|v\|^2+\beta_2|\omega(t)|\|\phi\|\|v\|+h'(\vartheta)\delta(u,v)\\
\le&-\beta_1\|v\|^2+\frac{\beta_1-\delta}{6}\|v\|^2+c|\omega(t)|^2\|\phi\|^2+h'(\vartheta)\delta(u,v),
\end{aligned}\tag{4.5}$$

where ϑ is between 0 and $v+\phi\omega(t)-\delta u$.
By (3.9) and (4.4), we get

$$\begin{aligned}
&h'(\vartheta)\delta(u,v)\\
=&h'(\vartheta)\delta(u,u_t-\phi\omega(t)+\delta u)\\
\le&\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2+\beta_2\delta^2\|u\|^2+\beta_2\delta|\omega(t)|\|\phi\|\|u\|\\
\le&\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2+\beta_2\delta^2\|u\|^2+\frac{1}{4}\delta(\lambda+\delta^2-\beta_2\delta)\|u\|^2+c|\omega(t)|^2\|\phi\|^2.
\end{aligned}\tag{4.6}$$

Substituting (4.4) into the third, fourth and fifth terms on the left-hand side of (4.3), we find that

$$\begin{aligned}
&(\lambda+\delta^2)(u,v)\\
=&(\lambda+\delta^2)(u,u_t-\phi\omega(t)+\delta u)\\
\ge&\frac{1}{2}(\lambda+\delta^2)\frac{d}{dt}\|u\|^2+\delta(\lambda+\delta^2)\|u\|^2-(\lambda+\delta^2)|\omega(t)|\|\phi\|\|u\|\\
\ge&\frac{1}{2}(\lambda+\delta^2)\frac{d}{dt}\|u\|^2+\delta(\lambda+\delta^2)\|u\|^2-\frac{1}{4}\delta(\lambda+\delta^2-\beta_2\delta)\|u\|^2-c|\omega(t)|^2\|\phi\|^2,
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
&(Au,v)=(\Delta u,\Delta v)=(\Delta u,\Delta u_t-\omega(t)\Delta\phi+\delta\Delta u)\\
&\ge\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2+\delta\|\Delta u\|^2-|\omega(t)|\|\Delta\phi\|\|\Delta u\|\\
&\ge\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2+\frac{\delta}{2}\|\Delta u\|^2-\frac{1}{2\delta}|\omega(t)|^2\|\Delta\phi\|^2,
\end{aligned}\tag{4.8}$$

$$\begin{aligned}
&(p-\|\nabla u\|^2)(\Delta u,v)\\
=&(p-\|\nabla u\|^2)\,(\Delta u,u_t-\phi\omega(t)+\delta u)\\
=&(\|\nabla u\|^2-p)\,(\nabla u,\nabla(u_t-\phi\omega(t)+\delta u))\\
=&\frac{1}{4}\frac{d}{dt}(\|\nabla u\|^2-p)^2+\frac{\delta}{2}(\|\nabla u\|^2-p)^2+\frac{\delta}{2}\|\nabla u\|^4-\frac{\delta p^2}{2}\\
&-\omega(t)(\|\nabla u\|_1^2-p)(\nabla u,\nabla\phi)\\
\ge&\frac{1}{4}\frac{d}{dt}(\|\nabla u\|^2-p)^2+\frac{\delta}{2}(\|\nabla u\|^2-p)^2+\frac{\delta}{2}\|\nabla u\|^4-\frac{\delta p^2}{2}\\
&-\frac{\delta}{4}(\|\nabla u\|^2-p)^2-\frac{\delta}{2}\|\nabla u\|^4-c|\omega(t)|^4\|\nabla\phi\|^4\\
\ge&\frac{1}{4}\frac{d}{dt}(\|\nabla u\|^2-p)^2+\frac{\delta}{4}(\|\nabla u\|^2-p)^2-\frac{\delta p^2}{2}-c|\omega(t)|^4\|\nabla\phi\|^4.
\end{aligned}\tag{4.9}$$

Using the Cauchy-Schwarz inequality and Young's inequality, we have

$$\delta(\phi\omega(t), v) \leq \delta|\omega(t)|\|\phi\|\|v\| \leq c\|\phi\|^2|\omega(t)|^2 + \frac{\beta_1 - \delta}{6}\|v\|^2, \tag{4.10}$$

and

$$(g, v) \leq \|g\|\|v\| \leq c\|g\|^2 + \frac{\beta_1 - \delta}{6}\|v\|^2. \tag{4.11}$$

Let $\widetilde{F}(x,u) = \int_{\mathbb{R}^n} F(x,u)dx$. Then for the last term on the left-hand side of (4.3) we have

$$\begin{aligned}(f(x,u), v) &= (f(x,u), u_t - \phi\omega(t) + \delta u)\\ &= \frac{d}{dt}\widetilde{F}(x,u) + \delta(f(x,u),u) - (f(x,u), \phi\omega(t)).\end{aligned} \tag{4.12}$$

By condition (3.5) we get

$$(f(x,u),u) \geq c_2\widetilde{F}(x,u) + \int_{\mathbb{R}^n} \eta_2(x)dx. \tag{4.13}$$

Following from condition (3.4) and (3.6), we obtain

$$\begin{aligned}&(f(x,u), \phi\omega(t))\\ \leq &\int_{\mathbb{R}^n} (c_1|u|^p + \eta_1(x))|\phi\omega(t)|dx\\ \leq &\|\eta_1(x)\|\|\phi\||\omega(t)| + c_1(\int_{\mathbb{R}^n} |u|^{p+1}dx)^{\frac{p}{p+1}}\|\phi\|_{p+1}|\omega(t)|\\ \leq &\|\eta_1(x)\|\|\phi\||\omega(t)| + c_1(\int_{\mathbb{R}^n} (F(x,u) + \eta_3(x))dx)^{\frac{p}{p+1}}\|\phi\|_{p+1}|\omega(t)|\\ \leq &\frac{1}{2}\|\eta_1(x)\|^2 + \frac{1}{2}\|\phi\|^2|\omega(t)|^2 + \frac{\delta c_2}{2}\widetilde{F}(x,u) + \frac{\delta c_2}{2}\int_{\mathbb{R}^n}\eta_3(x)dx + c\|\phi\|_{H^2}^{p+1}|\omega(t)|^{p+1}.\end{aligned} \tag{4.14}$$

By (4.12)-(4.14), we get

$$\begin{aligned}&\delta(f(x,u),u) - (f(x,u), \phi\omega(t))\\ \geq &\frac{\delta c_2}{2}\widetilde{F}(x,u) + \delta\int_{\mathbb{R}^n}\eta_2(x)dx - \frac{1}{2}\|\eta_1(x)\|^2 - \frac{1}{2}\|\phi\|^2|\omega(t)|^2\\ &- \frac{\delta c_2}{2}\int_{\mathbb{R}^n}\eta_3(x)dx - c\|\phi\|_{H^2}^{p+1}|\omega(t)|^{p+1}.\end{aligned} \tag{4.15}$$

Substitute (4.5)-(4.15) into (4.3) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + \frac{1}{2}(\|\nabla u\|^2 - p)^2 + 2\widetilde{F}(x,u))\\ &+ \frac{\delta}{2}\|v\|^2 + \frac{\delta}{2}(\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \frac{\delta}{2}\|\Delta u\|^2 + \frac{\delta}{4}(\|\nabla u\|^2 - p)^2 + \frac{\delta c_2}{2}\widetilde{F}(x,u)\\ \leq &\frac{2\delta - \beta_1}{2}\|v\|^2 + c(1 + |\omega(t)|^2 + |\omega(t)|^4 + |\omega(t)|^{p+1}) + c\|g\|^2.\end{aligned} \tag{4.16}$$

Let $\sigma = \min\{\delta, \frac{\delta c_2}{2}\}$, then

$$\frac{d}{dt}(\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + \frac{1}{2}(\|\nabla u\|^2 - p)^2 + 2\widetilde{F}(x,u))$$

$$
\begin{aligned}
&+\sigma(\|v\|^2+(\lambda+\delta^2-\beta_2\delta)\|u\|^2+\|\Delta u\|^2+\frac{1}{2}(\|\nabla u\|^2-p)^2+2\widetilde{F}(x,u)) \\
&\leq c\|g\|^2+c(1+|\omega(t)|^2+|\omega(t)|^4+|\omega(t)|^{p+1}).
\end{aligned} \tag{4.17}
$$

Multiplying (4.17) by $e^{\sigma t}$ and then integrating over $(\tau-t,\tau)$, we have

$$
\begin{aligned}
&e^{\sigma\tau}(\|v(\tau,\tau-t,\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\omega,u_0)\|^2 \\
&+\|\Delta u(\tau,\tau-t,\omega,u_0)\|^2+\frac{1}{2}(\|\nabla u(\tau,\tau-t,\omega,u_0)\|^2-p)^2+2\widetilde{F}(x,(\tau,\tau-t,\omega,u_0))) \\
&\leq e^{\sigma(\tau-t)}(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+\frac{1}{2}(\|\nabla u_0\|^2-p)^2+2\widetilde{F}(x,u_0)) \\
&+c\int_{\tau-t}^{\tau}e^{\sigma s}\|g(x,s)\|^2ds+c\int_{\tau-t}^{\tau}e^{\sigma s}(1+|\omega(s)|^2+|\omega(s)|^4+|\omega(s)|^{p+1})ds.
\end{aligned}
$$

Replacing ω by $\theta_{-\tau}\omega$ in the above we obtain, for every $t\in\mathbb{R}^+,\ \tau\in\mathbb{R},$ and $\omega\in\Omega$,

$$
\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\Delta u(\tau,\tau-t,\omega,u_0)\|^2 \\
&+\frac{1}{2}(\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2-p)^2+2\widetilde{F}(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)) \\
&\leq e^{-\sigma t}(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+\frac{1}{2}(\|\nabla u_0\|^2-p)^2+2\widetilde{F}(x,u_0)) \\
&+c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(1+|\theta_{-\tau}\omega(s)|^2+|\theta_{-\tau}\omega(s)|^4+|\theta_{-\tau}\omega(s)|^{p+1})ds \\
&+c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|g(x,s)\|^2ds.
\end{aligned} \tag{4.18}
$$

Again, by (3.8), we get

$$
\widetilde{F}(x,u_0)\leq c(1+\|u_0\|^2+\|u_0\|^{p+1}).
$$

Therefore, for the first term on the right-hand side of (4.18), we have

$$
\begin{aligned}
&e^{-\sigma t}(\|v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0\|^2+\|\Delta u_0\|^2+\frac{1}{2}(\|\nabla u_0\|^2-p)^2+2\widetilde{F}(x,u_0)) \\
&\leq ce^{-\sigma t}(1+\|v_0\|^2+\|u_0\|_{H^2}^2+\|u_0\|_{H^2}^{p+1}).
\end{aligned}
$$

Since that $(u_0,v_0)^\top\in D(\tau-t,\theta_{-t}\omega)$ and $D\in\mathcal{D}$, then we find

$$
\lim_{t\to+\infty}e^{-\sigma t}(\|v_0\|^2+\|u_0\|_{H^2}^2+\|u_0\|_{H^2}^{p+1})=0.
$$

Therefore, there exists $T=T(\tau,\omega,D)>0$ such that for all $t\geq T$,

$$
e^{-\sigma t}(1+\|v_0\|^2+\|u_0\|_{H^2}^2+\|u_0\|_{H^2}^{p+1})\leq 1. \tag{4.19}
$$

For the second term on the right-hand side of (4.18), we find

$$
\begin{aligned}
&c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(1+|\theta_{-\tau}\omega(s)|^2+|\theta_{-\tau}\omega(s)|^4+|\theta_{-\tau}\omega(s)|^{p+1})ds \\
&\leq c\int_{-t}^{0}e^{\sigma s}(1+|\omega(s)|^2+|\omega(s)|^4+|\omega(s)|^{p+1})ds \\
&\leq c\int_{-\infty}^{0}e^{\sigma s}(1+|\omega(s)|^2+|\omega(s)|^4+|\omega(s)|^{p+1})ds \\
&\leq \frac{c}{\sigma}+c\int_{-\infty}^{0}e^{\sigma s}(|\omega(s)|^2+|\omega(s)|^4+|\omega(s)|^{p+1})ds.
\end{aligned}
$$

It is worth mentioning that $\omega(s)$ has at most linear growth at $|s| \to \infty$, which combines (3.18), we can have

$$c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}(1+|\theta_{-\tau}\omega(s)|^2+|\theta_{-\tau}\omega(s)|^4+|\theta_{-\tau}\omega(s)|^{p+1})ds \to \frac{c}{\sigma}, \quad (t\to\infty). \tag{4.20}$$

In order to complete the proof, we still need to estimate the fifth term on the left-hand side of (4.18). Thanks to (3.6), we obtain that, for all $t \geq 0$,

$$-2\widetilde{F}(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)) \leq 2\int_{\mathbb{R}^n}\eta_3 dx. \tag{4.21}$$

Then it follows from (4.18)-(4.21), we find

$$\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|\Delta u(\tau,\tau-t,\omega,u_0)\|^2+\frac{1}{2}(\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2-p)^2\\
\leq& c+c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|^2ds+c\int_{-\infty}^{0} e^{\sigma s}(|\omega(s)|^2+|\omega(s)|^4+|\omega(s)|^{p+1})ds.
\end{aligned} \tag{4.22}$$

Thus the proof is completed. □

The following lemma will be used to show the uniform estimates of solutions as well as to establish pullback asymptotic compactness.

Lemma 4.2 *Under Assumptions I,II, for every $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \geq T$ the solution of problem (3.10)-(3.11) satisfies*

$$\|A^{\frac{1}{4}}Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|_E^2 \leq R_2(\tau,\omega),$$

and $R_2(\tau,\omega)$ is given by

$$\begin{aligned}
R_2(\tau,\omega)=&ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2+\|A^{\frac{1}{4}}u_0\|^2+\|A^{\frac{3}{4}}u_0\|^2+\|A^{\frac{1}{4}}u_0\|^2\|A^{\frac{1}{2}}u_0\|^2-p\|A^{\frac{1}{2}}u_0\|^2)\\
&+c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|_1^2ds+c\int_{-\infty}^{0} e^{\sigma s}(1+|\omega(s)|^2)ds+R_3(\tau,\omega).
\end{aligned} \tag{4.23}$$

Proof. Taking the inner product of the second equation of (3.10) with $A^{\frac{1}{2}}v$ in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}v\|^2-\delta\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2)(u,A^{\frac{1}{2}}v)+(Au,A^{\frac{1}{2}}v)\\
&-(p-\|\nabla u\|^2)(A^{\frac{1}{2}}u,A^{\frac{1}{2}}v)+(f(x,u),A^{\frac{1}{2}}v)\\
=&(g(x,t),A^{\frac{1}{2}}v)-(h(v+\phi\omega(t)-\delta u),A^{\frac{1}{2}}v)+\delta(\phi,A^{\frac{1}{2}}v)\omega(t).
\end{aligned} \tag{4.24}$$

Similar to the proof of Lemma 4.1, we have the following estimates:

$$\begin{aligned}
&-(h(v+\phi\omega(t)-\delta u),A^{\frac{1}{2}}v)\\
=&-(h(v+\phi\omega(t)-\delta u)-h(0),A^{\frac{1}{2}}v)\\
=&-(h'(\vartheta)(v+\phi\omega(t)-\delta u),A^{\frac{1}{2}}v)
\end{aligned}$$

$$
\begin{aligned}
&\le -\beta_1\|A^{\frac14}v\|^2-(h'(\vartheta)(\phi\omega(t)-\delta u),A^{\frac12}v)\\
&\le -\beta_1\|A^{\frac14}v\|^2+\beta_2|\omega(t)|\|A^{\frac14}\phi\|\|A^{\frac14}v\|+h'(\vartheta)\delta(u,A^{\frac12}v)\\
&\le -\beta_1\|A^{\frac14}v\|^2+\frac{\beta_1-\delta}{6}\|A^{\frac14}v\|^2+c|\omega(t)|^2\|A^{\frac14}\phi\|^2+h'(\vartheta)\delta(u,A^{\frac12}v),
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
&h'(\vartheta)\delta(u,A^{\frac12}v)\\
=&h'(\vartheta)\delta(u,A^{\frac12}u_t-\omega(t)A^{\frac12}\phi+\delta A^{\frac12}u)\\
\le&\beta_2\delta\cdot\frac12\frac{d}{dt}\|A^{\frac14}u\|^2+\beta_2\delta^2\|A^{\frac14}u\|^2+\beta_2\delta|\omega(t)|\|A^{\frac14}\phi\|\|A^{\frac14}u\|\\
\le&\beta_2\delta\cdot\frac12\frac{d}{dt}\|A^{\frac14}u\|^2+\beta_2\delta^2\|A^{\frac14}u\|^2+\frac16\delta(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u\|^2+c|\omega(t)|^2\|A^{\frac14}\phi\|^2,
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
&(\lambda+\delta^2)(u,A^{\frac12}v)\\
=&(\lambda+\delta^2)(u,A^{\frac12}u_t-\omega(t)A^{\frac12}\phi+\delta A^{\frac12}u)\\
\ge&\frac12(\lambda+\delta^2)\frac{d}{dt}\|A^{\frac14}u\|^2+\delta(\lambda+\delta^2)\|A^{\frac14}u\|^2-(\lambda+\delta^2)|\omega(t)|\|A^{\frac14}\phi\|\|A^{\frac14}u\|\\
\ge&\frac12(\lambda+\delta^2)\frac{d}{dt}\|A^{\frac14}u\|^2+\delta(\lambda+\delta^2)\|A^{\frac14}u\|^2-\frac16\delta(\lambda+\delta^2-\beta_2\delta)\|A^{\frac14}u\|^2\\
&-c|\omega(t)|^2\|A^{\frac14}\phi\|^2,
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
&(Au,A^{\frac12}v)=(Au,A^{\frac12}u_t-\omega(t)A^{\frac12}\phi+\delta A^{\frac12}u)\\
&\ge\frac12\frac{d}{dt}\|A^{\frac34}u\|^2+\delta\|A^{\frac34}u\|^2-|\omega(t)|\|A^{\frac34}\phi\|\|A^{\frac34}u\|\\
&\ge\frac12\frac{d}{dt}\|A^{\frac34}u\|^2+\frac{\delta}{2}\|A^{\frac34}u\|^2-c|\omega(t)|^2\|A^{\frac34}\phi\|^2,
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
&-(p-\|\nabla u\|^2)(A^{\frac12}u,A^{\frac12}v)\\
=&-(p-\|A^{\frac14}u\|^2)\left(A^{\frac12}u,A^{\frac12}u_t-\omega(t)A^{\frac12}\phi+\delta A^{\frac12}u\right)\\
=&\frac12\frac{d}{dt}\left(\|A^{\frac14}u\|^2\|A^{\frac12}u\|^2-p\|A^{\frac12}u\|^2\right)+\delta(\|A^{\frac14}u\|^2\|A^{\frac12}u\|^2-p\|A^{\frac12}u\|^2)\\
&+\omega(t)(p-\|A^{\frac14}u\|^2)\left(A^{\frac12}u,A^{\frac12}\phi\right)-\|A^{\frac12}u\|^2(A^{\frac12}u,u_t)\\
\ge&\frac12\frac{d}{dt}\left(\|A^{\frac14}u\|^2\|A^{\frac12}u\|^2-p\|A^{\frac12}u\|^2\right)+\frac{\delta}{2}(\|A^{\frac14}u\|^2\|A^{\frac12}u\|^2-p\|A^{\frac12}u\|^2)\\
&-\frac{\delta}{2}|p|\|A^{\frac12}u\|^2-|\omega(t)|(|p|+\|A^{\frac14}u\|^2)\|A^{\frac12}u\|\|A^{\frac12}\phi\|-\|A^{\frac12}u\|^2(A^{\frac12}u,u_t)\\
\ge&\frac12\frac{d}{dt}\left(\|A^{\frac14}u\|^2\|A^{\frac12}u\|^2-p\|A^{\frac12}u\|^2\right)+\frac{\delta}{2}\left(\|A^{\frac14}u\|^2\|A^{\frac12}u\|^2-p\|A^{\frac12}u\|^2\right)\\
&-\left(\frac{\delta}{2}|p|+\frac12(|p|+\|A^{\frac14}u\|^2)^2\right)\|A^{\frac12}u\|^2-\frac12\|A^{\frac12}\phi\|^2|\omega(t)|^2-\|A^{\frac12}u\|^2(A^{\frac12}u,u_t),
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
&\delta(\phi\omega(t),A^{\frac12}v)\\
\le&\delta|\omega(t)|\|A^{\frac14}\phi\|\|A^{\frac14}v\|\\
\le&c\|A^{\frac14}\phi\|^2|\omega(t)|^2+\frac{\beta_1-\delta}{6}\|A^{\frac14}v\|^2,
\end{aligned}
\tag{4.30}
$$

$$
(g,A^{\frac12}v)\le\|g\|_1\|A^{\frac14}v\|\le c\|g\|_1^2+\frac{\beta_1-\delta}{6}\|A^{\frac14}v\|^2.
\tag{4.31}
$$

For the last term on the left-hand side of (4.24), thanks to (3.7), we have

$$\begin{aligned}
&-(f(x,u),A^{\frac{1}{2}}v)\\
\le&|(f(x,u),A^{\frac{1}{2}}v)|\\
=&|\int_{\mathbb{R}^n}\frac{\partial}{\partial x}f(x,u)\cdot A^{\frac{1}{4}}vdx+\int_{\mathbb{R}^n}\frac{\partial}{\partial u}f(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}vdx|\\
\le&\int_{\mathbb{R}^n}|\frac{\partial}{\partial x}f(x,u)|\cdot|A^{\frac{1}{4}}v|dx+\int_{\mathbb{R}^n}|\frac{\partial}{\partial u}f(x,u)|\cdot|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}v|dx\\
\le&\int_{\mathbb{R}^n}|\eta_4|\cdot|A^{\frac{1}{4}}v|dx+\beta\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}v|dx\\
\le&\|\eta_4\|\|A^{\frac{1}{4}}v\|+\beta\|A^{\frac{1}{4}}u\|\|A^{\frac{1}{4}}v\|\\
\le&c+(\delta+\frac{3\beta^2}{2\delta(\lambda+\delta^2-\beta_2\delta)})\|A^{\frac{1}{4}}v\|^2+\frac{1}{6}\delta(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2.
\end{aligned}\tag{4.32}$$

Plugging (4.25)-(4.32) into (4.24) and together with (3.14) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+\|A^{\frac{1}{4}}u\|^2\|A^{\frac{1}{2}}u\|^2-p\|A^{\frac{1}{2}}u\|^2)\\
&+\frac{\delta}{2}(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+(\|A^{\frac{1}{4}}u\|^2\|A^{\frac{1}{2}}u\|^2-p\|A^{\frac{1}{2}}u\|^2))\\
\le&(\frac{\delta}{2}|p|+\frac{1}{2}(|p|+\|A^{\frac{1}{4}}u\|^2)^2)\|A^{\frac{1}{2}}u\|^2+\|A^{\frac{1}{2}}u\|^2(A^{\frac{1}{2}}u,u_t)\\
&+c(1+|\omega(t)|^2)+c\|g\|_1^2.
\end{aligned}\tag{4.33}$$

then according to Lemma 4.1, we have

$$\begin{aligned}
&\frac{d}{dt}(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+\|A^{\frac{1}{4}}u\|^2\|A^{\frac{1}{2}}u\|^2-p\|A^{\frac{1}{2}}u\|^2)\\
&+\sigma(\|A^{\frac{1}{4}}v\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u\|^2+\|A^{\frac{3}{4}}u\|^2+(\|A^{\frac{1}{4}}u\|^2\|A^{\frac{1}{2}}u\|^2-p\|A^{\frac{1}{2}}u\|^2)\\
\le&R_3(\tau,\omega)+c(1+|\omega(t)|^2)+c\|g\|_1^2,
\end{aligned}\tag{4.34}$$

where $R_3(\tau,\omega)=(\delta|p|+(|p|+R_1(\tau,\omega))^2)R_1(\tau,\omega)+2R_1^2(\tau,\omega)$.
Multiplying (4.34) by $e^{\sigma t}$ and then integrating over $(\tau-t,\tau)$, we have

$$\begin{aligned}
&e^{\sigma\tau}(\|A^{\frac{1}{4}}v(\tau,\tau-t,\omega,v_0)\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u(\tau,\tau-t,\omega,u_0)\|^2\\
&+\|A^{\frac{3}{4}}u(\tau,\tau-t,\omega,u_0)\|^2+\|A^{\frac{1}{4}}u(\tau,\tau-t,\omega,u_0)\|^2\\
&\cdot\|A^{\frac{1}{2}}u(\tau,\tau-t,\omega,u_0)\|^2-p\|A^{\frac{1}{2}}u(\tau,\tau-t,\omega,u_0)\|^2)\\
\le&e^{\sigma(\tau-t)}(\|A^{\frac{1}{4}}v_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u_0\|^2+\|A^{\frac{3}{4}}u_0\|^2\\
&+\|A^{\frac{1}{4}}u_0\|^2\|A^{\frac{1}{2}}u_0\|^2-p\|A^{\frac{1}{2}}u_0\|^2)+c\int_{\tau-t}^{\tau}e^{\sigma s}\|g(x,s)\|_1^2ds\\
&+c\int_{\tau-t}^{\tau}e^{\sigma s}(1+|\omega(s)|^2)ds+R_3(\tau,\omega)\int_{\tau-t}^{\tau}e^{\sigma s}ds.
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above we obtain, for every $t \in \mathbb{R}^+,\ \tau \in \mathbb{R},$ and $\omega \in \Omega$,

$$\begin{aligned}
&\|A^{\frac{1}{4}}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2 + (\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|A^{\frac{3}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 + \|A^{\frac{1}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&\cdot\|A^{\frac{1}{2}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 - p\|A^{\frac{1}{2}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
\le& e^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + (\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2\\
&+\|A^{\frac{1}{4}}u_0\|^2\|A^{\frac{1}{2}}u_0\|^2 - p\|A^{\frac{1}{2}}u_0\|^2) + c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|_1^2 ds\\
&+c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}(1+|\theta_{-\tau}\omega(s)|^2)ds + \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}R_3(s,\theta_{-\tau}\omega)ds\\
\le& ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2\|A^{\frac{1}{2}}u_0\|^2 - p\|A^{\frac{1}{2}}u_0\|^2)\\
&+c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|_1^2 ds + c\int_{-\infty}^{0} e^{\sigma s}(1+|\omega(s)|^2)ds + R_3(\tau,\omega)\int_{-\infty}^{0} e^{\sigma s}ds.
\end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned}
&\|A^{\frac{1}{4}}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\|^2 + (\lambda+\delta^2-\beta_2\delta)\|A^{\frac{1}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|A^{\frac{3}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
\le& ce^{-\sigma t}(\|A^{\frac{1}{4}}v_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2 + \|A^{\frac{3}{4}}u_0\|^2 + \|A^{\frac{1}{4}}u_0\|^2\|A^{\frac{1}{2}}u_0\|^2 - p\|A^{\frac{1}{2}}u_0\|^2)\\
&+c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\|g(x,s)\|_1^2 ds + c\int_{-\infty}^{0} e^{\sigma s}(1+|\omega(s)|^2)ds + R_3(\tau,\omega).
\end{aligned}$$

Thus the proof is completed. □

Next we conduct uniform estimates on the tail parts of the solutions for large space variables when time is sufficiently large in order to prove the pullback asymptotic compactness of the cocycle associated with Eqs.(3.10)-(3.11) on the unbounded domain $\mathbb{R}^n$.

Lemma 4.3 *Under Assumptions I and II, for every $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\eta) > 0, K = K(\tau,\omega,\eta) \ge 1$ such that for all $t \ge T,\ k \ge K$, the solution of problem (3.10)-(3.11) satisfies*

$$\|Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|^2_{E(\mathbb{R}^n\setminus\mathbb{B}_k)} \le \eta, \tag{4.35}$$

where for $k \ge 1$, $\mathbb{B}_k = \{x \in \mathbb{R}^n : |x| \le k\}$ and $\mathbb{R}^n \setminus \mathbb{B}_k$ is the complement of $\mathbb{B}_k$.

Proof. Choose a smooth function ρ, such that $0 \le \rho \le 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & \text{if } 0 \le |s| \le 1,\\ 1, & \text{if } |s| \ge 2,\end{cases} \tag{4.36}$$

and there exist constants $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ such that $|\rho'(s)| \le \mu_1,\ |\rho''(s)| \le \mu_2,\ |\rho'''(s)| \le \mu_3,\ |\rho''''(s)| \le \mu_4$ for $s \in \mathbb{R}$. Taking the inner product of the second equation of (3.10) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx - \delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx + (\lambda+\delta^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx+(p-\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Delta uvdx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx\\
=&\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\phi\omega(t)vdx-\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v+\phi\omega(t)-\delta u)vdx\\
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x,t)vdx. \qquad (4.37)
\end{aligned}
$$

Similar to (4.5), we have

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v+\phi\omega(t)-\delta u)vdx\\
=&-\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(h(v+\phi\omega(t)-\delta u)-h(0))vdx\\
\le&-\beta_1\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+h'(\vartheta)\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx+\beta_2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\phi||\omega(t)||v|dx. \qquad (4.38)
\end{aligned}
$$

Taking (4.38) into (4.37), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx-(\delta-\beta_1)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx\\
&+(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx+(p-\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Delta uvdx\\
&+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})f(x,u)vdx\\
\le&(\delta+\beta_2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\phi||\omega(t)||v|dx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x,t)vdx\\
\le&\frac{\beta_1-\delta}{3}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\phi|^2|\omega(t)|^2dx+\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g(x,t)vdx. \qquad (4.39)
\end{aligned}
$$

For the fourth term on the left-hand side of (4.39), we have

$$
\begin{aligned}
&(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})uvdx\\
=&(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})u(\frac{du}{dt}+\delta u-\phi\omega(t))dx\\
=&(\lambda+\delta^2-h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(\frac{1}{2}\frac{d}{dt}u^2+\delta u^2-\phi\omega(t)u)dx\\
\ge&(\lambda+\delta^2-\beta_2\delta)(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx)\\
&-(\lambda+\delta^2-\beta_1\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\phi||\omega(t)||u|dx\\
\ge&(\lambda+\delta^2-\beta_2\delta)(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx+\delta\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx)\\
&-\frac{\delta}{2}(\lambda+\delta^2-\beta_2\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx-c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\phi|^2|\omega(t)|^2dx. \qquad (4.40)
\end{aligned}
$$

For the third term on the left-hand side of (4.39), we have

$$
\int_{\mathbb{R}^n}(Au)\rho(\frac{|x|^2}{k^2})vdx
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} (Au)\rho(\frac{|x|^2}{k^2})(\frac{du}{dt} + \delta u - \phi\omega(t))dx \\
&= \int_{\mathbb{R}^n} (\Delta^2 u)\rho(\frac{|x|^2}{k^2})(\frac{du}{dt} + \delta u - \phi\omega(t))dx \\
&= \int_{\mathbb{R}^n} (\Delta u)\Delta(\rho(\frac{|x|^2}{k^2})(\frac{du}{dt} + \delta u - \phi\omega(t)))dx \\
&= \int_{\mathbb{R}^n} (\Delta u)((\frac{2}{k^2}\rho'(\frac{|x|^2}{k^2}) + \frac{4x^2}{k^4}\rho''(\frac{|x|^2}{k^2}))(\frac{du}{dt} + \delta u - \phi\omega(t)) \\
&\quad + 2\cdot\frac{2|x|}{k^2}\rho'(\frac{|x|^2}{k^2})\nabla(\frac{du}{dt} + \delta u - \phi\omega(t)) + \rho(\frac{|x|^2}{k^2})\Delta(\frac{du}{dt} + \delta u - \phi\omega(t)))dx \\
&\geq -\int_{k<x<\sqrt{2}k} (\frac{2\mu_1}{k^2} + \frac{4\mu_2 x^2}{k^4})|(\Delta u)v|dx - \int_{k<x<\sqrt{2}k} \frac{4\mu_1 x}{k^2}|(\Delta u)(\nabla v)|dx \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx + \delta\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u||\Delta\phi||\omega(t)|dx \\
&\geq -\int_{\mathbb{R}^n} (\frac{2\mu_1 + 8\mu_2}{k^2})|(\Delta u)v|dx - \int_{\mathbb{R}^n} \frac{4\sqrt{2}\mu_1}{k}|(\Delta u)(\nabla v)|dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx \\
&\quad + \delta\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u||\Delta\phi||\omega(t)|dx \\
&\geq -\frac{\mu_1 + 4\mu_2}{k^2}(\|\Delta u\|^2 + \|v\|^2) - \frac{4\sqrt{2}\mu_1}{k}\|\Delta u\|\|\nabla v\| + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx \\
&\quad + \delta\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u||\Delta\phi||\omega(t)|dx \\
&\geq -\frac{\mu_1 + 4\mu_2}{k^2}(\|\Delta u\|^2 + \|v\|^2) - \frac{2\sqrt{2}\mu_1}{k}(\|\Delta u\|^2 + \|\nabla v\|^2) + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx \\
&\quad + \delta\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - \frac{\delta}{4}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta u|^2dx - c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\Delta\phi|^2|\omega(t)|^2dx. \qquad (4.41)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x,u)vdx \\
&= \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x,u)(\frac{du}{dt} + \delta u - \phi\omega(t))dx \\
&= \frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})F(x,u)dx + \delta\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x,u)udx - \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x,u)\phi\omega(t)dx. \qquad (4.42)
\end{aligned}
$$

Similar to (4.12) and (4.13) in Lemma 4.1, we have

$$
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x,u)udx \geq c_2\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})F(x,u)dx + \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\eta_2dx, \qquad (4.43)
$$

$$
\begin{aligned}
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x,u)\phi\omega(t)dx &\leq \frac{1}{2}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\eta_1|^2dx + \frac{1}{2}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\phi|^2|\omega(t)|^2dx \\
&\quad + \frac{\delta c_2}{2}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(F(x,u) + \eta_3)dx + c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\phi|^{p+1}|\omega(t)|^{p+1}dx. \qquad (4.44)
\end{aligned}
$$

$$
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})g(x,t)vdx \leq c\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|g(x,t)|^2dx + \frac{\beta_1 - \delta}{6}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2dx. \qquad (4.45)
$$

For the fifth term on the left-hand side of (4.39), by Lemma 4.1 we have

$$
\begin{aligned}
&(p-\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\Delta uv dx\\
=&-(p-\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho'(\frac{|x|^2}{k^2})v\nabla u\frac{2x}{k^2}dx-(p-\|\nabla u\|^2)\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx\\
&-\delta(p-\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx+(p-\|\nabla u\|^2)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\nabla u\nabla\phi\omega(t)dx\\
\geq&-(\|\nabla u\|^2-p)\int_{k<x<\sqrt{2}k}|\rho'(\frac{|x|^2}{k^2})||v||\nabla u|\frac{2|x|}{k^2}dx+\frac{\|\nabla u\|^2-p}{2}\cdot\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx\\
&+\delta(\|\nabla u\|^2-p)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx-(\|\nabla u\|^2-p)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\nabla u\nabla\phi\omega(t)dx\\
\geq&\frac{-c(\|\nabla u\|^2-p)}{k}(\|\nabla u\|^2+\|v\|^2)+\frac{\|\nabla u\|^2-p}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx\\
&+\delta(\|\nabla u\|^2-p)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx-(\|\nabla u\|^2-p)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\nabla u\nabla\phi\omega(t)dx\\
\geq&\frac{-c(\|\nabla u\|^2-p)}{k}(\|\nabla u\|^2+\|v\|^2)+\frac{\|\nabla u\|^2-p}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx\\
&+\frac{\delta}{2}(\|\nabla u\|^2-p)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx-c(\|\nabla u\|^2-p)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla\phi|^2|\omega(t)|^2dx\\
\geq&\frac{-c(\|\nabla u\|^2-p)}{k}(\|\nabla u\|^2+\|v\|^2)-\frac{p}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx\\
&-\frac{\delta}{2}p\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx-c(\|\nabla u\|^2-p)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla\phi|^2|\omega(t)|^2dx.
\end{aligned}
\tag{4.46}
$$

By (4.39)-(4.46), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v|^2+(\lambda+\delta^2-\beta_2\delta)|u|^2+|\Delta u|^2-p|\nabla u|^2+2F(x,u))dx\\
&+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+\frac{\delta}{2}(\lambda+\delta^2-\beta_2\delta)\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|u|^2dx+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\Delta u|^2dx\\
&-\frac{\delta}{2}p\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla u|^2dx+\frac{\delta c_2}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx\\
\leq&\frac{2\delta-\beta_1}{2}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\eta_1|^2+|\eta_2|+|\eta_3|+|g|^2+|\omega(t)|^{p+1}|\phi|^{p+1})dx\\
&+c|\omega(t)|^2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(1+|\phi|^2+|\Delta\phi|^2)dx+c(\|\nabla u\|^2-p)|\omega(t)|^2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla\phi|^2dx\\
&+\frac{c(\|\nabla u\|^2-p)}{k}(\|\nabla u\|^2+\|v\|^2)+\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)+\frac{2\sqrt{2}\mu_1}{k}(\|\Delta u\|^2+\|\nabla v\|^2).
\end{aligned}
\tag{4.47}
$$

Since that $\eta_1(x)\in L^2(\mathbb{R}^n)$, $\eta_2(x)\in L^1(\mathbb{R}^n)$, $\eta_3(x)\in L^1(\mathbb{R}^n)$ and the embedding $H^2(\mathbb{R}^n)\hookrightarrow L^{p+1}(\mathbb{R}^n)$, we obtain that there exists $K_1=K_1(\tau,\eta)\geq 1$ such that for all $k\geq K_1$,

$$
c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|\eta_1|^2+|\eta_2|+|\eta_3|+|\omega(t)|^{p+1}|\phi|^{p+1})dx
$$

$$
\begin{aligned}
&+c|\omega(t)|^2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(1+|\phi|^2+|\Delta\phi|^2)dx+c(\|\nabla u\|^2-p)|\omega(t)|^2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|\nabla\phi|^2dx\\
=&c\int_{|x|\geq k}\rho(\frac{|x|^2}{k^2})(|\eta_1|^2+|\eta_2|+|\eta_3|+|\omega(t)|^{p+1}|\phi|^{p+1})dx\\
&+c|\omega(t)|^2\int_{|x|\geq k}\rho(\frac{|x|^2}{k^2})(1+|\phi|^2+|\Delta\phi|^2)dx+c(\|\nabla u\|^2-p)|\omega(t)|^2\int_{|x|\geq k}\rho(\frac{|x|^2}{k^2})|\nabla\phi|^2dx\\
\leq&c\int_{|x|\geq k}(|\eta_1|^2+|\eta_2|+|\eta_3|+|\omega(t)|^{p+1}|\phi|^{p+1})dx\\
&+c|\omega(t)|^2\int_{|x|\geq k}(1+|\phi|^2+|\Delta\phi|^2)dx+c(\|\nabla u\|^2-p)|\omega(t)|^2\int_{|x|\geq k}|\nabla\phi|^2dx\\
\leq&c\eta(1+|\omega(t)|^2+|\omega(t)|^{p+1}),
\end{aligned}
\tag{4.48}
$$

together with

$$
c\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})g^2(x,t)dx\leq c\int_{|x|\geq k}g^2(x,t)dx,
\tag{4.49}
$$

we have that for all $k\geq K_1$,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v|^2+(\lambda+\delta^2-\beta_2\delta)|u|^2+|\Delta u|^2-p|\nabla u|^2+2F(x,u))dx\\
&+\sigma\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v|^2+(\lambda+\delta^2-\beta_2\delta)|u|^2+|\Delta u|^2-p|\nabla u|^2+2F(x,u))dx\\
\leq&\frac{2\mu_1+8\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)+\frac{4\sqrt{2}\mu_1}{k}(\|\Delta u\|^2+\|\nabla v\|^2)+\frac{c(\|\nabla u\|^2-p)}{k}(\|\nabla u\|^2\\
&+\|v\|^2)+c\eta(1+|\omega(t)|^2+|\omega(t)|^{p+1})+c\int_{|x|\geq k}g^2(x,t)dx.
\end{aligned}
\tag{4.50}
$$

Multiplying (4.50) by $e^{\sigma t}$ and then integrating over $(\tau-t,\tau)$, we find

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\omega,u_0)|^2\\
&+|\Delta u(\tau,\tau-t,\omega,u_0)|^2-p|\nabla u(\tau,\tau-t,\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\omega,u_0)))dx\\
\leq&e^{-\sigma t}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v_0|^2+(\lambda+\delta^2-\beta_2\delta)|u_0|^2+|\Delta u_0|^2-p|\nabla u_0|^2+2F(x,u_0))dx\\
&+\frac{2\mu_1+8\mu_2}{k^2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\Delta u(s,\tau-t,\omega,u_0)|^2+|v(s,\tau-t,\omega,v_0)|^2)ds\\
&+\frac{4\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\Delta u(s,\tau-t,\omega,u_0)|^2+|\nabla v(s,\tau-t,\omega,v_0)|^2)ds\\
&+\frac{c}{k}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(\|\nabla u(s,\tau-t,\omega,u_0)\|^2-p)(\|\nabla u(s,\tau-t,\omega,u_0)\|^2+\|v(s,\tau-t,\omega,v_0)\|^2)ds\\
&+c\frac{\eta}{\sigma}+c\eta\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\omega(s)|^2+|\omega(s)|^{p+1})ds+c\int_{\tau-t}^{\tau}\int_{|x|\geq k}e^{\sigma(s-\tau)}g^2(x,s)dxds.
\end{aligned}
$$

Replacing ω by $\theta_{-\tau}\omega$, it then follows from above that

$$
\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2
$$

$$
\begin{aligned}
&+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2-p|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))dx\\
\leq& c\eta+e^{-\sigma t}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v_0|^2+(\lambda+\delta^2-\beta_2\delta)|u_0|^2+|\Delta u_0|^2-p|\nabla u_0|^2+2F(x,u_0))dx\\
&+\frac{2\mu_1+8\mu_2}{k^2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|v(s,\tau-t,\theta_{-\tau}\omega,v_0)|^2)ds\\
&+\frac{4\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)|^2)ds\\
&+\frac{c}{k}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2-p)(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds+c\eta\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\theta_{-\tau}\omega(s)|^2+|\theta_{-\tau}\omega(s)|^{p+1})ds\\
&+c\int_{\tau-t}^{\tau}\int_{|x|\geq k}e^{\sigma(s-\tau)}g^2(x,s)dxds\\
\leq& c\eta+e^{-\sigma t}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v_0|^2+(\lambda+\delta^2-\beta_2\delta)|u_0|^2-p|\nabla u_0|^2+|\Delta u_0|^2+2F(x,u_0))dx\\
&+\frac{2\mu_1+8\mu_2}{k^2}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|v(s,\tau-t,\theta_{-\tau}\omega,v_0)|^2)ds\\
&+\frac{4\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)|^2)ds\\
&+\frac{c}{k}\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2-p)(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2\\
&+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)ds+c\eta\int_{-\infty}^{0}e^{\sigma s}(|\omega(s)|^2+|\omega(s)|^{p+1})ds\\
&+c\int_{-\infty}^{\tau}\int_{|x|\geq k}e^{\sigma(s-\tau)}g^2(x,s)dxds. \qquad (4.51)
\end{aligned}
$$

By (3.16), we see that there exists $K_2=K_2(\tau,\eta)\geq K_1$ such that for all $k\geq K_2$,

$$
c\int_{-\infty}^{\tau}\int_{|x|\geq k}e^{\sigma(s-\tau)}g^2(x,s)dxds\leq\eta. \qquad (4.52)
$$

Following from (4.51)-(4.52), Lemma 4.1 and Lemma 4.2 that there exists $T_1=T_1(\tau,\omega,D,\eta)>0$ such that for all $t\geq T_1$, $k\geq K_2$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\\
&+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2-p|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))dx\\
\leq& c\eta(1+\int_{-\infty}^{0}e^{\sigma s}(|\omega(s)|^2+|\omega(s)|^{p+1})ds+\int_{-\infty}^{\tau}\int_{|x|\geq k}e^{\sigma(s-\tau)}g^2(x,s)dxds), \qquad (4.53)
\end{aligned}
$$

where $(u_0,v_0)^\top\in D(\tau-t,\theta_{-t}\omega)$.

Note that (3.6) holds, then we find

$$
-2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})F(x,u)dx\leq 2\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\eta_3dx\leq 2\int_{|x|\geq k}\rho(\frac{|x|^2}{k^2})\eta_3dx,
$$

which along with $\eta_3 \in L^1(\mathbb{R}^n)$, we obtain that there exists $K_3 = K_3(\tau, \eta) \geq K_2$ such that for all $k \geq K_3$,

$$-2\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})F(x,u)dx \leq \eta. \tag{4.54}$$

Then from (4.53)-(4.54), we get that there exists $T_2 = T_2(\tau, \omega, D, \eta) > T_1$ such that for all $t \geq T_2$ and $k \geq K_3$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2 + (\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2 \\
&\quad + |\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)dx \\
\leq& c\eta(1+\int_{-\infty}^0 e^{\sigma s}(|\omega(s)|^2+|\omega(s)|^{p+1})ds + \int_{-\infty}^\tau \int_{|x|\geq k} e^{\sigma(s-\tau)}g^2(x,s)dxds),
\end{aligned} \tag{4.55}$$

which completes the proof. □

We now derive uniform estimates of solutions in bounded domains. These estimates will be used to establish pullback asymptotic compactness. Let $\widehat{\rho} = 1-\rho$ with ρ given by (4.36). Fix $k \geq 1$, and set

$$\begin{cases} \widehat{u}(t,\tau,\omega,\widehat{u_0}) = \widehat{\rho}(\frac{|x|^2}{k^2})u(t,\tau,\omega,u_0), \\ \widehat{v}(t,\tau,\omega,\widehat{v_0}) = \widehat{\rho}(\frac{|x|^2}{k^2})v(t,\tau,\omega,v_0), \end{cases} \tag{4.56}$$

By (3.10)-(3.11) we find that $\widehat{u}$ and $\widehat{v}$ satisfy the following system in $\mathbb{B}_{2k} = \{x \in \mathbb{R}^n : |x| < 2k\}$:

$$\frac{d\widehat{u}}{dt} = \widehat{v} + \widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t) - \delta\widehat{u}, \tag{4.57}$$

$$\begin{aligned}
&\frac{d\widehat{v}}{dt} - \delta\widehat{v} + (\delta^2+\lambda+A)\widehat{u} + (p-\|\nabla u\|^2)\Delta\widehat{u} + \widehat{\rho}(\frac{|x|^2}{k^2})f(x,u) \\
=&\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t) - \widehat{\rho}(\frac{|x|^2}{k^2})h(v+\phi\omega(t)-\delta u) + (1+\delta)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t) \\
&+ 4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\nabla u + 6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\Delta u + 4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\Delta\nabla u + u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}) \\
&+ (p-\|\nabla u\|^2)u\Delta\widehat{\rho}(\frac{|x|^2}{k^2}) + 2(p-\|\nabla u\|^2)\nabla u\nabla\widehat{\rho}(\frac{|x|^2}{k^2}),
\end{aligned} \tag{4.58}$$

with boundary conditions

$$\widehat{u} = \widehat{v} = 0 \quad \text{for} \quad |x| = 2k. \tag{4.59}$$

Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(\mathbb{B}_{2k})$ such that $Ae_n = \lambda_n e_n$ with zero boundary condition in $\mathbb{B}_{2k}$. Given n, let $X_n = \text{span}\{e_1, \cdots, e_n\}$ and $P_n : L^2(\mathbb{B}_{2k}) \to X_n$ be the projection operator.

Lemma 4.4 *Under Assumptions I and II, for every $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D,\eta) > 0, K = K(\tau,\omega,\eta) \geq 1$ and $N = N(\tau,\omega,\eta) \geq 1$ such that for all $t \geq T$, $k \geq K$ and $n \geq N$, the solution of problem (4.57)-(4.59) satisfies*

$$\|(I-P_n)\widehat{Y}(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-\tau}\omega))\|^2_{E(\mathbb{B}_{2k})} \leq \eta.$$

Proof. Let $\widehat{u}_{n,1} = P_n\widehat{u}$, $\widehat{u}_{n,2} = (I-P_n)\widehat{u}$, $\widehat{v}_{n,1} = P_n\widehat{v}$, $\widehat{v}_{n,2} = (I-P_n)\widehat{v}$. Applying $I-P_n$ to (4.57), we obtain

$$\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - (I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t). \tag{4.60}$$

Then applying $I-P_n$ to (4.58) and taking the inner product with $\widehat{v}_{n,2}$ in $L^2(\mathbb{B}_{2k})$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2-\delta\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2+A)(\widehat{u}_{n,2},\widehat{v}_{n,2})+(p-\|\nabla u\|^2)(\Delta\widehat{u}_{n,2},\widehat{v}_{n,2})\\
&+(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{v}_{n,2})\\
=&((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t),\widehat{v}_{n,2})+\delta(\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t),\widehat{v}_{n,2})\\
&-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v+\phi\omega(t)-\delta u),\widehat{v}_{n,2})\\
&+(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\nabla u+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\Delta u+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\Delta\nabla u+u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})\\
&+((p-\|\nabla u\|^2)u\Delta\widehat{\rho}(\frac{|x|^2}{k^2})+2(p-\|\nabla u\|^2)\nabla u\nabla\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2}). \qquad (4.61)
\end{aligned}$$

Substituting $\widehat{v}_{n,2}$ in (4.60) into the third term on the left-hand side of (4.61), we have

$$\begin{aligned}
(\lambda+\delta^2)(\widehat{u}_{n,2},\widehat{v}_{n,2})&=(\widehat{u}_{n,2},\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t))\\
&\geq\frac{1}{2}(\lambda+\delta^2)\frac{d}{dt}\|\widehat{u}_{n,2}\|^2+\delta(\lambda+\delta^2)\|\widehat{u}_{n,2}\|^2\\
&-\frac{1}{4}\delta(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2-c\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2, \qquad (4.62)
\end{aligned}$$

and then

$$\begin{aligned}
(A\widehat{u}_{n,2},\widehat{v}_{n,2})&=(\Delta\widehat{u}_{n,2},\Delta(\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t))\\
&\geq\frac{1}{2}\frac{d}{dt}\|\Delta\widehat{u}_{n,2}\|^2+\frac{3\delta}{4}\|\Delta\widehat{u}_{n,2}\|^2-c\|(I-P_n)\Delta(\widehat{\rho}(\frac{|x|^2}{k^2})\phi)\|^2|\omega(t)|^2. \qquad (4.63)
\end{aligned}$$

For the fourth term on the left-hand side of (4.61), we have

$$\begin{aligned}
&(p-\|\nabla u\|^2)(\Delta\widehat{u}_{n,2},\widehat{v}_{n,2})\\
=&(\|\nabla u\|^2-p)(\nabla\widehat{u}_{n,2},\nabla(\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t)))\\
\geq&\frac{\|\nabla u\|^2-p}{2}\frac{d}{dt}\|\nabla\widehat{u}_{n,2}\|^2+\frac{(\|\nabla u\|^2-p)\delta}{2}\|\nabla\widehat{u}_{n,2}\|^2\\
&-c(\|\nabla u\|^2-p)\|(I-P_n)\nabla(\widehat{\rho}(\frac{|x|^2}{k^2})\phi)\|^2|\omega(t)|^2\\
\geq&-\frac{p}{2}\frac{d}{dt}\|\nabla\widehat{u}_{n,2}\|^2-\frac{p\delta}{2}\|\nabla\widehat{u}_{n,2}\|^2-c(\|\nabla u\|^2-p)\|(I-P_n)\nabla(\widehat{\rho}(\frac{|x|^2}{k^2})\phi)\|^2|\omega(t)|^2. \qquad (4.64)
\end{aligned}$$

For the fifth term on the left-hand side of (4.61), we have

$$\begin{aligned}
&(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{v}_{n,2})\\
=&(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t))\\
=&\frac{d}{dt}(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})-(\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t,\widehat{u}_{n,2})
\end{aligned}$$

$$+\delta(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})-(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t)). \tag{4.65}$$

For the third term on the right-hand side of (4.61), we have

$$\begin{aligned}
&-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v+\phi\omega(t)-\delta u),\widehat{v}_{n,2})\\
=&-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})(h(v+\phi\omega(t)-\delta u)-h(0),\widehat{v}_{n,2})\\
\leq&-\beta_1\|\widehat{v}_{n,2}\|^2+h'(\vartheta)\delta(\widehat{u}_{n,2},\widehat{v}_{n,2})+\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2\\
\leq&-\beta_1\|\widehat{v}_{n,2}\|^2+h'(\vartheta)\delta(\widehat{u}_{n,2},\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t))+\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2\\
&+c\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2\\
\leq&-\beta_1\|\widehat{v}_{n,2}\|^2+\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2+\beta_2\delta^2\|\widehat{u}_{n,2}\|^2+\beta_2\delta\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\||\omega(t)|\|\widehat{u}_{n,2}\|\\
&+\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2\\
\leq&-\beta_1\|\widehat{v}_{n,2}\|^2+\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2+\beta_2\delta^2\|\widehat{u}_{n,2}\|^2+\frac{1}{4}\delta(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2\\
&+\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2.
\end{aligned}\tag{4.66}$$

Using the Cauchy-Schwarz inequality and Young's inequality, we get

$$\delta(\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t),\widehat{v}_{n,2})\leq\frac{\beta_1-\delta}{9}\|\widehat{v}_{n,2}\|^2+c\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2, \tag{4.67}$$

$$((I-P_n)\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t),\widehat{v}_{n,2})\leq\frac{\beta_1-\delta}{9}\|\widehat{v}_{n,2}\|^2+c\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2. \tag{4.68}$$

Now, we estimate the last two terms on the right-hand side of (4.61),

$$\begin{aligned}
&(4\Delta\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\nabla u+6\Delta\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta u+4\nabla\widehat{\rho}(\frac{|x|^2}{k^2})\cdot\Delta\nabla u+u\Delta^2\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})\\
=&(4\nabla u\cdot(\frac{12|x|}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{8|x|^3}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2}))+6\Delta u\cdot(\frac{2}{k^2}\widehat{\rho}\,'(\frac{|x|^2}{r^2})+\frac{4x^2}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2}))\\
&+\frac{8|x|}{k^2}\Delta\nabla u\cdot\widehat{\rho}\,'(\frac{|x|^2}{k^2})+u(\frac{12}{k^4}\widehat{\rho}\,''(\frac{|x|^2}{k^2})+\frac{48x^2}{k^6}\widehat{\rho}\,'''(\frac{|x|^2}{k^2})+\frac{16x^4}{k^8}\widehat{\rho}\,''''(\frac{|x|^2}{k^2})),\widehat{v}_{n,2})\\
\leq&\frac{16\sqrt{2}(3\mu_2+4\mu_3)}{k^3}\|\nabla u\|\cdot\|\widehat{v}_{n,2}\|+\frac{12(\mu_1+4\mu_2)}{k^2}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|\\
&+\frac{8\sqrt{2}\mu_1}{k}\|A^{\frac{3}{4}}u\|\cdot\|\widehat{v}_{n,2}\|+\frac{4(3\mu_2+24\mu_3+16\mu_4)}{k^4}\|u\|\cdot\|\widehat{v}_{n,2}\|\\
\leq&\frac{8(48\mu_2+64\mu_3)^2}{(\beta_1-\delta)k^6}\|\nabla u\|^2+\frac{4(12\mu_1+48\mu_2)^2}{(\beta_1-\delta)k^4}\|\Delta u\|^2+\frac{512\mu_1^2}{(\beta_1-\delta)k^2}\|A^{\frac{3}{4}}u\|^2\\
&+\frac{4(12\mu_2+96\mu_3+64\mu_4)^2}{(\beta_1-\delta)k^8}\|u\|^2+\frac{\beta_1-\delta}{4}\|\widehat{v}_{n,2}\|^2.
\end{aligned}\tag{4.69}$$

Similar to the estimates of (4.69) and by Lemma 4.1, we get

$$(p-\|\nabla u\|^2)(u\Delta\widehat{\rho}(\frac{|x|^2}{k^2})+2\nabla u\nabla\widehat{\rho}(\frac{|x|^2}{k^2}),\widehat{v}_{n,2})$$

$$\begin{aligned}
&\leq (\|\nabla u\|^2 - p)\frac{2\mu_1 + 8\mu_2}{k^2}\|u\|\|\widehat{v}_{n,2}\| + (\|\nabla u\|^2 - p)\frac{2\sqrt{2}\mu_1}{k}\|\nabla u\|\|\widehat{v}_{n,2}\| \\
&\leq \frac{\beta_1 - \delta}{9}\|\widehat{v}_{n,2}\|^2 + \frac{18(\|\nabla u\|^2 - p)^2(\mu_1 + 4\mu_2)^2}{(\beta_1 - \delta)k^4}\|u\|^2 + \frac{36\mu_1^2(\|\nabla u\|^2 - p)^2}{(\beta_1 - \delta)k^2}\|\nabla u\|^2. \qquad (4.70)
\end{aligned}$$

Assemble together (4.61)-(4.70) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\widehat{v}_{n,2}\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|\widehat{u}_{n,2}\|^2 + \|\Delta\widehat{u}_{n,2}\|^2 - p\|\nabla\widehat{u}_{n,2}\|^2 + 2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), \widehat{u}_{n,2})] \\
&+ \delta[\|\widehat{v}_{n,2}\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|\widehat{u}_{n,2}\|^2 + \|\Delta\widehat{u}_{n,2}\|^2 - p\|\nabla\widehat{u}_{n,2}\|^2] + \delta(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), \widehat{u}_{n,2}) \\
\leq& \frac{\delta}{2}\|\widehat{v}_{n,2}\|^2 + \frac{3\delta - \beta_1}{4}\|\widehat{v}_{n,2}\|^2 + \frac{\delta}{2}(\lambda + \delta^2 - \beta_2\delta)\|\widehat{u}_{n,2}\|^2 + \frac{\delta}{4}\|\Delta\widehat{u}_{n,2}\|^2 \\
&- \frac{p\delta}{2}\|\nabla\widehat{u}_{n,2}\|^2 + \frac{2}{\beta_1 - \delta}((\frac{4(48\mu_2 + 64\mu_3)^2}{k^6} + \frac{18\mu_1^2(\|\nabla u\|^2 - p)^2}{k^2})\|\nabla u\|^2 \\
&+ \frac{2(12\mu_1 + 48\mu_2)^2}{k^4}\|\Delta u\|^2 + \frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2 + (\frac{2(12\mu_2 + 96\mu_3 + 64\mu_4)^2}{k^8} \\
&+ \frac{9(\|\nabla u\|^2 - p)^2(\mu_1 + 4\mu_2)^2}{k^4})\|u\|^2) + c\|(I - P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2 \\
&+ c\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2 + c(\|\nabla u\|^2 - p)\|(I - P_n)\nabla(\widehat{\rho}(\frac{|x|^2}{k^2})\phi)\|^2|\omega(t)|^2 \\
&+ c\|(I - P_n)\Delta(\widehat{\rho}(\frac{|x|^2}{k^2})\phi)\|^2|\omega(t)|^2 + (\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t, \widehat{u}_{n,2}) \\
&+ (\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), (I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t)), \qquad (4.71)
\end{aligned}$$

For the nonlinear terms in (4.71), by (3.7), using Hölder inequality and Gagliardo-Nirenberg inequality, we obtain

$$(\widehat{\rho}(\frac{|x|^2}{k^2})f'_u(x,u)u_t, \widehat{u}_{n,2}) \leq \frac{\delta}{4}\|\Delta\widehat{u}_{n,2}\|^2 + c\lambda_{n+1}^{-1}\|u_t\|^2. \qquad (4.72)$$

By (3.4), we know

$$\begin{aligned}
&(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u), (I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\omega(t)) \\
\leq& c\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\||\omega(t)| + c\|u\|^p_{H^2(\mathbb{R}^n)}\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\||\omega(t)|. \qquad (4.73)
\end{aligned}$$

Since $1 \leq p \leq \frac{n+4}{n-4}$ and $\lambda_n \to \infty$, by Lemma 4.1 and 4.2, there are $N_1 = N(\eta),\ K_1 = K(\eta)$ such that for all $n \geq N_1,\ k \geq K_1$,

$$\begin{aligned}
&\frac{2}{\beta_1 - \delta}((\frac{4(48\mu_2 + 64\mu_3)^2}{k^6} + \frac{18\mu_1^2(\|\nabla u\|^2 - p)^2}{k^2})\|\nabla u\|^2 + \frac{2(12\mu_1 + 48\mu_2)^2}{k^4}\|\Delta u\|^2 \\
&+ \frac{256\mu_1^2}{k^2}\|A^{\frac{3}{4}}u\|^2 + (\frac{2(12\mu_2 + 96\mu_3 + 64\mu_4)^2}{k^8} + \frac{9(\|\nabla u\|^2 - p)^2(\mu_1 + 4\mu_2)^2}{k^4})\|u\|^2) \\
&+ c\lambda_{n+1}^{-\frac{1}{2}}\|u_t\|^2 + c\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\||\omega(t)| + c\|u\|^p_{H^2(\mathbb{R}^n)}\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\||\omega(t)| \\
&+ c\|(I - P_n)\widehat{\rho}(\frac{|x|^2}{k^2})\phi\|^2|\omega(t)|^2 + c\|(I - P_n)\nabla(\widehat{\rho}(\frac{|x|^2}{k^2})\phi)\|^2|\omega(t)|^2
\end{aligned}$$

$$
\begin{aligned}
&+c\|(I-P_n)\Delta(\widehat{\rho}(\frac{|x|^2}{k^2})\phi)\|^2|\omega(t)|^2\\
\leq & c\eta(1+|\omega(t)|^2+\|u_t\|^{18}+\|u\|^{18}_{H^2(\mathbb{R}^n)}).
\end{aligned}
\tag{4.74}
$$

Then by (4.71)-(4.74), we obtain

$$
\begin{aligned}
&\frac{d}{dt}[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2-p\|\nabla\widehat{u}_{n,2}\|^2+2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})]\\
&+\sigma[\|\widehat{v}_{n,2}\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2-p\|\nabla\widehat{u}_{n,2}\|^2+2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2})]\\
\leq & c\eta(1+|\omega(t)|^2+\|u_t\|^{18}+\|u\|^{18}_{H^2(\mathbb{R}^n)})+c\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,t))\|^2.
\end{aligned}
\tag{4.75}
$$

Multiplying (4.75) by $e^{\sigma t}$ and then integrating over $(\tau-t,\tau)$, we have for all $n>N_1$ and $k>K_1$,

$$
\begin{aligned}
&\|\widehat{v}_{n,2}(\tau,\tau-t,\omega)\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\omega)\|^2+\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\omega)\|^2\\
&-p\|\nabla\widehat{u}_{n,2}(\tau,\tau-t,\omega)\|^2+2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\omega))\\
\leq & e^{-\sigma t}(\|v_0\|^2+\|u_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|\Delta u_0\|^2-p\|\nabla u_0\|^2\\
&+2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),u_0))+c\eta\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(1+|\omega(s)|^2+\|u_t(s,\tau-t,\omega,u_0)\|^{18}\\
&+\|u(s,\tau-t,\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)})ds+c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,s)\|^2ds.
\end{aligned}
\tag{4.76}
$$

Replacing ω by $\theta_{-\tau}\omega$, by a similar process as in Lemma 4.1, we get,

$$
\begin{aligned}
&\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2-p\|\nabla\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2\\
&+\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega))\\
\leq & e^{-\sigma t}(1+\|v_0\|^2+\|u_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|\Delta u_0\|^2-p\|\nabla u_0\|^2+\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_0\|^{p+1}_{H^2(\mathbb{R}^n)})\\
&+c\eta\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(1+|\theta_{-\tau}\omega(s)|^2+\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{10}\\
&+\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{10}_{H^2(\mathbb{R}^n)})ds+c\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,s)\|^2ds\\
\leq & e^{-\sigma t}(1+\|v_0\|^2+\|u_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|\Delta u_0\|^2-p\|\nabla u_0\|^2+\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_0\|^{p+1}_{H^2(\mathbb{R}^n)})\\
&+c\eta\int_{-\infty}^{0}e^{\sigma s}(1+|\omega(s)|^2)ds+c\eta\int_{\tau-t}^{\tau}e^{\sigma(s-\tau)}(\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}\\
&+\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)})ds+c\int_{-\infty}^{0}e^{\sigma s}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,s+\tau)\|^2ds.
\end{aligned}
\tag{4.77}
$$

By the first equation of (3.10) and $\phi\in H^3(\mathbb{R}^n)$ as well as the Minkowski inequality, we can obtain

$$
\begin{aligned}
&\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}\\
=&\|-\delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)+v(s,\tau-t,\theta_{-\tau}\omega,v_0)+\phi\theta_{-\tau}\omega\|^{18}\\
\leq & c(\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^{18}+|\theta_{-\tau}\omega|^{18})
\end{aligned}
$$

$$\leq cR_1^9(\tau,\omega)+c|\theta_{-\tau}\omega|^{18}, \tag{4.78}$$

and

$$\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^{18}_{H^2(\mathbb{R}^n)}\leq cR_1^9(\tau,\omega), \tag{4.79}$$

where $c=\max\{\delta,\|\phi\|^{18},1\}$ and $R_1(\tau,\omega)$ is given in Lemma 4.1. Hence, it follows from (4.77)-(4.79),

$$\begin{aligned}
&\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+(\lambda+\delta^2-\beta_2\delta)\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2-p\|\nabla\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2\\
&+\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\|^2+2(\widehat{\rho}(\frac{|x|^2}{k^2})f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega))\\
\leq& e^{-\sigma t}(1+\|v_0\|^2+\|u_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|\Delta u_0\|^2-p\|\nabla u_0\|^2+\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_0\|^{p+1}_{H^2(\mathbb{R}^n)})\\
&+c\eta R_1^9(\tau,\omega)+c\eta\int_{-\infty}^0 e^{\sigma s}(1+|\omega(s)|^2+|\omega(s)|^{18})ds\\
&+c\int_{-\infty}^0 e^{\sigma s}\|(I-P_n)\widehat{\rho}(\frac{|x|^2}{r^2})g(x,s+\tau)\|^2ds.
\end{aligned} \tag{4.80}$$

Since that $(u_0,v_0)^\top\in D(\tau-t,\theta_{-t}\omega)$ and $D\in\mathcal{D}$, then

$$\begin{aligned}
&e^{-\sigma t}(1+\|v_0\|^2+\|u_0\|^2+(\lambda+\delta^2-\beta_2\delta)\|\Delta u_0\|^2-p\|\nabla u_0\|^2\\
&\qquad+\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_0\|^{p+1}_{H^2(\mathbb{R}^n)})\to 0,\qquad t\to\infty.
\end{aligned} \tag{4.81}$$

For the last term on the right-hand side of (4.80), by (3.15), there exists $N_2=N_2(\tau,\omega,\eta)\geq N_1$, such that for all $n\geq N_2$,

$$\int_{-\infty}^0 e^{\sigma s}\|(I-P_n)(\widehat{\rho}(\frac{|x|^2}{k^2})g(x,s+\tau))\|^2ds<\eta. \tag{4.82}$$

The proof is completed by (3.4), (4.81)-(4.82) and Lemma 4.1. □

5 Random attractors

In this section, we prove existence and uniqueness of $\mathcal{D}$- pullback attractors for the stochastic system (3.10)-(3.11). First we apply the lemmas shown in Section4 to prove the asymptotic compactness of solutions of (3.10)-(3.11) in E.

Lemma 5.1 *Under Assumptions I and II, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, the sequence of weak solutions of (3.10)-(3.11), $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\}_{m=1}^\infty$ has a convergent subsequence in E whenever $t_m\to\infty$ and $Y_0(\theta_{-t_m}\omega)\in D(\tau-t_m,\theta_{-t_m}\omega)$ with $D\in\mathcal{D}$.*

Proof. Let $t_m\to\infty$ and $Y_0(\theta_{-t_m}\omega)\in D(\tau-t_m,\theta_{-t_m}\omega)$ with $D\in\mathcal{D}$. By Lemma 4.1, there exists $m_1=m_1(\tau,\omega,D)>0$ such for all $m\geq m_1$, we have

$$\|Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\|^2_E\leq R_1(\tau,\omega). \tag{5.1}$$

By Lemma 4.3, for every $\eta>0$, there exist $k_0=r_0(\tau,\omega,\eta)\geq 1$ and $m_2=m_2(\tau,\omega,D,\eta)\geq m_1$ such for all $m\geq m_2$,

$$\|Y(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\|^2_{E(\mathbb{R}^n\setminus\mathbb{B}_{k_0})}\leq\eta, \tag{5.2}$$

By Lemma 4.4, there exist $k_1 = k_1(\tau,\omega,\eta) \geq k_0$ and $m_3 = m_3(\tau,\omega,D,\eta) \geq m_2$ and $n_1 = n_1(\tau,\omega,\eta) \geq 0$ such for all $m \geq m_3$,

$$\|(I-P_n)\widehat{Y}(\tau,\tau-t,\theta_{-\tau}\omega, D(\tau-t,\theta_{-\tau}\omega))\|^2_{E(\mathbb{B}_{2k_1})} \leq \eta. \tag{5.3}$$

Using (4.56) and (5.1), we get

$$\|P_n\widehat{Y}(\tau,\tau-t,\theta_{-\tau}\omega, D(\tau-t,\theta_{-\tau}\omega))\|^2_{P_nE(\mathbb{B}_{2k_1})} \leq cR_1(\tau,\omega), \tag{5.4}$$

which together with (5.3) implies that $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega, Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{2k_1})$. Note that $\widehat{\rho}(\frac{|x|^2}{k_1^2}) = 1$ for $|x| \leq k_1$. Therefore, $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega, Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{k_1})$, which along with (5.2)shows the precompactness of this sequence in E. □

Theorem 5.1 *Under Assumptions I and II, the random dynamical system Φ generated by the stochastic plate equation (3.10)-(3.11) has a unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$ in the space E.*

Proof. Note that the cocycle Φ is pullback $\mathcal{D}$-asymptotically compact in E by Lemma 5.1. On the other hand, the cocycle Φ has a pullback $\mathcal{D}$-absorbing set by Lemma 4.1. Then the existence and uniqueness of a pullback D-attractor of Φ follow from Proposition 2.1 immediately. □

Conflict of interest

The author declare that there is no conflict of interests regarding the publication of this paper.

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