Review Paper

Modules Whose Endomorphism Rings Are Right Rickart

Abstract

In this paper, we study modules whose endomorphism rings are right Rickart (or right p.p.) rings, which we call R-endoRickart modules. We provide some characterizations of R-endoRickart modules. Some classes of rings are characterized in terms of R-endoRickart modules. We prove that an R-endoRickart module with no infinite set of nonzero orthogonal idempotents in its endomorphism ring is precisely an endoBaer module. We show that a direct summand of an R-endoRickart modules inherits the property, while a direct sum of R-endoRickart modules does not. Necessary and sufficient conditions for a finite direct sum of R-endoRickart modules to be an R-endoRickart module are provided.

Keywords: R-endoRickart module; endoBaer module; Rickart module; right Rickart ring; Baer ring.
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1 Introduction

It is well known that Baer rings and Rickart rings (also known as p.p. rings) play an important role in providing a rich supply of idempotents and hence in the structure theory for rings. Rickart rings and Baer rings have their roots in functional analysis with close links to C^*-algebras and von Neumann algebras. Kaplansky [1] introduced the notion of Baer rings, which was extended to Rickart rings in ([2],[3]), and to quasi-Baer rings in [4], respectively. A number of research papers have been devoted to the study of Baer, quasi-Baer, and Rickart rings (see e.g [1], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]). A ring R is said to be Baer if the right annihilator of any nonempty subset of R is generated by an idempotent as a right ideal of R. The notion of Baer rings was generalized to a module theoretic version and studied in recent years (see [18],[19]). An R-module M is called a Baer module if for each left ideal I of $S = \mathrm{End}_R(M)$, $r_M(I) = eM$ for $e^2 = e \in S$. A more general notion of a Baer ring is that of a right Rickart ring. A ring R is called a right Rickart ring if the right annihilator of any element in R is generated by an idempotent as a right ideal of R. It is clear that any Baer ring is a right Rickart ring. A module M_R is called Rickart if the right annihilator of each left principal ideal of End$_R(M)$ is generated by an idempotent, i.e, for each $\varphi \in S = \mathrm{End}_R(M)$, there exists $e = e^2$ in S such that $r_M(\varphi) = eM$. In this paper, we introduce the notion of R-endoRickart module, investigate some basic properties of these modules.

In section 2, we introduce the notion of R-endoRickart module, investigate some basic properties of these modules. It is shown that a direct summand of an R-endoRickart modules inherits the property. The classes of hereditary rings and von Neumann regular rings are characterized in terms of R-endoRickart R-modules.

In Section 3, we investigate when a direct sum of R-endoRickart modules is also R-endoRickart. We obtain necessary and sufficient conditions for a finite direct sum of copies of R-endoRickart modules to be R-endoRickart.

In Section 4, We show that if the endomorphism ring $\mathrm{End}_R M$ of an R-endoRickart module M has no infinite set of nonzero orthogonal idempotents, then M is an endoBaer module (a module whose endomorphism ring is a Baer), and obtain that every R-endoRickart module with only countably many direct summands is an endoBaer module. We also prove that a module M is an R-endoRickart with the endomorphism ring $\mathrm{End}_R M$ has the $SSIP$ if and only if M is an endoBaer module.

Throughout this paper, all rings are associative with unity. All modules are unital right R-modules unless otherwise indicated and $S = \mathrm{End}_R(M)$ is the ring of endomorphisms of M_R. Mod-R denotes the category of all right R-modules, and M_R a right R-module. By $N \subseteq M$, $N_R \leq M_R$ and $N_R \leq^{\oplus} M_R$ denote that N is a subset, submodule and direct summand of M, respectively. By $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ we denote the ring of real, integer and natural numbers, respectively. Z_n denotes Z/nZ, $M^{(n)}$ denotes the direct sum of n copies of M. The notations $r_R(.)$ and $r_M(.)$ denote the right annihilator of a subset of M with elements from R and the right annihilator of a subset of R with elements from M, respectively.

2 R-endoRickart Modules

In this section, we introduce the notion of R-endoRickart module, investigate some basic properties of these modules. It is shown that a direct summand of an R-endoRickart modules inherits the property. The classes of hereditary rings and von Neumann regular rings are characterized in terms of R-endoRickart R-modules.

Definition 2.1. An R-module M is called R-endoRickart if $\mathrm{End}_R(M)$ is a right Rickart ring.

Recall that R is a hereditary ring if all submodules of projective modules over R are again projective. If this is required only for finitely generated submodules, it is called semihereditary. Also recall R is a von Neumann regular ring if for every $a \in R$ there exists an $x \in R$ such that $a = axa$.

Remark 2.1. (1) Obviously, R_R is an R-endoRickart module if R is a right Rickart ring, a Baer ring, a von Neumann regular ring or a hereditary ring.

(2) Every semisimple module is an R-endoRickart module.

(3) Any Rickart module is an R-endoRickart since the endomorphism ring of a Rickart module is right Rickart [18, Proposition 3.2].

(4) Any Baer module is R-endoRickart since the endomorphism ring of a Baer module is a Baer. (see [20, Theorem 4.1]).

Recall that a sequence $(a_0, a_1, a_2, ...)$ is a p-adic number if for all $n \geq 0$ we have $a_n \in \mathbb{Z}/p^{n+1}\mathbb{Z}$ and $a_{n+1} \equiv a_n$ (mod p^n). The set of p-adic numbers is denoted $\mathbb{Z}_p$ and is called the ring of p-adic integers. In the next example we shows that not every R-endoRickart module ia a Rickart (i.e, the converse of Remark 4.1 (3) does not hold in general).

Example 2.1. *Consider the module $M = \mathbb{Z}_{p^\infty}$, as a $\mathbb{Z}$-module. We know that the endomorphism ring $S = \mathrm{End}_{\mathbb{Z}}(M)$ is the ring of p-adic integers (see [21, Example 3, p. 216]). Since S is a Baer ring, it is a Rickart ring, and then $M = \mathbb{Z}_{p^\infty}$ is an R-endoRickart module. However M is not a Rickart module.*

Recall that a module M is k-local retractable if $r_M(\varphi) = r_S(\varphi)(M)$ for any $\varphi \in S = \mathrm{End}_R(M)$.

Proposition 2.1. *Let M be a k-local retractable module and $S = \mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(i) M *is an Rickart module.*
(ii) M *is an R-endoRickart module.*

Proof. (i) $\Rightarrow$ (ii) follows from Remark 2.1.
(ii) $\Rightarrow$ (i) Let M be an R-endoRickart module, since $S = \mathrm{End}_R(M)$ is a right Rickart ring and M is k-local retractable module, then M is an Rickart module by [18, Theorem 3.9]. □

Recall that a module M is said to have D_2 condition if for any $N \leq M$ with $M/N \cong M' \leq^{\oplus} M$, we have $N \leq^{\oplus} M$.

Corollary 2.1. *The following conditions are equivalent for a k-local retractable module M and $S = \mathrm{End}_R(M)$:*
(i) M *is an R-endoRickart module.*
(ii) M *is an Rickart module.*
(iii) M *satisfies the D_2 condition, and $\mathrm{Im}\varphi$ is isomorphic to a direct summand of M for any $\varphi \in S$.*

Proof. Follows from Proposition 2.1 and [18, Proposition 2.11] . □

If M is an R-module, N a direct summand of M, and e the projection of M onto N, then it is easy to see that e is an idempotent of $S = \mathrm{Hom}_R(M, M)$ and $\mathrm{Hom}_R(N, N) = eSe$. This fact will be used in the next proposition.

Proposition 2.2. *Every direct summand of an R-endoRickart module is R-endoRickart.*

Proof. Let M be an R-endoRickart module, N a direct summand of M, $S = \mathrm{Hom}_R(M, M)$, and e the projection onto N. Then $\mathrm{Hom}_R(N, N) = eSe$. But for any right Rickart ring S and any idempotent $e \in S$, eSe is a right Rickart ring by [18, Corollary 3.3]. Thus N is R-endoRickart. □

Recall that a morphism $f : M \to N$, (M and N are right R-modules) is a regular morphism (or regular map) if there exists $g : N \to M$ such that $f = fgf$.

Remark 2.2. If M is an R-endoRickart module, then so are $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ for every regular $\varphi \in \mathrm{End}_R(M)$.

Proof. This follows from the fact that $\varphi \in \mathrm{End}_R(M)$ is regular if and only if $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of M by [22, Theorem 16]. □

Corollary 2.2. *If R is a right Rickart ring, then eR is an R-endoRickart R-module for every $e^2 = e \in R$.*

Corollary 2.2 also follows from the fact that if R is a right Rickart ring then so is eRe for every $e^2 = e \in R$ by [18, Corollary 3.3].

The next example shows an application of Proposition 2.2.

Example 2.2. *(Example 1.7, [23]) Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Consider $T = \{(a_n)_{n=1}^{\infty} \in A | a_n$ is eventually constant$\}$, $I = \{(a_n)_{n=1}^{\infty} \in A | a_n$ = 0 is eventually $\}$ = $\bigoplus_{n=1}^{\infty} \mathbf{Z}_2$. Now, consider the ring $R = \begin{pmatrix} T & T/I \\ 0 & T/I \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} (1,1,...) & 0+I \\ 0 & 0+I \end{pmatrix}$ in R. Note that R is a right hereditary ring, but R is not a Baer ring. Since R is a right Rickart ring (being right hereditary), $M = R_R$ is an R-endoRickart module, and the modules $M_1 = eR$ and $M_2 = (1-e)R$ are endoRickart R-modules by Proposition 2.2.*

The next example shows that the submodule of a module can be an R-endoRickart however the module is not.

Example 2.3. *The $\mathbb{Z}$-module $\mathbb{Z}_4$ is not R-endoRickart since $S = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_4)$ is not right Rickart ring. However, the submodule $2\mathbb{Z}_4$ of $\mathbb{Z}_4$ is an R-endoRickart $\mathbb{Z}$-module because $2\mathbb{Z}_4 \cong_{\mathbb{Z}} \mathbb{Z}_2$ ($\mathbb{Z}_2$ is a Rickart module).*

Proposition 2.3. *If $\mathrm{End}_R(M)$ is a von Neumann regular ring, then M is an R-endoRickart module.*

Proof. Since $\mathrm{End}_R(M)$ is a von Neumann regular ring, then it is a right Rickart ring. Hence M is an R-endoRickart module. □

Recall that a right R-module M is retractable if $\mathrm{Hom}_R(M, N) \neq 0$ whenever N is a non-zero submodule of M. Also recall that a module M is quasi-retractable if $\mathrm{Hom}_R(M, r_M(I)) \neq 0$ for every $I \leq S_S$ with $r_M(I) \neq 0$.

Proposition 2.4. *Let M be a (quasi-) retractable module and $S = \mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(i) *M is an Rickart module.*
(ii) *M is an R-endoRickart module.*

Proof. (i) ⇒ (ii) follows from Remark 2.1.

(ii) ⇒ (i) Let M be an R-endoRickart module, since $S = \mathrm{End}_R(M)$ is a right Rickart ring and M is (quasi-) retractable module, then M is an Rickart module by [18, Proposition 3.5]. □

Recall that a module M is said to have C_2 condition if any submodule N of M which is isomorphic to a direct summand of M is a direct summand of M.

Proposition 2.5. *Let M be either a (quasi-) retractable or a k-local retractable module and $S = \mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(i) *M is an R-endoRickart module with C_2 condition.*
(ii) *S is a von Neumann regular ring.*
(iii) *For each $\varphi \in S$, $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of M.*

Proof. Follows from [18, Theorem 3.17], Proposition 2.1, Proposition 2.3 and Proposition 2.4. □

Corollary 2.3. *Let M be either a (quasi-) retractable or a k-local retractable module with C_2 condition. If M is an R-endoRickart module, then $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are R-endoRickart for each $\varphi \in S$.*

Proof. $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of M for each $\varphi \in S$ by Proposition 2.5. Thus they are R-endoRickart modules by Proposition 2.2. □

Next, we characterize several classes of rings in terms of R-endoRickart modules.

Theorem 2.1. *The following conditions are equivalent for a ring R:*

(i) *Every free module M_R is an R-endoRickart module.*
(ii) *Every free module M_R is a Rickart module.*

Proof. (i) ⇒ (ii) This follows from the fact that the endomorphism ring of a free module M_R is a right Rickart ring if and only if M_R is a Rickart module by [18, Corollary 5.3].

(ii) ⇒ (i) It is clear. □

Recall that a module M is endoregular if $\mathrm{End}_R(M)$ is a von Neumann regular ring.

Proposition 2.6. *Every endoregular module M is an R-endoRickart module.*

Proof. Let M be an endoregular module. Then $\mathrm{End}_R(M)$ is a von Neumann regular ring, thus M is an R-endoRickart module by Proposition 2.3. □

Proposition 2.7. *Let M be either a (quasi-) retractable or a k-local retractable module with C_2 condition and $S = \mathrm{End}_R(M)$, Then the following conditions are equivalent:*

(i) *M is an endoregular module.*
(ii) *M is an R-endoRickart module.*
(iii) *For each $\varphi \in S$, $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ are direct summands of M.*

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 2.6.

(ii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follows from Proposition 2.5. □

Recall that a module M has the (strong) summand intersection property, SIP $(SSIP)$, if the intersection of any two (any family of) direct summands is a direct summand of M. M is said to have the (strong) summand sum property, SSP $(SSSP)$, if the sum of any two (any family of) direct summands is a direct summand of M.

Corollary 2.4. *Let M be either a (quasi-) retractable or a k-local retractable module with C_2 condition, Then the following statements hold:*

(i) *Every R-endoRickart module M satisfies the SIP and the SSP.*

(ii) *For every R-endoRickart module M, $\bigcap_{i=1}^{n} \mathrm{Ker}\varphi_i$ and $\sum_{i=1}^{n} \mathrm{Im}\varphi_i$ are R-endoRickart modules for every finite set $\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ in $\mathrm{End}_R(M)$.*

Proof. (i) Note that every R-endoRickart module is an endoregular by Proposition 2.7. This is a direct consequence of [24, Proposition 2.28].

(ii) For each $\varphi_i \in \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$, $\mathrm{Ker}\varphi_i$ and $\mathrm{Im}\varphi_i$ are direct summands of M by Proposition 2.7. Then $\bigcap_{i=1}^{n} \mathrm{Ker}\varphi_i$ and $\sum_{i=1}^{n} \mathrm{Im}\varphi_i$ are direct summands of M by (i). Thus R-endoRickart modules by Proposition 2.2. □

Proposition 2.8. *Let M be an R-module and $S = End_R(M)$, if for every $0 \neq \varphi \in S$, φ is a monomorphism, then M is an indecomposable R-endoRickart module.*

Proof. Assume that M is not indecomposable. Then $M = N_1 \oplus N_2$ with $N_1, N_2 \neq 0$. Take $\varphi = \pi_1$ the canonical projection of M onto N_1. Then $\mathrm{Ker}(\varphi) = N_2 \neq 0$, a contradiction (as φ is a monomorphism), and so M is indecomposable. It is clear that for every $\varphi \in S$, $Ker\varphi \leq^{\oplus} M$, M is a Rickart module, and hence an R-endoRickart module. □

Proposition 2.9. *If the $End(M)$ is a domain, then a module M is an indecomposable R-endoRickart.*

Proof. Every domain is trivially a right Rickart ring, then M is an R-endoRickart module. Since there are no idempotents other than 0 and 1 in a domain, M is also indecomposable. □

Proposition 2.10. *If M is an R-endoRickart module, with only countably many direct summands, then M contains no infinite direct sums of disjoint summands.*

Proof. Since M has only countably many direct summands, S has no infinite set of nonzero orthogonal idempotents, hence there exist no infinite sets of mutually disjoint direct summands in M. □

Corollary 2.5. *If M is an R-endoRickart module, with only countably many direct summands, then M is a finite direct sum of indecomposable summands.*

Proof. By Proposition 2.10, S has no infinite sets of orthogonal idempotents, hence any direct sum decomposition of M must be finite, thus M is a finite direct sum of indecomposable submodules. □

Recall that a ring is regular in the sense of commutative algebra if it is a commutative unit ring such that all its localizations at prime ideals are regular local rings.

Corollary 2.6. *Let M be an R-endoRickart module with only countably many direct summands and the endomorphism ring $S = End_R(M)$ is a regular. Then M is a semisimple Artinian.*

Proof. S is a regular Baer ring with only countably many idempotents by Theorem [25, Theorem 7.55]. Then S is a semisimple Artinian ring, by [26, Theorem 2 and Theorem 3]. It is easy to check that M is also a semisimple Artinian module. □

Corollary 2.7. *Let M be R-module with only countably many direct summands and $S = End_R(M)$ is a regular ring. Then M is an R-endoRickart module if and only if M is a semisimple Artinian.*

Proof. The proof follows directly from Remark 2.1 and Corollary 2.6. □

Proposition 2.11. *The following conditions are equivalent for a ring R:*

(i) *Every free R-module M is an R-endoRickart module.*

(ii) *R is a right hereditary ring.*

Proof. Since that a free module is a retractable, M is R-endoRickart module if and only if it is a Rickart by Proposition 2.4. Thus every free R-module M is an R-endoRickart module if and only if R is a right hereditary ring by [18, Theorem 2.26] and Remark 2.1. □

Corollary 2.8. *Let R be a right hereditary ring, then every projective right R-module is an R-endoRickart module.*

Proof. From Proposition 2.11 every free R-module is an R-endoRickart module, since that every projective module is a direct summand of a free module, then every projective module is an R-endoRickart by Proposition 2.2. □

Proposition 2.12. *Let R be a von Neumann regular ring. Then a free module $R^{(n)}$ is an R-endoRickart R-module for some $n \in \mathbb{N}$.*

Proof. This follows from the well-known fact that R is von Neumann regular if and only if so is $Mat_n(R)$. since $Mat_n(R) = End_R(R^n)$ is a von Neumann regular ring. Thus R^n is R-endoRickart by Proposition 2.3. □

Recall that a ring R is a principal ideal domain or PID if R is an integral domain in which every ideal is principal, i.e., can be generated by a single element.

Proposition 2.13. *Let M be a free module M of countable rank over a principal ideal domain (PID) R, then M is an R-endoRickart and has the $SSIP$.*

Proof. Since R is a principal ideal domain (PID), then M has the $SSIP$ (see [26, Exercise 51(c)], and it is a Rickart R-module by [18, Theorem 2.26]. Thus it is an R-endoRickart by Remark 2.1. □

Corollary 2.9. *Let M be a projective module. Then the following statements hold:*

(i) *Every submodule of M over a hereditary ring is an R-endoRickart module.*

(ii) *Every finitely generated submodule of M over a von Neumann regular ring is an R-endoRickart module.*

Proof. (i) Since that all submodules of projective modules over a hereditary ring R are again projective. Thus they are R-endoRickart modules by Corollary 2.8.

(ii) Let I be a finitely generated submodule of M. It is well-known that a von Neumann regular ring is left and right semihereditary, and every finitely generated submodule of a projective module over a von Neumann regular ring R is isomorphic to a direct summand of a finitely generated free R-module by [27]. Hence $I \cong K \leq^{\oplus} R^{(n)}$. Therefore, I is an R-endoRickart module by Proposition 2.2 and Propositions 2.12 . □

3 Direct Sums Of R-endoRickart Modules

It is shown that a direct sum of R-endoRickart modules may not be R-endoRickart. In this section, we investigate when a direct sum of R-endoRickart modules is also R-endoRickart. We obtain necessary and sufficient conditions for a finite direct sum of copies of (k-local) retractable R-endoRickart module to be R-endoRickart.

The next example shows that a direct sum of R-endoRickart modules may not inherit the R-endoRickart property.

Example 3.1. *A finite direct sum of R-endoRickart modules is not necessarily an R-endoRickart module. For example, the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ is not R-endoRickart while $\mathbb{Z}$ and $\mathbb{Z}_2$ are both R-endoRickart $\mathbb{Z}$-modules ($\mathbb{Z}$ and $\mathbb{Z}_2$ are both Rickart modules). We note that the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$ is a retractable module (Any direct sum of $\mathbb{Z}_{p^i}$ is retractable, where p is a prime number). For the endomorphism $f(x, \bar{y}) = \bar{x}$ where $x \in \mathbb{Z}$ and $y \in \mathbb{Z}_2$, $\mathrm{Ker} f = 2\mathbb{Z} \oplus \mathbb{Z}_2$ which is not a direct summand of $\mathbb{Z} \oplus \mathbb{Z}_2$. So $\mathbb{Z} \oplus \mathbb{Z}_2$ is not a Rickart module [see ([20], Example 2.24)]. Thus $\mathbb{Z} \oplus \mathbb{Z}_2$ is not an R-endoRickart module by Proposition 2.4.*

Recall that a module M is a quasi-continuous if every complement in M is a direct summand of M, and for any direct summands M_1 and M_2 of M such that $M_1 \cap M_2 = 0$, the submodule $M_1 \oplus M_2$ is also a direct summand of M.

Proposition 3.1. *Let M_i be a direct summand of a quasi-continuous R-endoRickart module M for all $i = 1, .., n$, such that $M_i \cap M_j = 0$ for $i \neq j$. Then M_i is an R-endoRickart module for all i and $\bigoplus_{i=1}^{n} M_i$ is an R-endoRickart module.*

Proof. Since M is a quasi-continuous module and $M_i \cap M_j = 0$ for all $i \neq j$, $\bigoplus_{i=1}^{n} M_i$ is a direct summand of M, Therefore, it is an R-endoRickart module by Proposition 2.2. □

Proposition 3.2. *Let M be an artinian R-endoRickart module. Then there exists a decomposition*

$$M = N_1 \oplus N_2 \oplus N_3 \oplus \cdots \oplus N_n,$$

where N_i is an indecomposable R-endoRickart module for each i.

Proof. From [28, Proposition 19.20] Since M is artinian, there exists a decomposition

$$M = N_1 \oplus N_2 \oplus N_3 \oplus \cdots \oplus N_n,$$

where each N_i is an indecomposable. Also, each N_i is an R-endoRickart module by Proposition 2.2. □

Proposition 3.3. *Let R be a commutative ring and $M = \bigoplus_{i \in I} M_i$ a direct sum of cyclic R-endoRickart modules M_i over an arbitrary index set I. If $S = \mathrm{End}_R(M)$ is a domain, then M is an R-endoRickart module.*

Proof. Note that M is a k-local retractable Rickart module by [18, Proposition 4.9] and [18, Proposition 5.1]. Thus M is an R-endoRickart module by Proposition 2.1. □

The following result study finite direct sums of copies of an arbitrary R-endoRickart module M.

Theorem 3.1. *Let M be a finitely generated R-endoRickart module and $S = \mathrm{End}(M)$, Then the following conditions are equivalent:*

(i) The arbitrary direct sum of copies of M is an R-endoRickart module.

(ii) $S = \mathrm{End}(M)$ is a hereditary ring.

Proof. (i)⇒(ii) For a finitely generated module M and $S = End(M)$, we have that $End(M^{(f)}) \cong End(S^{(f)})$ as rings, where f is an arbitrary set. Hence, if an arbitrary direct sum of copies of M is R-endoRickart, its endomorphism ring $End(M^{(f)})$ is a right Rickart ring, hence $End(S^{(f)})$ is also a right Rickart ring, thus $S^{(f)}$ is an R-endoRickart module. Hence By Proposition 2.11, S is hereditary.

(ii)⇒(i) let $S = \mathrm{End}(M)$ is hereditary, for an arbitrary set f, Since $S^{(f)}$ is a free S-module, we obtain that $S^{(f)}$ is an R-endoRickart S-module By Proposition 2.11, hence $End(S^{(f)})$ is a right Rickart ring, thus $End(M^{(f)})$ is a right Rickart ring, and $M^{(f)}$ is an R-endoRickart module. □

The following result studies finite direct sums of copies of an arbitrary (k-local) retractable R-endoRickart module M.

Proposition 3.4. *Let M be a (k-local) retractable R-endoRickart module with C_2 condition. Then any finite direct sum of copies of M is an R-endoRickart module.*

Proof. Since that a finite direct sum of copies of M is a Rickart module by [29, Corollary 2.31], Proposition 2.1 and Proposition 2.4. Thus it is an R-endoRickart by Remark 2.1. □

The next example shows an application of Proposition 3.4.

Recall that an element $m \in M$ is singular if $r_R(m) \leq^{ess} R_R$. We denote the set of all singular elements of M by $Z(M)$. Then we say a module M nonsingular if $Z(M) = 0$ and singular if $Z(M) = M$. A ring R is right nonsingular if R_R is nonsingular.

Example 3.2. *Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and the R-module $M = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$. Since M is a nonsingular quasi-injective R-module, M is a Rickart module with C_2 condition(see [29], Example 2.32), thus M is an R-endoRickart module with C_2 condition. Thus $M^{(n)}$ is an R-endoRickart module by Proposition 3.4.*

Recall that a ring R is a Prüfer domain if R is a commutative ring without zero divisors in which every non-zero finitely generated ideal is invertible.

Theorem 3.2. *([30, Corollary 15]). If R is a commutative integral domain, then $M_n(R)$ is a Baer ring (for some $n > 1$) if and only if every finitely generated ideal of R is invertible, i.e., if R is a Prüfer domain.*

Theorem 3.3. *Let M be a free R-module of finite $rank > 1$ with only countably many direct summands. Then the following conditions are equivalent for a commutative integral domain R :*

(i)M is R-endoRickart.

(ii) R is a Prüfer domain.

Proof. Consider R is a Prüfer domain, then $M_n(R)$ is a Baer ring by Theorem 3.2. but $End(M) \cong M_n(R)$ is a Baer ring, thus $End(M)$ is a right Rickart ring, so we obtain that M is an R-endoRickart module.

Conversely, if M is an R-endoRickart module, $End(M)$ is a right Rickart ring has no infinite set of nonzero orthogonal idempotents (as M is R-module with only countably many direct summands), then it is a Baer ring by [25, Theorem 7.55]Theorem, hence $M_n(R)$ for $n > 1$ is a Baer ring, thus R must be a Prüfer domain. □

We now characterize the semisimple artinian rings in terms of free R-endoRickart modules.

Proposition 3.5. *Let M be a (quasi-) retractable module. Then the following conditions are equivalent for a ring R :*

(i) All R-module M is an R-endoRickart module;

(ii)All R-module M is a Rickart module;

(iii)All injective R-module M is a Rickart module;

(iv)All extending R-module M is a Rickart module;

(v)All (injective) R-module M is a Baer module;

(vi) R is a semisimple artinian ring.

Proof. $(i) \Leftrightarrow (ii)$ Since M is a retractable module, then the result follows from Proposition 2.4.

$(ii) \Rightarrow (iii) \Rightarrow (iv)$ It is clear.

$(iv) \Leftrightarrow (v)$ Is easy to see because every injective Rickart module is Baer (see [18], Remark 2.13).

$(v) \Rightarrow (vi)$ Since M is a Baer module, thus R is a semisimple artinian ring by in [31, Theorem 2.20].

$(vi) \Rightarrow (1)$ Every right R-module M is a Rickart module, thus an R-endoRickart by Remark 2.1. □

Recall that a module over a ring is torsion free if 0 is the only element annihilated by a regular element (nonzero divisor) of the ring.

Proposition 3.6. *Let M be a finite direct sum of copies of some finite rank, torsion-free module and $S = End(M)$ is a PID. Then M is R-endoRickart module.*

Proof. By [32] $Ker\varphi \leq^{\oplus} M, \forall \varphi \in S$, hence M is a Rickart module, thus it is an R-endoRickart by our Remark 2.1. □

Recall that a ring R is a right n-fir if any right ideal that can be generated with $\leqslant n$ elements is free of unique rank (i.e., for every $I \leq R_R$, $I \cong R^k$ for some $k \leqslant n$, and if $I \cong R^l \Rightarrow k = l$) (for alternate definitions see , [33, Theorem 1.1]).

The definition of (right) n-firs is left-right symmetric, thus we will call such rings simply n-firs.

Proposition 3.7. *Let M be a module with endomorphism ring S is n-fir, then M is an R-endoRickart module and S^n is a Baer module. Consequently, $M_n(S)$ is a Baer ring*

Proof. Since S is an n-fir, it is in particular an integral domain (see page 45, [33]), then trivially a right Rickart ring. Thus M is an R-endoRickart module. S^n is a Baer module by [19, Theorem 3.16]. Consequently, $M_n(S)$ is a Baer ring. □

Next we study finite direct sums of copies of a finitely generated R-endoRickart module M.

Proposition 3.8. *Let M be a finitely generated module with endomorphism ring S is n-fir, then M is an R-endoRickart module and a finite direct sum of copies of M is an R-endoRickart module.*

Proof. We note that, for a finitely generated module M and $S = End(M)$, we have that $End(M^n) \cong End(S^n)$ as rings, where $n \in \mathbf{N}$. Since S is n-fir, then M is an R-endoRickart module and S^n is a Baer module by Proposition 3.7, and so $End(S^n)$ is a Baer ring (the endomorphism ring of a Baer module is a Baer). Thus S^n is an R-endoRickart S-module by Remark 2.1, hence $End(S^n)$ is a right Rickart ring (being a Baer ring), thus $End(M^n)$ is a right Rickart ring, and M^n is an R-endoRickart. □

4 R-endoRickart Modules Versus EndoBaer Modules

In this section, we show that if the endomorphism ring $\mathrm{End}_R M$ of an R-endoRickart module M has no infinite set of nonzero orthogonal idempotents, then M is an endoBaer module, and obtain that every R-endoRickart module with only countably many direct summands is an endoBaer module. We also prove that a module M is R-endoRickart with the endomorphism ring $\mathrm{End}_R M$ has the $SSIP$ if and only if M is an endoBaer module.

Definition 4.1. An R-module M is called endoBaer if $\mathrm{End}_R(M)$ is a Baer ring.

Remark 4.1. Any Baer module is an endoBaer, since the endomorphism ring of a Baer module is a Baer. (see [20, Theorem 4.1]).

Proposition 4.1. *Let M be a (quasi-) retractable module. Then the following conditions are equivalent:*
(i) M *is an endoBaer module.*
(ii) M *is a Baer module.*

Proof. (i) ⇒ (ii) Since M is an endoBaer module, $S = \mathrm{End}_R(M)$ is a Baer ring, Also M is a (quasi-) retractable, thus M is a Baer module by [20, Proposition 4.6] and [19, Theorem 2.5].

(ii) ⇒ (i) follows from Remark 4.1. □

Remark 4.2. It is clear any endoBaer module is an R-endoRickart, since that any Baer ring is a right Rickart ring. But the converse does not hold in general.

The following examples exhibit an R-endoRickart module which is not an endoBaer module with the property that its endomorphism ring has an infinite set of nonzero orthogonal idempotents.

Example 4.1. *Let $R=\prod_{n=1}^{\infty}\mathbb{Z}_2$ be a commutative ring, R is a von Neumann regular, and Baer. Consider $T=\{(a_n)_{n=1}^{\infty}\in R|a_n$ is eventually constant$\}$, a subring of R. Then T is a right Rickart ring, while T is not a Baer ring by ([23, Example 7.54] and it has an infinite set of nonzero orthogonal idempotents, $\{\alpha_i=(a_k)\in T\mid a_k=1$ if $k=i$, otherwise, $a_k=0\}$. Consider $M=T_T$. Then M is an R-endoRickart module, which is not an endoBaer module.*

Example 4.2. *From example 2.2, note that R is a right hereditary ring, but R is not a Baer ring. Since R is a right Rickart ring (being right hereditary), $M=R_R$ is an R-endoRickart module, which is not an endoBaer module.*

Example 4.3. *([10], Example 1.6). Let A be a field, take $A_n=A$ for $n=1,2,....$ and let*

$$R=\begin{pmatrix}\prod_{n=1}^{\infty}A_n & \bigoplus_{n=1}^{\infty}A_n\\ \bigoplus_{n=1}^{\infty}A_n & \langle\bigoplus_{n=1}^{\infty}A_n,1\rangle\end{pmatrix}$$

which is a subring of the 2×2 matrix ring over the ring $\prod_{n=1}^{\infty}A_n$, where $\langle\bigoplus_{n=1}^{\infty}A_n,1\rangle$ is the A-algebra generated by $\bigoplus_{n=1}^{\infty}A_n$ and 1. Then R is a von Neumann regular ring which is not a Baer ring. thus $M=R_R$ is an R-endoRickart module, which is not an endoBaer module. Denote the idempotent $e=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Then $M=eR$ is a R-endoRickart R-module by Proposition 2.2. However, M is not an endoBaer R-module because $End_R(M)\cong\langle\bigoplus_{n=1}^{\infty}A_n,1\rangle$ is not a Baer ring (see ([18], Example 2.19)).

Example 4.4. *Since that a free modules $\mathbf{Z}^{\mathbf{N}}$ and $\mathbf{Z}^{\mathbf{R}}$ are R-endoRickart $\mathbf{Z}$-modules ($\mathbf{Z}^{\mathbf{N}}$ and $\mathbf{Z}^{\mathbf{R}}$ are both Rickart modules,see Example 2.2.12 in [34]), then $End_Z(\mathbf{Z}^{\mathbf{N}})$ and $End_Z(\mathbf{Z}^{\mathbf{R}})$ are right Rickart rings. Note that $End_Z(\mathbf{Z}^{\mathbf{N}})$ is also a Baer ring, but $End_Z(\mathbf{Z}^{\mathbf{R}})$ is not a Baer ring. This, because $\mathbf{Z}^{\mathbf{R}}$ is retractable but is not a Baer $\mathbf{Z}$-module (see [19, Proposition 2.5]. Thus $\mathbf{Z}^{\mathbf{N}}$ is an endoBaer module, but $\mathbf{Z}^{\mathbf{R}}$ is not.*

Theorem 4.1. *Let M be a (quasi-)retractable module, Then the following conditions are equivalent for a ring R :*

(i) All R-module M is an R-endoRickart module;
(ii)All R-module M is a Rickart module;
(iii)All R-module M is a Baer module;
(iv)All R-module M is an endoBaer module;
(v) R is a semisimple artinian ring.

Proof. Follows from Proposition 4.1 and Proposition 3.5. □

Proposition 4.2. *Let $M=\bigoplus_{i\in I}M_i$ be a direct sum of finitely generated R-endoRickart modules M_i, where I is a countable index set over a principal ideal domain R. Then the following conditions are equivalent:*

(i) *M is a semisimple module.*
(ii) *M is an R-endoRickart module.*
(iii) *M is an endoBaer module.*

Proof. (i)⇒ (ii) By Remark 2.1 (1).
(iii) ⇒ (ii) It is clear.
(ii) ⇒ (iii) and (iii) ⇒ (i) follows from [18, Corollary 5.8]. □

Proposition 4.3. *The following conditions are equivalent for a (quasi-) retractable module M:*

(i) *M is an indecomposable R-endoRickart module.*
(ii) *M is an endoBaer module.*

Proof.(i) $\Rightarrow$ (ii) Since M is an indecomposable R-endoRickart module, then M is a Baer module by [18, Corollary 4.6] and Proposition 2.4. Thus an endoBaer module by Remark 4.1.

(ii) $\Rightarrow$ (i) M is a Baer module by Propsition 4.1 and indecomposable Rickart module by [18, Corollary 4.6]. Thus an R-endoRickart module by Remark 2.1. □

Recall that a module M is quasi-injective if every homomorphism of a submodule of M into M may be realized by an endomorphism of M.

Corollary 4.1. *Let M be a quasi-injective R-module. The following statements are equivalent:*

(i) M is an endoBaer module.

(ii) M is an R-endoRickart module.

Proof. $(i) \Rightarrow (ii)$ It is clear.

$(ii) \Rightarrow (i)$ M is a Baer module by [35, Theorem 3.11] , thus it is an endoBaer by Remark 4.1. □

Theorem 4.2. *Let M be a right R-module, and let $S = \mathrm{End}_R M$ have no infinite set of nonzero orthogonal idempotents. Then the following conditions are equivalent:*

(i) *M is an R-endoRickart module.*

(ii) *M is an endoBaer module.*

Proof.(i) $\Rightarrow$ (ii) Since M is an R-endoRickart module, R is a right Rickart ring has no infinite set of nonzero orthogonal idempotents. Thus R is a right Rickart ring if and only if R is a Baer ring by [25, Theorem 7.55].

(ii) $\Rightarrow$ (i) It is clear. □

Proposition 4.4. *Let M be a right R-module with only countably many direct summands. Then the following conditions are equivalent:*

(i) *M is an R-endoRickart module.*

(ii) *M is an endoBaer module.*

Proof. (i) $\Rightarrow$ (ii) Since M has only countably many direct summands, $\mathrm{End}_R(M)$ has no infinite set of nonzero orthogonal idempotents. Hence M is an endoBaer module by Theorem 4.2.

(ii) $\Rightarrow$ (i) It is clear. □

Theorem 4.3. *An R-module M is an R-endoRickart and $S = \mathrm{End}_R(M)$ has the $SSIP$ if and only if M is an endoBaer module.*

Proof. Let N be any submodule of S. Since M is R-endoRickart, S is a right Rickart ring and for each $n \in N$, there exists $e_n^2 = e_n \in S$ such that $r_S(n) = e_n S$. Thus, there exists $e^2 = e \in S$ such that $r_S(N) = \bigcap_{n \in N} r_S(n) = \bigcap_{n \in N} e_n S = eS$ by the $SSIP$. Thus, S is a Baer ring and M is an endoBaer module. Conversely, suppose M is an endoBaer module. Hence M is an R-endoRickart module by Remark 4.2, and S is a Baer ring. Thus, S has the $SSIP$. □

Corollary 4.2. *Let M be a retractable module and $S = \mathrm{End}_R(M)$ has the $SSIP$. Then the following conditions are equivalent:*

(i) *M is an R-endoRickart module.*

(ii) *M is an endoBaer module.*

(iii) *φ splits in M for any $\varphi \in \mathrm{End}_R(M)$.*

Proof. $(i) \Leftrightarrow (ii)$ Follows from Theorem 4.3.

(ii) $\Rightarrow$ (iii) For $\varphi \in \mathrm{End}_R(M)$, consider the short exact sequence

$$0 \to \mathrm{Ker}\varphi = r_M(\varphi) \to M \to \varphi M \to 0.$$

Since M is a retractable module and S is a Baer ring, M is a Baer module by [20, Proposition 4.6]. Thus M is a Rickart module and $\mathrm{Ker}\varphi \leq^{\oplus} M$. So the short exact sequence splits.

(iii) ⇔ (i) φ splits in M for any $\varphi \in \mathrm{End}_R(M)$ if and only if $\mathrm{Ker}\varphi \leq^{\oplus} M$ if and only if M is a Rickart module if and only if M is an R-endoRickart module by Proposition 2.4. □

Proposition 4.5. *Let M be a (quasi-) retractable module and $S = \mathrm{End}_R(M)$ with only two idempotents, 0 and 1. Then the following conditions are equivalent:*

(i) *M is an R-endoRickart module.*

(ii) *M is an endoBaer module.*

Proof. $(i) \Rightarrow (ii)$ Since S is a right Rickart ring with only two idempotents, 0 and 1, then S is a domain by [18, Remark 4.10]. and then M is an indecomposable R-endoRickart module by [18, Proposition 4.9] and Remark 2.1. Thus M is an endoBaer module by Proposition 4.3.

(ii) ⇒ (i) It is clear. □

Recall that a ring R is a right (left) self injective ring if it is injective over itself as a right (left) module. If a von Neumann regular ring R is also right or left self injective, then R is Baer.

Proposition 4.6. *Let M be an R-module and $S = \mathrm{End}_R(M)$ be any right self-injective ring. Then the following conditions are equivalent:*

(i) *M is an R-endoRickart module.*

(ii) *M is an endoBaer module.*

Proof. $(i) \Rightarrow (ii)$ Let M be an R-endoRickart module, S is a right Rickart ring. Since S is right self-injective ring, then S is a right Rickart ring if and only if it is a Baer ring by [25, Theorem 7.52]. Thus M is an endoBaer module.

(ii) ⇒ (i) It is clear. □

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