On Some Inequalities for the Chaudhry-Zubair Extension of the Gamma Function

Original
Research Article

Abstract

By applying the classical Holder's inequality, Young's inequality, Minkowski's inequality and some other analytical tools, we establish some inequalities involving the Chaudhry-Zubair extension of the gamma function. The established results serve as generalizations to some known results in the literature.

1   Introduction

The classical Euler's gamma function, which is an extension of the factorial notation to non-integer values, is usually defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0. \tag{1}$$

It was first defined by Leonhard Euler and later studied by several renowned mathematicians. Since its introduction, the gamma has found and continue find useful applications in almost all branches of mathematics. As a result of its importance and wide applicable areas, it has been studied extensively. Several extesions and generalizations have also been established. See for example [1], [2], [6], [7], [8], [11], [12], [13], [15], [29], [33]. Closely associated with the gamma function, is the digamma function which is defined as

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

In this paper, our focus is on the Chaudhry-Zubair extension of the gamma function which is defined as [2]

$$\Gamma_p(x) = \int\limits_0^\infty t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt, \quad p > 0, x > 0, \tag{2}$$

where, $\Gamma_p(x)$ tends to the classical gamma function $\Gamma(x)$ when $p = 0$. It satisfies the following identities

$$\Gamma_p(x+1) = x\Gamma_p + p\Gamma_p(x-1),$$
$$\Gamma_p(-x) = p^{-x}\Gamma_p(x).$$

The Chaudhry-Zubair extension of the gamma function has attracted the attention of several researchers and it has be investigated in diverse ways (see [4], [3], [5], [9], [10], [14], [16], [19], [20], [21], [31], [32], and the related references therein).

By differentiating (2) repeatedly, one obtains

$$\Gamma_p^{(n)}(x) = \int\limits_0^\infty (\ln t)^n t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt, \quad p > 0 \tag{3}$$

where $n \in \mathbb{N}_0$. It is clear that $\Gamma_p^{(n)}(x)$ returns to $\Gamma_p(x)$ when $n = 0$. Here, and for the rest of the paper, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$.

Our aim in this paper is, to establish some properties such as log-convexity, monotonicity and inequalities concerning the Chaudhry-Zubair extension of the gamma function. The results established serve as generalizations of some known results in the literature.

2 Preliminaries

The following results are well known in the scientific community.

Lemma 2.1 (Holder's Inequality). *Let $r > 1$, $s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a, b]$, then inequality*

$$\int_a^b |f(t)g(t)| \; dt \leq \left(\int_a^b |f(t)|^r \; dt\right)^{\frac{1}{r}} \left(\int_a^b |g(t)|^s \; dt\right)^{\frac{1}{s}}, \tag{4}$$

holds.

Lemma 2.2 (Young's Inequality). *If $u \geq 0$, $v \geq 0$, $\alpha, \beta \in (0,1)$ such that $\alpha + \beta = 1$, then inequality*

$$u^\alpha v^\beta \leq \alpha u + \beta v, \tag{5}$$

holds.

Lemma 2.3 (Minkowski's Inequality). *Let $u \geq 1$. If $f(t)$ and $g(t)$ are continuous real-valued functions on $[a, b]$, then inequality*

$$\left(\int_a^b |f(t) + g(t)|^u \; dt\right)^{\frac{1}{u}} \leq \left(\int_a^b |f(t)|^u \; dt\right)^{\frac{1}{u}} + \left(\int_a^b |g(t)|^u \; dt\right)^{\frac{1}{u}}, \tag{6}$$

holds .

Definition 2.4. A function $f : I \to (0, \infty)$ is said to be log-convex if $\ln f$ is convex on I. That is if

$$\ln f(\alpha x + \beta y) \leq \alpha \ln f(x) + \beta \ln f(y)$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^\alpha (f(y))^\beta$$

for each $x, y \in I$ and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$.

3 Results and Discussion

Theorem 3.1. *The Chaudhry-Zubair extended gamma function satisfies the inequality*

$$\Gamma_p^{(\alpha m + \beta n)}(\alpha x + \beta y) \leq \left[\Gamma_p^{(m)}(x)\right]^{\alpha} \left[\Gamma_p^{(n)}(y)\right]^{\beta}, \tag{7}$$

where, $x > 0$, $y > 0$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$, and $m, n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. Let $x > 0$, $y > 0$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$, and $m, n \in \{2s : s \in \mathbb{N}_0\}$. Then by using (3), we obtain

$$\Gamma_p^{(\alpha m + \beta n)}(\alpha x + \beta y) = \int_0^\infty (\ln t)^{\alpha m + \beta n} t^{(\alpha x + \beta y) - (\alpha + \beta)} e^{\left(-t - \frac{p}{t}\right)(\alpha + \beta)} dt$$

$$= \int_0^\infty (\ln t)^{m\alpha} (\ln t)^{n\beta} t^{\alpha(x-1)} t^{\beta(y-1)} e^{\alpha\left(-t - \frac{p}{t}\right)} e^{\beta\left(-t - \frac{p}{t}\right)} dt$$

$$= \int_0^\infty (\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha\left(-t - \frac{p}{t}\right)} (\ln t)^{\beta n} t^{\beta(y-1)} e^{\beta\left(-t - \frac{p}{t}\right)} dt,$$

By virtue of the Holder's inequality, we have

$$\int_0^\infty (\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha\left(-t - \frac{p}{t}\right)} (\ln t)^{\beta n} t^{\beta(y-1)} e^{\beta\left(-t - \frac{p}{t}\right)} dt$$

$$\leq \left[\int_0^\infty \left[(\ln t)^{\alpha m} t^{\alpha(x-1)} e^{\alpha\left(-t - \frac{p}{t}\right)}\right]^{\frac{1}{\alpha}} dt\right]^{\alpha} \left[\int_0^\infty \left[(\ln t)^{\beta n} t^{\beta(y-1)} e^{\beta\left(-t - \frac{p}{t}\right)}\right]^{\frac{1}{\beta}} dt\right]^{\beta}$$

$$= \left[\int_0^\infty (\ln t)^m t^{x-1} e^{\left(-t - \frac{p}{t}\right)} dt\right]^{\alpha} \left[\int_0^\infty (\ln t)^n t^{(y-1)} e^{\left(-t - \frac{p}{t}\right)} dt\right]^{\beta}$$

$$= \left[\Gamma_p^{(m)}(x)\right]^{\alpha} \left[\Gamma_p^{(n)}(y)\right]^{\beta}.$$

Hence,

$$\Gamma_p^{(\alpha m + \beta n)}(\alpha x + \beta y) \leq \left[\Gamma_p^{(m)}(x)\right]^{\alpha} \left[\Gamma_p^{(n)}(y)\right]^{\beta}.$$

$\square$

Corollary 3.2. *The Chaudhry-Zubair's extended function satisfies the inequality*

$$\Gamma_p^{(n)}(\alpha x + \beta y) \leq \left[\Gamma_p^{(n)}(x)\right]^{\alpha} \left[\Gamma_p^{(n)}(y)\right]^{\beta}, \tag{8}$$

where, $x > 0$, $y > 0$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$ and $n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. Let $m = n$ in Theorem 3.1. $\square$

Corollary 3.3. *The inequality*

$$\Gamma_p^{\left(\frac{m+n}{2}\right)}\left(\frac{x+y}{2}\right) \leq \sqrt{\Gamma_p^{(m)}(x)\Gamma_p^{(n)}(y)}, \tag{9}$$

holds for $x > 0$, $y > 0$ and $m, n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. Let $\alpha = \beta = \frac{1}{2}$ in Theorem 3.1. $\qquad\qquad\square$

Remark 3.4. Corollary 3.3 is a genralization of the results of Mortici as presented in Theorem 2.1 of [17].

Corollary 3.5. *The function, $\Gamma_p(x)$ satisfies the inequality*

$$\Gamma_p\left(\alpha x + \beta y\right) \leq \left[\Gamma_p(x)\right]^\alpha \left[\Gamma_p\left(y\right)\right]^\beta, \tag{10}$$

where, $x > 0$, $y > 0$, $\alpha, \beta \in (0, 1)$ and $\alpha + \beta = 1$.

Proof. Let $m = n = 0$ in Theorem 3.1. $\qquad\qquad\square$

Remark 3.6. Corollary 3.2 is another way of saying that the Chuahdry-Zubair gamma function is log-convex. This fact has been established in Theorem 4.1 of [30].

Corollary 3.7. *The function, $\Gamma_p(x)$ satisfies the inequality*

$$\Gamma_p\left(x\right)\Gamma_p''\left(x\right) \geq \left[\Gamma_p'\left(x\right)\right]^2. \tag{11}$$

In other words,

$$\det \left[\begin{array}{cc} \Gamma_p\left(x\right) & \Gamma_p'\left(x\right) \\ \Gamma_p'\left(x\right) & \Gamma_p''\left(x\right) \end{array} \right] \geq 0. \tag{12}$$

Proof. Since $\Gamma_p(x)$ is log-convex, then $\left[\ln \Gamma_p\left(x\right)\right]'' \geq 0$ for all $x > 0$. Then

$$\left[\ln \Gamma_p\left(x\right)\right]'' = \left[\frac{\Gamma_p'\left(x\right)}{\Gamma_p\left(x\right)}\right]' = \frac{\Gamma_p\left(x\right)\Gamma_p''\left(x\right) - \left[\Gamma_p'\left(x\right)\right]^2}{\left[\Gamma_p\left(x\right)\right]^2} \geq 0.$$

Hence

$$\Gamma_p\left(x\right)\Gamma_p''\left(x\right) - \left[\Gamma_p'\left(x\right)\right]^2 \geq 0,$$

which completes the proof. $\qquad\qquad\square$

Corollary 3.8. *Let $\psi_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}, x > 0$ be the Chaudhry-Zubair extension of the digamma. Then $\psi_p(x)$ is increasing.*

Proof. By direct differentiation and by Corollary 3.7, we have,

$$\psi_p'(x) = \left[\frac{\Gamma_p'\left(x\right)}{\Gamma_p\left(x\right)}\right]' \geq 0,$$

which concludes the proof. $\qquad\qquad\square$

Theorem 3.9. *The function $\Gamma_p(x)$ satisfies the inequality*

$$\Gamma_p(x + y) \leq \left[\Gamma_p\left(\frac{x}{\alpha}\right)\right]^\alpha \left[\Gamma_p\left(\frac{y}{\beta}\right)\right]^\beta, \tag{13}$$

for $x > 0$, $y > 0$, $\alpha, \beta \in (0, 1)$ and $\alpha + \beta = 1$.

Proof. By definition (2), we have

$$\Gamma_p(x+y) = \int_0^\infty t^{(x+y)-1} e^{(-t-\frac{p}{t})} dt$$

$$= \int_0^\infty t^{(x+y)-(\alpha+\beta)} e^{(-t-\frac{p}{t})(\alpha+\beta)} dt$$

$$= \int_0^\infty t^{x-\alpha} e^{(-t-\frac{p}{t})\alpha} t^{y-\beta} e^{(-t-\frac{p}{t})\beta} dt,$$

and by employing the Holders inequality, we obtain

$$\int_0^\infty t^{x-\alpha} e^{(-t-\frac{p}{t})\alpha} t^{y-\beta} e^{(-t-\frac{p}{t})\beta} dt$$

$$\leq \left[\int_0^\infty \left[t^{x-\alpha} e^{(-t-\frac{p}{t})\alpha} \right]^{\frac{1}{\alpha}} dt \right]^\alpha \left[\int_0^\infty \left[t^{y-\beta} e^{(-t-\frac{p}{t})\beta} \right]^{\frac{1}{\beta}} dt \right]^\beta$$

$$= \left[\int_0^\infty t^{(\frac{x}{\alpha})-1} e^{(-t-\frac{p}{t})} dt \right]^\alpha \left[\int_0^\infty t^{(\frac{y}{\beta})-1} e^{(-t-\frac{p}{t})} dt \right]^\beta$$

$$= \left[\Gamma_p \left(\frac{x}{\alpha} \right) \right]^\alpha \left[\Gamma_p \left(\frac{y}{\beta} \right) \right]^\beta.$$

Hence,

$$\Gamma_p(x+y) \leq \left[\Gamma_p \left(\frac{x}{\alpha} \right) \right]^\alpha \left[\Gamma_p \left(\frac{y}{\beta} \right) \right]^\beta.$$

$\square$

Corollary 3.10. *The inequality*

$$\Gamma_p(x+y) \leq \alpha \Gamma_p \left(\frac{x}{\alpha} \right) + \beta \Gamma_p \left(\frac{y}{\beta} \right), \tag{14}$$

holds where $x > 0$, $y > 0$, $\alpha, \beta \in (0,1)$ and $\alpha + \beta = 1$.

Proof. From Theorem 3.9 it is obtained that,

$$\Gamma_p(x+y) \leq \left[\Gamma_p \left(\frac{x}{\alpha} \right) \right]^\alpha \left[\Gamma_p \left(\frac{y}{\beta} \right) \right]^\beta, \tag{15}$$

and then by Young's inequality (5), we have

$$\left[\Gamma_p \left(\frac{x}{\alpha} \right) \right]^\alpha \left[\Gamma_p \left(\frac{y}{\beta} \right) \right]^\beta \leq \alpha \Gamma_p \left(\frac{x}{\alpha} \right) \beta \Gamma_p \left(\frac{y}{\beta} \right). \tag{16}$$

Now, combining equations (15) and (16) gives (14). $\square$

Theorem 3.11. *For $a \geq 1$, the function*

$$f(x) = \frac{\Gamma_p(ax)}{[\Gamma_p(x)]^a}, \tag{17}$$

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is increasing on $(0, \infty)$ and as a result, the inequality

$$\left(\frac{\Gamma_p(y)}{\Gamma_p(x)}\right)^a \leq \frac{\Gamma_p(ay)}{\Gamma_p(ax)} \tag{18}$$

is satisfied for $0 < x < y$.

Proof. Let $g(x) = \ln f(x) = \ln \Gamma_p(ax) - a \ln \Gamma_p(x)$ for $x > 0$. Then

$$g(x)' = a\frac{\Gamma_p'(ax)}{\Gamma_p(ax)} - a\frac{\Gamma_p'(x)}{\Gamma_p(x)},$$
$$= a\left[\psi_p(ax) - \psi_p(x)\right] > 0,$$

since $\psi_p(x)$ is increasing. Hence $g(x)$ is increasing. Consequently $f(x)$ is also increasing and for $0 < x < y$, we have $f(x) \leq f(y)$ which gives

$$\frac{\Gamma_p(ax)}{[\Gamma_p(x)]^a} \leq \frac{\Gamma_p(ay)}{[\Gamma_p(y)]^a},$$

and by rearrangement, we obtain (18). $\qquad\square$

Theorem 3.12. *The inequality*

$$\left[\Gamma_p^{(m)}(x) + \Gamma_p^{(n)}(y)\right]^{\frac{1}{k}} \leq \left[\Gamma_p^{(m)}(x)\right]^{\frac{1}{k}} + \left[\Gamma_p^{(n)}(y)\right]^{\frac{1}{k}}, \tag{19}$$

holds for $x > 0$, $y > 0$, $k \geq 1$ and $m, n \in \{2s : s \in \mathbb{N}_0\}$.

Proof. We apply the Minkowski's inequality and also use the fact that $A^k + B^k \leq (A + B)^k$, for $A, B \geq 0$ and $k \geq 1$. By (3), we obtain

$$\left[\Gamma_p^{(m)}(x) + \Gamma_p^{(n)}(y)\right]^{\frac{1}{k}}$$
$$= \left[\int_0^\infty (\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt + \int_0^\infty (\ln t)^n t^{y-1} e^{\left(-t-\frac{p}{t}\right)} dt\right]^{\frac{1}{k}}$$
$$= \left[\int_0^\infty \left(\left[(\ln t)^{\frac{m}{k}} t^{\frac{x-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right]^k + \left[(\ln t)^{\frac{n}{k}} t^{\frac{y-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right]^k\right) dt\right]^{\frac{1}{k}}$$
$$\leq \left[\int_0^\infty \left(\left[(\ln t)^{\frac{m}{k}} t^{\frac{x-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right] + \left[(\ln t)^{\frac{n}{k}} t^{\frac{y-1}{k}} e^{\frac{1}{k}\left(-t-\frac{p}{t}\right)}\right]\right)^k dt\right]^{\frac{1}{k}}$$
$$\leq \left[\int_0^\infty (\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt\right]^{\frac{1}{k}} + \left[\int_0^\infty (\ln t)^n t^{y-1} e^{\left(-t-\frac{p}{t}\right)} dt\right]^{\frac{1}{k}}$$
$$= \left[\Gamma_p^{(m)}(x)\right]^{\frac{1}{k}} + \left[\Gamma_p^{(n)}(y)\right]^{\frac{1}{k}},$$

which gives the desired result. $\qquad\square$

Theorem 3.13. *The inequality*

$$\exp\left\{\Gamma_p^{(m-r)}(x)\right\} \exp\left\{\Gamma_p^{(m+r)}(x)\right\} \geq \left(\exp\left\{\Gamma_p^{(m)}(x)\right\}\right)^2, \tag{20}$$

holds for $x > 0$ and $m, r \in \{2s : s \in \mathbb{N}_0\}$ such that $m \geq r$.

Proof. By using (3), we make the following estimation.

$$
\begin{aligned}
&\frac{1}{2}\left(\Gamma_p^{(m-r)}(x) + \Gamma_p^{(m+r)}(x)\right) - \Gamma_p^{(m)}(x) \\
&= \frac{1}{2}\left(\int_0^\infty (\ln t)^{m-r} t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt + \int_0^\infty (\ln t)^{m+r} t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt\right) \\
&\quad - \int_0^\infty (\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt \\
&= \frac{1}{2}\int_0^\infty \left[\frac{1}{(\ln t)^r} + (\ln t)^r - 2\right](\ln t)^m t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt \\
&= \frac{1}{2}\int_0^\infty [1 - (\ln t)^r]^2 (\ln t)^{m-r} t^{x-1} e^{\left(-t-\frac{p}{t}\right)} dt \\
&\geq 0.
\end{aligned}
$$

Thus,

$$
\Gamma_p^{(m-r)}(x) + \Gamma_p^{(m+r)}(x) \geq 2\Gamma_p^{(m)}(x),
$$

and by taking exponents, we obtain (20). $\qquad\square$

Remark 3.14. Theorem 3.13 is a genralization of the results of Mortici as given in Theorem 3.1 of [18].

Remark 3.15. For similar results concerning other generalizations and other special functions, one may refer to the recent works [22], [23], [24], [25], [26], [27] and [28].

4   Conclusion

By applying the classical Holder's inequality, Young's inequality, Minkowski's inequality and some other analytical tools, we have established some inequalities involving the Chaudhry-Zubair extension of the gamma function. The established results are generalizations of some known results in the literature.

Competing Interests

The authors declare that there is no competing interests regarding the publication of this paper.

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