Original Research Article

ON SIMPLICIAL POLYTOPIC NUMBERS

ABSTRACT. The ultimate goal of this work is to provide in a concise manner old and new results relating to the simplicial polytopic numbers.

1. Introduction

Pythagoras and the other Greek mathematicians of his school studied intensively the class of figurate numbers named the polygonal numbers [5], [6]. On the other hand, they made little research on the the class of figurate numbers named the simplicial polytopic numbers or, as they are sometimes called, the generalized triangular numbers.

The simplicial polytopic numbers are a family of sequences of figurate numbers corresponding to the r-dimensional simplex for each dimension r, where r is an integer. Little is known of these numbers. The goal of this article is, therefore, to provide in a concise manner new as well as old results pertaining to the simplicial polytopic numbers.

The remainder of the article is divided into seven sections. Section 2 deals with the basis of the polytopic numbers. Section 3 is concerned with the sums of the polytopic numbers and section 4 deals with the alternating sums of the polytopic numbers. In sections 5 and 6 we discuss respectively the sums and alternating sums of reciprocals of the polytopic numbers. Section 7 covers the products of the polytopic numbers. The final section, that is section 8, provides interesting identities relating to the polytopic numbers.

We omit proofs of easily verified results and those that can be found in readily available sources. One proving technique that may be of great service for the reader is the mathematical induction.

2. Basis of Polytopic Numbers

This section treats the special class of numbers called the polytopic numbers. A **polytopic number** is a number represented as dots or points arranged in the shape of a regular polytope. It is sometimes called the **Generalized Triangular Number**. It can be regarded as a family of sequences of figurate numbers with the general formula

$$P_r(n) = \binom{n+r-1}{r} = \frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}$$

where $P_r(n)$ is the nth term of a sequence of r-polytopic numbers, $\binom{n}{r}$ is a binomial coefficient, and $r!$ is factorial r [1], [14], [15].

Based on the different values of r, we have the various types of polytopic numbers:

Key words and phrases. Polytopic numbers, Triangular Numbers, Harmonic numbers, Alternating Harmonic numbers, Eulerian numbers, Pascal triangle numbers.

- Point numbers where $r = 0$
- Linear (natural) numbers where $r = 1$
- Triangular numbers where $r = 2$
- Tetrahedral numbers where $r = 3$
- Pentatopic numbers where $r = 4$
- Hexateron numbers where $r = 5$
- Heptapeton numbers where$r = 6$
- Octahexon numbers where $r = 7$
- Nonahepton numbers where $r = 8$
- Decaocton numbers where $r = 9$
- Hendecaenneon numbers where $r = 10$

The sequence of point numbers has the nth term

$$P_0(n) = \binom{n-1}{0} = 1$$

and is

$$1, 1, 1, 1, 1, \ldots$$

obtained by setting $n = 1, 2, 3, \ldots$.

The sequence of linear (natural) numbers has the nth term

$$P_1(n) = \binom{n}{1} = n$$

and is

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

obtained by letting $n = 1, 2, 3, \ldots$.

The sequence of triangular numbers has the nth term

$$P_2(n) = \binom{n+1}{2} = \frac{n(n+1)}{2!}$$

and is

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

obtained by putting $n = 1, 2, 3, \ldots$.

The sequence of tetrahedral numbers has the nth term

$$P_3(n) = \binom{n+2}{3} = \frac{n(n+1)(n+2)}{3!}$$

and is

$$1, 4, 10, 20, 35, 56, 84, 120, 165, 220, \ldots$$

obtained by letting $n = 1, 2, 3, \ldots$.

This pattern continues indefinitely.

2.0.1. *Pascal's Triangles.* One fascinating array of numbers that are important in the study of counting techniques and probability and that arise in algebra is the famous Pascal's Triangle. *Pascal's Triangle*, named in honour of Blaise Pascal for writing a treatise about it's many interesting properties and applications in 1653, is an unending triangular array of numbers in which each number is the sum of the two numbers immediately above it (except for the 1s) [14], [15]:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}$$

and so on. The numbers in each row are the coefficients of the expansion of the binomial $(1+x)^n$ where n is a whole number. Thus,

$$\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1+x \\
(1+x)^2 &= 1+2x+x^2 \\
(1+x)^3 &= 1+3x+3x^2+x^3 \\
&\vdots \\
(1+x)^n &= \binom{n}{o} + \binom{n}{1}x + \binom{n}{2}x^2 + + \binom{n}{n}x^n
\end{aligned}$$

We will not give the proof of the above result. The proof can be found in many algebraic texts. Letting $x = 1$, we get

$$\begin{aligned}
2^n &= \binom{n}{o} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \\
&= \sum_{j=0}^{n} \binom{n}{j}
\end{aligned}$$

We state the above result as follows: *the sum of the numbers in the nth row of the Pascal's Triangle is* 2^n. The Pascal's Triangle can be re-arranged as follows:

1												
1	1											
1	2	1										
1	3	3	1									
1	4	6	4	1								
1	5	10	10	5	1							
1	6	15	20	15	6	1						
1	7	21	35	35	21	7	1					
1	8	28	56	70	56	28	8	1				
1	9	36	84	126	126	84	36	9	1			
1	10	45	120	210	252	210	120	45	10	1		
1	11	55	165	330	462	462	330	165	55	11	1	
1	12	66	220	495	792	924	792	495	220	66	12	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The $(r+1)$th column of the Pascal's Triangle where $r = 0, 1, 2, 3, \ldots$ gives the r-polytopic numbers. Thus,

- The point numbers are the column 1 numbers
- The linear numbers are the column 2 numbers
- The triangular numbers are the column 3 numbers
- The tetrahedral numbers are the column 4 numbers
- The pentatope numbers are the column 5 numbers

and so on.The numbers in each column are the coefficients of the expansion of the binomial $(1-x)^{-r-1}$ where r is a non- negative integer. Thus,

$$\begin{aligned}
(1-x)^{-1} &= 1 + x + x^2 + x^3 + \ldots \\
(1-x)^{-2} &= 1 + 2x + 3x^2 + 4x^3 + \ldots \\
(1-x)^{-3} &= 1 + 3x + 6x^2 + 10x^3 + \ldots \\
&\vdots \\
(1-x)^{-r-1} &= P_r(1) + P_r(2)x + P_r(3)x^2 + P_r(4)x^3 + \ldots
\end{aligned}$$

2.0.2. *Generalized Polytopic Numbers.* One enchanting feature of these figurate numbers is that if the nth term of a sequence of any given r- figurate numbers be added to the $(n+1)$th term of the sequence of the preceding r- figurate numbers, the sum will be equal to the $(n+1)$th term of the sequence of the given r- figurate numbers. As an instance, let us take two sequences of the triangular numbers and the tetrahedral numbers:

$$1, \quad 3, \quad 6, \quad 10, \quad 15, \quad \ldots$$

$$1, \quad 4, \quad 10, \quad 20, \quad 35, \quad \ldots.$$

Here, if we add to any term in the upper sequence that term in the lower which stands one place to the left, the sum is the next term in the lower sequence. Starting with 6 sequences of 1's, all of the sequences of figurate numbers may be deduced in succession by the aid of this principle:

$r=0$	:	1	1	1	1	1	1	1	1
$r=1$	:	1	2	3	4	5	6	7	8
$r=2$	:	1	3	6	10	15	21	28	36
$r=3$	:	1	4	10	20	35	56	84	120
$r=4$	:	1	5	15	35	70	126	210	330
$r=5$	:	1	6	21	56	126	252	462	792.

The author, employing Bhaskara's arithmetic of zero and infinity to include negative integers of r [14], [15] , generalises the polytopic numbers so that the pathern above becomes

$r=-6$	:	1	-5	10	-10	5	-1		
$r=-5$	:	1	-4	6	-4	1			
$r=-4$	:	1	-3	3	-1				
$r=-3$	:	1	-2	1					
$r=-2$	:	1	-1						
$r=-1$	:	1							
$r=0$	:	1	1	1	1	1	1	1	1
$r=1$	:	1	2	3	4	5	6	7	8
$r=2$	:	1	3	6	10	15	21	28	36
$r=3$	:	1	4	10	20	35	56	84	120
$r=4$	:	1	5	15	35	70	126	210	330
$r=5$	:	1	6	21	56	126	252	462	792.

3. Sum of Polytopic Numbers

It is a well known fact that the sum of the first n terms of a sequence of r-polytopic numbers is the nth term of a sequence of $(r+1)$-polytopic numbers. Thus,

$$\sum_{k=1}^{n} P_r(n) = P_{r+1}(n).$$

Thus, we have

$$\sum_{k=1}^{n} P_0(k) = P_1(n)$$
$$\sum_{k=1}^{n} P_1(k) = P_2(n)$$
$$\sum_{k=1}^{n} P_2(k) = P_3(n)$$
$$\sum_{k=1}^{n} P_3(k) = P_4(n)$$
$$\sum_{k=1}^{n} P_4(k) = P_5(n)$$

and so on.

4. Alternating Sum of Polytopic Numbers

The general formula for finding the alternating sum of the first n r-polytopic numbers is

$$\sum_{k=1}^{n}(-1)^{k-1}P_r(k) = \frac{1}{2^{r+1}}\left(1-(-1)^n\sum_{j=0}^{r}2^j P_j(n)\right).$$

Thus, we have the results:

$$\sum_{k=1}^{n}(-1)^{k-1}P_0(k) = \frac{1}{2}\left[1-(-1)^n\right]$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_1(k) = \frac{1}{4}\left\{1-(-1)^n[1+2P_1(n)]\right\}$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_2(k) = \frac{1}{8}\left\{1-(-1)^n[1+2P_1(n)+4P_2(n)]\right\}$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_3(k) = \frac{1}{16}\left\{1-(-1)^n[1+2P_1(n)+4P_2(n)+8P_3(n)]\right\}$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_4(k) = \frac{1}{32}\left\{1-(-1)^n[1+2P_1(n)+4P_2(n)+8P_3(n)+16P_4(n)]\right\}$$

and so on. The coefficients of the polytopic numbers are given in Table 1.

r	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
0	1									
1	1	2								
2	1	2	4							
3	1	2	4	8						
4	1	2	4	8	16					
5	1	2	4	8	16	32				
6	1	2	4	8	16	32	64			
7	1	2	4	8	16	32	64	128		
8	1	2	4	8	16	32	64	128	256	
9	1	2	4	8	16	32	64	128	256	512

TABLE 1. Coefficients of Polytopic Numbers

The following results follow from the above results:

$$\sum_{k=1}^{n}(-1)^{k-1}P_0(k) = \frac{1}{2}\left[1-(-1)^nP_0(n)\right]$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_1(k) = \frac{1}{4}\left\{1-(-1)^n[3P_1(n)-P_1(n-1)]\right\}$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_2(k) = \frac{1}{8}\left\{[1-(-1)^n[7P_2(n)-4P_2(n-1)+P_2(n-2)]\right\}$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_3(k) = \frac{1}{16}\left\{1-(-1)^n[15P_3(n)-11P_3(n-1)+5P_3(n-2)-P_3(n-3)]\right\}$$
$$\sum_{k=1}^{n}(-1)^{k-1}P_4(k) = \frac{1}{32}\left\{1-(-1)^n[31P_4(n)-26P_4(n-1)+16P_4(n-2)-6P_4(n-3)+P_4(n-4)]\right\}$$

The coefficients of the polytopic numbers in the results above are shown in Table 1

r	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
0	1									
1	3	-1								
2	7	-4	1							
3	15	-11	5	-1						
4	31	-26	16	-6	1					
5	63	-57	42	-22	7	-1				
6	127	-120	99	-64	28	-8	1			
7	255	-247	219	-163	93	37	-9	1		
8	511	-502	466	-382	256	-130	46	-10	1	
9	1023	-1013	968	-848	638	-386	176	-56	11	-1

TABLE 2. Coefficients of Polytopic Numbers

5. Sum of Reciprocals of Polytopic Numbers

The sum of the reciprocals of the first n r-polytopic numbers is

$$\sum_{k=1}^{n} \frac{1}{P_r(n)} = \frac{r}{r-1}\left(1 - \frac{1}{Pn(r)}\right)$$

Thus, we have the following results:

$$\sum_{k=1}^{n} \frac{1}{P_2(k)} = \frac{2}{1}\left(1 - \frac{1}{P_n(2)}\right)$$

$$\sum_{k=1}^{n} \frac{1}{P_3(k)} = \frac{3}{2}\left(1 - \frac{1}{P_n(3)}\right)$$

$$\sum_{k=1}^{n} \frac{}{1} P_4(k) = \frac{4}{3}\left(1 - \frac{1}{P_n(4)}\right)$$

$$\sum_{k=1}^{n} \frac{1}{P_5(k)} = \frac{5}{4}\left(1 - \frac{1}{P_n(5)}\right)$$

$$\sum_{k=1}^{n} \frac{1}{P_6(k)} = \frac{6}{5}\left(1 - \frac{1}{P_n(6)}\right)$$

and so forth.

When n increases indefinitely, the general formula becomes

$$\sum_{k=1}^{\infty} \frac{1}{P_r(k)} = \frac{r}{r-1}$$

which is the sum to infinity of the reciprocals of the r-polytopic numbers. Thus, the sum to infinity of the reciprocals of the triangular numbers is 2; of the tetrahedral numbers is 3/2; of the pentatope numbers is 4/3; and so on.

The sum of the first n reciprocals of the r-polytopic numbers can also be expressed in partial fractions as follows:

$$\sum_{k=1}^{n} \frac{1}{P_r(n)} = \frac{r}{r-1} - r\sum_{j=0}^{r-2} \frac{(-1)^{j}\,{}^{r-2}C_j}{n+j+1}.$$

Setting $r = 2, 3, 4, \ldots$, we get the sum of the first n reciprocals of the

$$\sum_{k=1}^{n} \frac{1}{P_2(k)} = 2 - 2\left(\frac{1}{n+1}\right)$$
$$\sum_{k=1}^{n} \frac{1}{P_3(n)} = \frac{3}{2} - 3\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$
$$\sum_{k=1}^{n} \frac{1}{P_4(n)} = \frac{4}{3} - 4\left(\frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+1}\right)$$

and so forth.

The bracketed alternating partial fractions form an easily remembered pattern; their unsigned numerators are numbers in the $r-1$th row of the famous Pascal's triangle and their denominators align themselves in the order $n+1, n+2, n+3, \ldots n+r-1$. Thus the next sum is

$$\sum_{k=1}^{n} \frac{1}{P_5(n)} = \frac{5}{4} - 5\left(\frac{1}{n+1} - \frac{3}{n+2} + \frac{3}{n+3} - \frac{1}{n+4}\right).$$

6. Alternating Sum of Reciprocals of Polytopic Numbers

6.1. Alternating sum of the first n reciprocals of r-polytopic numbers. The general formula for the alternating sum of the first n reciprocals of r-polytopic numbers is

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{P_r(k)} = r\left[2^{r-1} H'_n - (-1)^r \sum_{k=1}^{r-1} \left(\frac{t_k}{k} + (-1)^n \frac{t_k}{n+k}\right)\right]$$

where t_k is obtained from the recursive relation

$$t_k = t_{k-1} -^{r-1} C_{k-1}, t_1 = 2^{r-1} - 1.$$

The symbol H'_n represents the nth alternating harmonic number and is,therefore, given as

$$H'_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n}.$$

An asymptotic formula for the nth alternating harmonic number is

$$H'_n = \ln 2 + (-1)^{n-1}\frac{1}{2q} - \frac{1}{4}\sum_{k=1}^{\infty} \frac{G_{2k}}{kq^2}$$

where

$$q = \frac{1}{2}\left(2p - 1 + (-1)^p\right)$$

and G_p is the pth Genocchi number [8], [16], [17], [18]. Computing the altrnating sum for $r = 1, 2, 3, \ldots$ gives

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{P_1(k)} = H'_n$$

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{P_2(k)} = 2\left(2H'_n - 1 + (-1)^n\frac{1}{n+1}\right)$$

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{P_3(k)} = 3\left[4H'_n - \frac{5}{2} + (-1)^n\left(\frac{3}{n+1} - \frac{1}{n+2}\right)\right]$$

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{P_4(k)} = 4\left[8H' - \frac{16}{3} + (-1)^n\left(\frac{7}{n+1} - \frac{4}{n+2} + \frac{1}{n+3}\right)\right]$$

and so forth.

6.1.1. *Proof of the Asymptotic Expansion for the nth Alternating Harmonic Number.* Here we derive the asymptotic formula for the alternating harmonic number H'_p. We begin with the sum of the reciprocals of the first even numbers from 2 to n:

$$\sum_{k=2,4,6,\ldots}^{n}\frac{1}{k} = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\frac{n}{2}}\right) = \frac{1}{2}H_{\frac{n}{2}}.$$

The sum of the reciprocals of the first odd numbers from 1 to n is therefore

$$\sum_{k=1,3,5,\ldots}^{n}\frac{1}{k} = \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n-1}\right) = H_n - \frac{1}{2}H_{\frac{n-1}{2}}.$$

If n is odd, the alternating sum of the reciprocals of the first n natural numbers is thus:

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} &= \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{n-1}\right)\\ &= \left(H_n - \frac{1}{2}H_{\frac{n-1}{2}}\right) - \left(\frac{1}{2}H_{\frac{n-1}{2}}\right) = H_n - H_{\frac{n-1}{2}}.\end{aligned}$$

Similarly, if n is even, the alternating sum of the reciprocals of the first n natural numbers is

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} &= \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{n}\right)\\ &= \left(H_{n-1} - \frac{1}{2}H_{\frac{n-2}{2}}\right) - \left(\frac{1}{2}H_{\frac{n}{2}}\right)\\ &= \left(H_n - \frac{1}{n} - \left(\frac{1}{2}H_{\frac{n}{2}} - \frac{1}{n}\right)\right) - \left(\frac{1}{2}H_{\frac{n}{2}}\right) = H_n - H_{\frac{n}{2}}.\end{aligned}$$

In general, if n is any natural number, the alternating sum of the reciprocals of the first n natural numbers is

$$H'_n = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} = \begin{cases} H_n - H_{\frac{n}{2}} & \text{if } n \text{ is even}\\ H_n - H_{\frac{n-1}{2}} & \text{if } n \text{ is odd}\end{cases}. \tag{6.1}$$

Now we are in a position to prove our Theorem 1. We begin with (6.1) and let

$$H'_n = H_n - H_N$$

where $N = \frac{n}{2}$ if n is even and $N = \frac{n-1}{2}$ if n is odd. We have

$$\begin{aligned} H'_n &= \left(\ln n + \gamma + \frac{1}{2n} + \sum_{k=1}^{\infty} \frac{\zeta(-k)}{n^{k+1}}\right) - \left(\ln N + \gamma + \frac{1}{2N} + \sum_{k=1}^{\infty} \frac{\zeta(-k)}{N^{k+1}}\right) \\ &= \ln n - \ln N + \frac{1}{2n} - \frac{1}{2N} + \sum_{k=1}^{\infty} \frac{\zeta(-k)}{n^{k+1}} - \sum_{k=1}^{\infty} \frac{\zeta(-k)}{N^{k+1}} \\ &= \ln \frac{n}{N} + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{N}\right) + \sum_{k=1}^{\infty} \zeta(-k)\left(\frac{1}{n^{k+1}} - \frac{1}{N^{k+1}}\right). \end{aligned}$$

If n is even, then $N = \frac{n}{2}$ and

$$\begin{aligned} H'_n &= \ln\left(\frac{n}{\frac{n}{2}}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{\frac{n}{2}}\right) + \sum_{k=1}^{\infty} \zeta(-k)\left(\frac{1}{n^{k+1}} - \frac{1}{\left(\frac{n}{2}\right)^{k+1}}\right) \\ &= \ln 2 + \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n}\right) + \sum_{k=1}^{\infty} \zeta(-k)\left(\frac{1}{n^{k+1}} - \frac{2^{k+1}}{n^{k+1}}\right) \\ &= \ln 2 - \frac{1}{2n} + \sum_{k=1}^{\infty} \left(1 - 2^{k+1}\right)\frac{\zeta(-k)}{n^{k+1}}. \end{aligned}$$

If n is even, then $n-1$ is odd and

$$\begin{aligned} H'_{n-1} &= H'_n + \frac{1}{n} \\ &= \left(\ln 2 - \frac{1}{2n} + \sum_{k=1}^{\infty} \left(1 - 2^{k+1}\right)\frac{\zeta(-k)}{n^{k+1}}\right) + \frac{1}{n} \\ &= \ln 2 + \frac{1}{2n} + \sum_{k=1}^{\infty} \left(1 - 2^{k+1}\right)\frac{\zeta(-k)}{n^{k+1}}. \end{aligned} \tag{6.2}$$

Letting $n - 1 = m$ in (6.2) gives $n = m + 1$. Thus,

$$H'_m = \ln 2 + \frac{1}{2(m+1)} + \sum_{k=1}^{\infty} \left(1 - 2^{k+1}\right)\frac{\zeta(-k)}{(m+1)^{k+1}}.$$

In general, if p is any non-zero positive integer, whether even or odd, the alternating harmonic number is

$$H'_p = \ln 2 + (-1)^{p-1}\frac{1}{2q} + \sum_{k=1}^{\infty} \left(1 - 2^{k+1}\right)\frac{\zeta(-k)}{q^{k+1}} \tag{6.3}$$

where

$$q = \begin{cases} p & \text{if } p \text{ is even} \\ p+1 & \text{if } p \text{ is odd} \end{cases}.$$

Let us employ the following relation to obtain a single formula for q:

$$B + \frac{A-B}{2}\left(1 + (-1)^n\right) = \begin{cases} A & \text{if } n \text{ is even} \\ B & \text{if } n \text{ is odd} \end{cases}.$$

Thus,

$$q = (p+1) + \frac{(p-(p+1))}{2}\left(1+(-1)^p\right).$$

Simplifying gives

$$q = \frac{1}{2}\left(2p+1-(-1)^p\right).$$

It is well–known fact that the nth Genocchi Number G_n is related to the nth Bernoulli Number B_n [**?**], [**?**], which in turn is related to the zeta constant $\zeta(1-n)$ [**?**], [**?**], [**?**]. The relations are given as

$$G_n = 2\left(1-2^n\right)B_n \tag{6.4}$$

and

$$B_n = -n\zeta(1-n). \tag{6.5}$$

Eliminating B_n from (6.4) and (6.5) gives $G_n = -2\left(1-2^n\right)n\zeta(1-n)$. Let $1-n = -k$. Therefore, $n = k+1$. So,

$$G_{k+1} = -2\left(1-2^{k+1}\right)(k+1)\,\zeta(-k). \tag{6.6}$$

From this we have

$$\zeta(-k) = \frac{G_{k+1}}{-2\,(k+1)\left(1-2^{k+1}\right)}. \tag{6.7}$$

Substituting (6.7) into (6.3), we obtain

$$\begin{aligned} H'_p &= \ln 2 + (-1)^{p-1}\frac{1}{2q} + \sum_{k=1}^{\infty}\left(1-2^{k+1}\right)\frac{G_{k+1}}{-2(k+1)(1-2^{k+1})q^{k+1}} \\ &= \ln 2 + (-1)^{p-1}\frac{1}{2q} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{G_{k+1}}{(k+1)q^{k+1}}. \end{aligned}$$

Letting $k+1 = 2k$, we get

$$H'_p = \ln 2 + (-1)^{p-1}\frac{1}{2q} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{G_{2k}}{kq^{2k}}.$$

6.2. **Alternating sum to infinity of the reciprocals of r–polytopic numbers.** The alternating sum to infinity of the reciprocals of r–polytopic numbers is

$$\sum_{k=}^{\infty}(-1)^{k-1}\frac{1}{P_r(k)} = r\left[N_{r-1} + K_{r-1}\right]$$

where N_p is the naira function in p defined by

$$N_p = 2^p\left(\ln 2 - H_p\right)$$

and K_p is the kobo function in p defined by

$$K_p = \sum_{k=0}^{p-1}\frac{\sum_{j=0}^{k}\binom{j}{p}}{p-k}$$

For $p = 1, 2, 3, \ldots, 10$ the values of the naira and kobo functions are given in Table 3. Thus, we get the following results:

p	N$_p$	K$_p$
0	$\ln 2$	0
1	$2\ln 2 - 2$	1
2	$4\ln 2 - 6$	7/3
3	$8\ln 2 - 44/3$	7/3
4	$16\ln 2 - 100/3$	269/12
5	$32\ln 2 - 1096/15$	1531/30
6	$64\ln 2 - 784/5$	3377/30
7	$128\ln 2 - 11616/35$	25544/105
8	$256\ln 2 - 24352/35$	435479/840
9	$512\ln 2 - 456256/315$	1377977/9260
10	$1024\ln 2 - 944768/315$	721213/315

TABLE 3. Values of N_p and K_p

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{P_1(k)} = \ln 2$$
$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{P_2(k)} = 2(2\ln 2 - 1)$$
$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{P_3(k)} = 3\left(4\ln 2 - \frac{5}{2}\right)$$
$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{P_4(k)} = 4\left(8\ln 2 - \frac{2}{3}\right)$$
$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{P_5(k)} = 5\left(16\ln 2 - \frac{131}{12}\right)$$

and so on.

7. Product of Polytopic Numbers

The product of the first n r–polytopic numbers is

$$\prod_{k=1}^{n} P_r(k) = \frac{(n!)^r}{n^r (r!)^{n-1}} \prod_{m=1}^{r} P_m(n).$$

Hence, we have the following results:

$$\prod_{k=1}^{n} P_2(k) = \frac{n+1}{1!!(2!)^n}(n!)^2$$
$$\prod_{k=1}^{n} P_3(k) = \frac{(n+1)^2(n+2)}{2!!(3!)^n}(n!)^3$$
$$\prod_{k=1}^{n} P_4(k) = \frac{(n+1)^3(n+2)^2(n+3)}{3!!(4!)^n}(n!)^4$$
$$\prod_{k=1}^{n} P_5(k) = \frac{(n+1)^4(n+2)^3(n+3)^2(n+4)}{4!!(5!)^n}(n!)^5$$
$$\prod_{k=1}^{n} P_6(k) = \frac{(n+1)^5(n+2)^4(n+3)^3(n+4)^2(n+5)}{5!!(6!)^n}(n!)^6$$

and so forth.

8. Identities Relating to Polytopic Numbers

8.1. **Sums of Powers.**

8.1.1. *Type 1.*

$$\sum_{k=1}^{n} k = P_2(n)$$
$$\sum_{k=1}^{n} k^2 = 2P_3(n) - P_2(n)$$
$$\sum_{k=1}^{n} k^3 = 6P_4(n) - 6P_3(n) + P_2(n)$$
$$\sum_{k=1}^{n} k^4 = 24P_5(n) - 36P_4(n) + 14P_3(n) - P_2(n)$$
$$\sum_{k=1}^{n} k^5 = 120P_6(n) - 240P_5(n) + 150P_4(n) - 30P_3(n) + P_2(n)$$

and so forth.

8.1.2. *Type 2.*

$$\sum_{k=1}^{n} k = P_2(n)$$
$$\sum_{k=1}^{n} k^2 = P_3(n) + P_3(n-1)$$
$$\sum_{k=1}^{n} k^3 = P_4(n) + 4P_4(n-1) + P_4(n-2)$$
$$\sum_{k=1}^{n} k^4 = P_5(n) + 11P_5(n-1) + 11P_5(n-2) + P_5(n-3)$$
$$\sum_{k=1}^{n} k^5 = P_6(n) + P_6(n-1) + P_6(n-2) + P_6(n-3) + P_6(n-4)$$

and so on.

8.2. **Alternating Sums of Integer Powers.**

$$\sum_{k=1}^{n} (-1)^{k-1} k = \frac{1}{4}\left\{1 - (-1)^n [1 + 2P_1(n)]\right\}$$
$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n-1} P_2(n)$$
$$\sum_{k=1}^{n} (-1)^{k-1} k^3 = -\frac{1}{8} = \left\{1 + (-1)^n [1 + 2P_1(n) + 24P_2(n) - 24P_3(n)]\right\}$$
$$\sum_{k=1}^{n} (-1)^{k-1} k^4 = (-1)^n [P_2(n) - 12P_3(n) + 12P_4(n)]$$

and so on.

8.3. **Sums of Powers of Reciprocals of Natural Numbers.**

$$\sum_{k=1}^{\infty} \frac{1}{k} = \ln(-1)! = -\ln 0$$
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}$$
$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3) = 1.2021\ldots$$
$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \zeta(4) = \frac{\pi^4}{90}$$

and so forth.

We can obtain approximate formulas for the sums of powers of the reciprocals of the first n natural numbers. The first few cases are as follows:

$$\sum_{k=1}^{n} \frac{1}{k} \approx \gamma + \ln n$$
$$\sum_{k=1}^{n} \frac{1}{k^2} \approx \zeta(2) - \frac{\zeta(2)}{\zeta(2)n+1}$$
$$\sum_{k=1}^{n} \frac{1}{k^3} \approx \zeta(3) - \frac{1}{2}\left\{\frac{4\zeta(3)^2}{4\zeta(3)^2 n+1}\right\}^2$$
$$\sum_{k=1}^{n} \frac{1}{k^4} \approx \zeta(4) - \frac{1}{3}\left\{\frac{27\zeta(4)^3}{27\zeta(4)^3 n+1}\right\}^3.$$

The general approximate formula for the sums of powers of the reciprocals of the first n natural numbers is given as

$$\sum_{k=1}^{n} \frac{1}{k^r} \approx \zeta(r) - \frac{1}{r-1}\left\{\frac{[(r-1)\zeta(r)]^{r-1}}{[(r-1)\zeta(r)]^{r-1} n+1}\right\}^{r-1}$$

8.4. **Alternating Sums of Reciprocal of Powers of Natural Numbers.** These include

$$\sum_{k=1}^{\infty} \frac{1}{k} = \ln 2$$
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2}\zeta(2)$$
$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{3}{4}\zeta(3)$$
$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{7}{8}\zeta(4)$$

and so forth.

8.5. **Sums of Reciprocal Powers of Triangular Numbers.** These can be obtained as follows. We begin with the familiar sums of reciprocal powers:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \zeta(1),$$
$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2),$$
$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3),$$

where $\zeta(n)$ is a zeta constant. We can, by a stylish little manipulation of the sums above, show that

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1,$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+\frac{1}{(k+1)^2}\right)=2\zeta(1)-1,$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^3}-\frac{1}{(k+1)^3}\right)=1.$$

To find the sum $\sum_{k=1}^{\infty}\frac{1}{P_2(k)^2}$ where $P_2(k)$ is the kth triangular number, we first resolve $\frac{1}{k^2(k+1)^2}$ into partial fractions and get

$$\frac{1}{k^2(k+1)^2}=1\left(\frac{1}{k^2}+\frac{1}{(k+1)^2}\right)-2\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

which becomes, after finding the sum to infinity of both sides,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2}&=1\sum_{k=1}^{\infty}\left(\frac{1}{k^2}+\frac{1}{(k+1)^2}\right)-2\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\&=1\left(2\zeta(2)-1\right)-2\left(1\right)\\&=2\zeta(2)-3\end{aligned}$$

We multiply both sides of the above equation by 2^2 and get the sum of reciprocals of squares of all the triangular numbers as

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{4}{k^2(k+1)^2}&=4\left(2\zeta(2)-3\right)\\&=4\left(2\cdot\frac{\pi^2}{6}-3\right)\\\sum_{k=1}^{\infty}\frac{1}{P_2(k)^2}&=\frac{4}{3}\left(\pi^2-9\right).\end{aligned}$$

Similarly, we find the sum of reciprocals of cubes of all the triangular numbers as follows:

$$\frac{1}{k^3(k+1)^3} = 1\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) - 3\left(\frac{1}{k^2} + \frac{1}{(k+1)^2}\right) + 6\left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^3(k+1)^3} = 1\sum_{k=1}^{\infty}\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right) - 3\sum_{k=1}^{\infty}\left(\frac{1}{k^2} + \frac{1}{(k+1)^2}\right) + 6\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= 1\,(1) - 3\,(2\zeta(2) - 1) + 6\,(1)$$

$$= 10 - 6\zeta(2).$$

We multiply both sides of the above equation by 2^3 and get the sum of reciprocals of cubes of all the triangular numbers as

$$\sum_{k=1}^{\infty} \frac{8}{k^3(k+1)^3} = 8\,(10 - 6\zeta(2))$$

$$= 8\left(10 - 6\left(\frac{\pi^2}{6}\right)\right)$$

$$\sum_{k=1}^{\infty} \frac{1}{P_2(k)^3} = 8\left(10 - \pi^2\right).$$

We can obtain the sum of reciprocals of any power of all the triangular numbers in a similar fashion. For example, the sum of reciprocals of the 4th power of all the triangular numbers is

$$\sum_{k=1}^{\infty} \frac{1}{P_2(k)^4} = \frac{16}{15}\left(\pi^4 + 50\pi^2 - 525\right).$$

In general, the sum of reciprocals of any power r of all the triangular numbers is

$$\sum_{k=1}^{\infty} \frac{1}{P_2(k)^r} = 2^r \sum_{j=1}^{r} (-1)^{j-1}\binom{r+j-2}{r-1}\left(\zeta(r-j+1) + (-1)^{r-j+1}\zeta(r-j+1) - (-1)^{r-j+1}\right)$$

where $\zeta(s)$ is the Riemann zeta function and $\binom{n}{r}$ is a binomial coefficient.

8.6. **Alternating Sums of Reciprocal of Powers of Triangular Numbers.**

(1) The alternating sum of the reciprocals of all the triangular numbers converges to $2(2\ln 2 - 1)$:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{P_2(k)} = 2(2\ln 2 - 1)$$

(2) The alternating sum of the reciprocals of the squares of all the triangular numbers converges to $4(3 - 4\ln 2)$:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{P_2(k)^2} = 4(3 - 4\ln 2)$$

(3) The alternating sum of the reciprocals of the first n triangular numbers is related to the alternating harmonic number H'_n by

$$\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{P_2(k)} = 4H'_n - (-1)^{n+1}\frac{2}{n+1} - 2.$$

8.7. **Relations of $P_r(kn)$.** For $r = 2$:

$$\begin{aligned}
P_2(2n) &= 3P_2(n) + P_2(n-1)\\
P_2(3n) &= 6P_2(n) + 3P_2(n-1)\\
P_2(4n) &= 10P_2(n) + 6P_2(n-1)\\
P_2(5n) &= 15P_2(n) + 21P_2(n)\\
&\vdots\\
P_2(kn) &= P_2(k)P_2(n) + P_2(k-1)P_2(n-1)
\end{aligned}$$

For $r = 3$:

$$\begin{aligned}
P_3(2n) &= 4P_3(n) + 4P_3(n-1)\\
P_3(3n) &= 10P_3(n) + 16P_3(n-1) + P_3(n-2)\\
P_3(4n) &= 20P_3(n) + 40P_3(n-1) + 4P_3(n-2)\\
P_3(5n) &= 35P_3(n) + 80P_3(n-1) + 10P_3(n-2)\\
&\vdots\\
P_3(kn) &= P_3(k)P_3(n) + 4P_3(k-1)P_3(n-1) + P_3(k-2)P_3(n-2)
\end{aligned}$$

8.8. **Relations of $P_r(n-m)$.**

$$\begin{aligned}
P_r(n-1) &= P_r(n) - P_{r-1}(n)\\
P_r(n-2) &= P_r(n) - 2P_{r-1}(n) + P_{r-2}(n)\\
P_r(n-3) &= P_r(n) - 3P_{r-1}(n) + 3P_{r-2}(n) - P_{r-3}(n)\\
P_r(n-4) &= P_r(n) - 4P_{r-1}(n) + 6P_{r-2}(n) - 4P_{r-3}(n) + P_{r-4}(n)
\end{aligned}$$

and so forth. In general, we have

$$P_r(n-m) = \sum_{k=0}^{m}\binom{k}{m}P_{r-k}(n).$$

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