THE TOTAL OUTER INDEPENDENT MONOPHONIC DOMINATING PARAMETERS IN GRAPHS

ABSTRACT

In this paper the concept of total outer independent monophonic domination number of a graph is introduced. A monophonic set $S \subseteq V$ is said to be total outer independent monophonic domination set if $\langle S \rangle$ has no isolated vertex and $\langle V - S \rangle$ is an independent set.

Keywords: Monophonic set, monophonic number, monophonic dominating set, monophonic domination number, Crown graph, independent monophonic domination number, total outer independent monophonic domination number.

1. INTRODUCTION

Let $G = (V, E)$ be a graph and n be the number of vertices and m be the number of edges. Thus the cardinality of $V(G) = m$ and the cardinality of $E(G) = n$. we consider a finite undirected graph without loops or multiple edges. For the basic

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graph theoretic notations and terminology we refer to Buckley and Harary. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G . A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic.

The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an extreme vertex if the subgraph induced by its neighbourhood is complete. A vertex v in a connected graph G is a cut vertex of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be a semi-extreme vertex if $\Delta(< N(v) >) = |N(v)| - 1$. A graph G is said to be semi-extreme graph if every vertex of G is a semi-extreme vertex. An acyclic graph is called a tree [1].

A subset of $V(G)$ is independent if there is no edge between any two vertices of this set. A *monophonic set* of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The *monophonic number* $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* of G. For a subset D of vertices, we call D a dominating set if for each $x \in V(G) - D$, x is adjacent to at least one vertex of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A set of vertices M in G is called a monophonic dominating set if M is both a monophonic set and a dominating set. The minimum cardinality of a monophonic dominating set of G is its monophonic domination number and is denoted by $\gamma_m(G)$. A monophonic dominating set of size $\gamma_m(G)$ is said to be a γ_m set. A total monophonic domination set of a graph G is a monophonic domination set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality among all the total monophonic domination set of G is called the total monophonic domination number and is denoted by $\gamma_{m_t}(G)$.

2. TOTAL OUTER INDEPENDENT MONOPHONIC DOMINATION NUMBER OF A GRAPH:

Definition 2.1: A monophonic set $S \subseteq V$ is said to be total outer independent monophonic dominating set, abbreviated TOIMDS if it is a monophonic dominating set and $\langle V - S \rangle$ is an independent set. The minimum cardinality of a total outer independent monophonic dominating set, denoted by $\gamma_{mt}^{oi}(G)$ is called the total outer independent monophonic domination number of G.

Example 2.2: For the graph given in Fig.2.1, it is clear that $M_1 = \{v_1, v_9\}$ is the monophonic set of G so that $m(G) = 2$. It is verified that the set $M_2 = \{v_1, v_5, v_9\}$ is the minimum monophonic domination set so that $\gamma_m(G) = 3$. Also, the set $M_3 = \{v_1, v_2, v_4, v_5, v_7, v_9\}$ is the total monophonic domination set and so $\gamma_{mt}(G) = 6$ and the set $M_4 = \{v_1, v_2, v_4, v_5, v_6, v_7, v_9\}$ is the total outer independent monophonic dominating set of G so that $\gamma_{mt}^{oi}(G) = 7$.

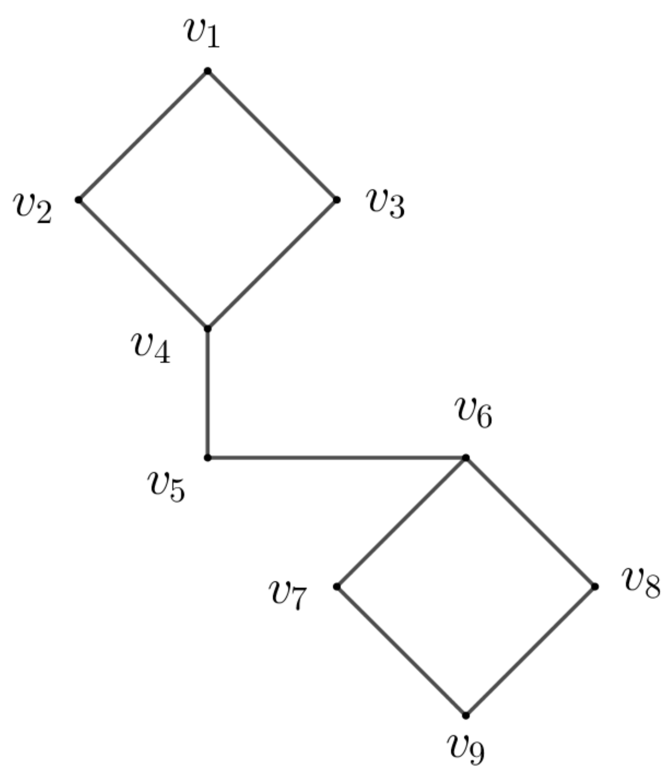

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Fig 2.1

Observation 2.3 : For the graph G given in Fig 2.1, $m(G) = 3$. Thus the $m(G)$ and $\gamma_{mt}^{oi}(G)$ are different.

Observation 2.4 : For any connected graph G, $\gamma_m(G) \leq \gamma_{mt}^{oi}(G)$.

Observation 2.5 : For any connected graph G, $\gamma_{mt}(G) \leq \gamma_{mt}^{oi}(G)$.

Observation 2.6: For any connected graph G, $2 \leq m(G) \leq \gamma_m(G) \leq \gamma_{mt}^{oi}(G)$.

Observation 2.7 : For any complete graph K_n $(n \geq 2)$, $\gamma_{mt}^{oi}(K_n) = n$.

Observation 2.8: Each support vertex of a connected graph G belongs to every total outer independent monophonic domination set of G.

3 Main Results

Corollary 3.1 For any graph G, we have $2 \leq \gamma_{mt}^{oi} \leq n$.

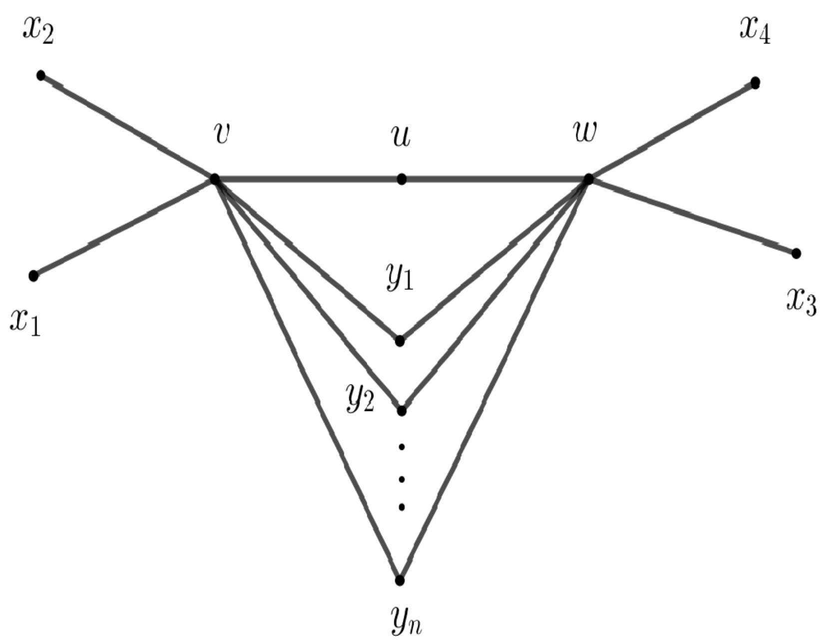

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Fig 2.2

From the figure 2.2 it is obvious that $m(G) = 4, \gamma_m(G) = 5, \gamma_{mt}^{oi} = 7$.

Theorem 3.2 For any connected graph $G, \gamma_{mt}^{oi}(G) = n$, if and only if $G = mK_2$, when $2m = n$.

Proof. We have $\gamma_{mt}^{oi}(mK_2) = m. \gamma_{mt}^{oi}(K_2) = m. 2 = n$. Now for some graph G, we have $\gamma_{mt}^{oi}(G) = n$. Let K be any connected component of G. The graph G has no isolated vertices, thus $|V(K)| \geq 2$. If $|V(K)| \geq 3$, then x, some vertex of K is not a support vertex. Therefore $\gamma_{mt}^{oi}(G) \leq n - 1 < n$, a contradiction. Hence $K = K_2$ and so $G = mK_2$.

Observation 3.3 For any connected graph G, and $e \notin E(G)$, then $\gamma_{mt}^{oi}(G) - 2 \leq \gamma_{mt}^{oi}(G + e) \leq \gamma_{mt}^{oi}(G) + 1$.

From the figure 2.2, $\gamma_{mt}^{oi} = 7$, Thus, $5 \leq 8 \leq 8$.

Theorem 3.4 For any connected graph $G, \gamma_{mt}^{oi}(G) + \gamma_{mt}^{oi}(\bar{G}) = 2n - 1$ if and only if $G = C_4$ or $\bar{G} = C_4$.

Proof. Obviously, $\gamma_{mt}^{oi}(C_4) + \gamma_{mt}^{oi}(\overline{C_4}) = 7 = 2n - 1$. Now assume that for some graph G, we have $\gamma_{mt}^{oi}(G) + \gamma_{mt}^{oi}(\bar{G}) = 2n - 1$. We assume that $\gamma_{mt}^{oi}(G) = n$. By theorem 3.2, we have $G = mK_2$. Clearly $m \geq 2$. If $m = 2$, then $G = K_2 \cup K_2$, therefore, $\bar{G} = C_4$. Now assume that $m \geq 3$. Let $E(mK_2) = \{u_1v_1, u_2v_2, \ldots, u_mv_m\}$. Clearly, $\{u_1v_1, u_2v_2, \ldots, u_{m-1}v_{m-1}\}$ is a TOIMDS of the graph mK_2. Hence, γ_{mt}^{oi} $(\overline{mK_2}) \leq n - 2$. Thus $\gamma_{mt}^{oi}(mK_2) + \gamma_{mt}^{oi}(\overline{mK_2}) \leq n + n - 2 < 2n - 1$, a contradiction.

Theorem 3.5 For the wheel W_n of order $n \geq 5$,then

$$\gamma_{mt}^{oi}(W_n) = \begin{cases} \frac{n+2}{2} & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}.$$

Proof. Let W_n be the wheel of order $n \geq 5$. The vertices of W_n are $v_1, v_2, \dots, v_{n-1,}v_n$ where $\deg(v_1) = n - 1$ and $\deg(v_i) = 3$ where $i \in \{2,3, \dots, n\}$.

Case (i) Suppose n is even. Consider $S = M \cup K_1$, where M is a total monophonic dominating set of W_n, $|M| = \frac{n}{2}$ and $K_1 = v_1$. Hence $\langle S \rangle$ has no isolated vertex and $\langle V - S \rangle$ is an independent set. Thus S is a total outer independent monophonic dominating set in W_n and so $\gamma_{mt}^{oi}(W_n) = \frac{n+2}{2}$.

Case (ii) Suppose n is odd. Consider $S = M \cup K_1$, where M is a total monophonic dominating set of W_n, $|M| = \frac{n-1}{2}$ and $K_1 = v_1$. Hence $\langle S \rangle$ has no isolated vertex and $\langle V - S \rangle$ is an independent set. Thus S is a total outer independent monophonic dominating set in W_n and so $\gamma_{mt}^{oi}(W_n) = \frac{n+1}{2}$.

Theorem 3.6: For the wheel W_n of order $n \geq 5$,then

$$\gamma_{mt}^{oi}(W_n) = \begin{cases} \frac{\Delta+\delta}{2} & if\ n\ is\ even \\ \frac{\Delta+\delta-1}{2} & if\ n\ is\ odd \end{cases}$$

Proof. Let W_n be the wheel of order $n \geq 5$. The vertices of W_n are $v_1, v_2, \dots, v_{n-1,}v_n$ where $\deg(v_1) = n - 1$ and $\deg(v_i) = 3$ where $i \in \{2,3, \dots, n\}$. In a wheel, $\Delta = n - 1$ and $\delta = 3$.

Case (i) If n is even, From theorem 3.5, $\gamma_{mt}^{oi}(W_n) = \frac{n+2}{2} = \frac{(n-1)+3}{2} = \frac{\Delta+\delta}{2}$

Case (ii) If n is odd, From theorem 3.5, $\gamma_{mt}^{oi}(W_n) = \frac{n+1}{2} = \frac{(n-1)+2}{2} = \frac{\Delta+\delta-1}{2}$.

Theorem 3.7: For a complete graph K_n, remove an non adjacent edges $\{e_i, e_j\}$ to obtain $G = K_n - \{e_i, e_j\}$, $n \geq 5$ then $\gamma_{mt}^{oi}(G) = n - 1$.

Proof. Let $G = K_n - \{e_i, e_j\}$ where $e_i\ and\ e_j$ are the non adjacent edges of K_n. Let $e_i = xy$ and $e_j = vw$ where x, y, v, w are some of the vertices of G, then $S = \{x, y\} = \{v, w\}$ is the monophonic dominating set, $S_1 = S \cup \{u\}$ is the total monophonic dominating set of G. Also S_1 has isolated vertex and $\langle V - S_1 \rangle$ is not an independent set. Consider $S_2 = S_1 \cup H$ where H is in $V(G) - S_1$ having $n - 4$ vertices. Now S_2 has no isolated vertices and $\langle V - S_2 \rangle$ is an independent set. Therefore, S_2 is the total outer independent monophonic domination set. Hence $|S_2| = |S_1 \cup H| = 3 + n - 4 = n - 1$. Therefore, $\gamma_{mt}^{oi}(G) = n - 1$.

Theorem 3.8: For any connected graph G of order ≥ 4, $2 \leq m(G) \leq \gamma_{mt}^{oi}(G) \leq n - 2$.

Proof. A monophonic set needs atleast two vertices and therefore $m(G) \geq 2$. Since every monophonic set is a total outer independent monophonic dominating set we have $m(G) \leq \gamma_{mt}^{oi}(G)$. Hence $2 \leq m(G) \leq \gamma_{mt}^{oi}(G) \leq n - 2$.

Theorem 3.9: For any connected graph G of order ≥ 4, $2 \leq \gamma_m(G) \leq \gamma_{mt}^{oi}(G) \leq n - 2$.

Proof. Every monophonic dominating set need at least two vertices, so $2 \leq \gamma_m(G)$. Sine every monophonic domination set is a total outer independent monophonic dominating set , we have $\gamma_m(G) \leq \gamma_{mt}^{oi}(G)$. Also since G is connected, $\gamma_{mt}^{oi}(G) \leq n - 2$. Hence $2 \leq \gamma_m(G) \leq \gamma_{mt}^{oi}(G) \leq n - 2$.

Theorem 3.10: For any connected graph G of order ≥ 4, $3 \leq \gamma_{mt}(G) \leq \gamma_{mt}^{oi}(G) \leq n - 2$.

Proof. Every total monophonic dominating set need at least 3 vertices, $3 \leq \gamma_{mt}(G)$. Since every total monophonic dominating set is a total outer independent monophonic dominating set, we have $\gamma_{mt}(G) \leq \gamma_{mt}^{oi}(G)$. Also $\gamma_{mt}^{oi}(G) \leq n - 2$. Hence $3 \leq \gamma_{mt}(G) \leq \gamma_{mt}^{oi}(G) \leq n - 2$.

Theorem 3.11: For any positive integers $2 \leq p \leq q$, there exists a connected graph G such that $\gamma_{mt}(G) = p$ and $\gamma_{mt}^{oi}(G) = q$.

Proof. Let $P: u, v, w, x$ be a path on 4 vertices. Add new vertices $x_1, x_2, \ldots, x_{p-3}$ and $y_1, y_2, \ldots, y_{q-p}$ and join each $x_i (1 \leq i \leq p-3)$ with x and each $y_i (1 \leq i \leq q-p)$ with v, x . Also, join y_i's with each y_i's $(1 \leq i \leq q-p)$ thereby obtaining the graph G in Fig. 2.3

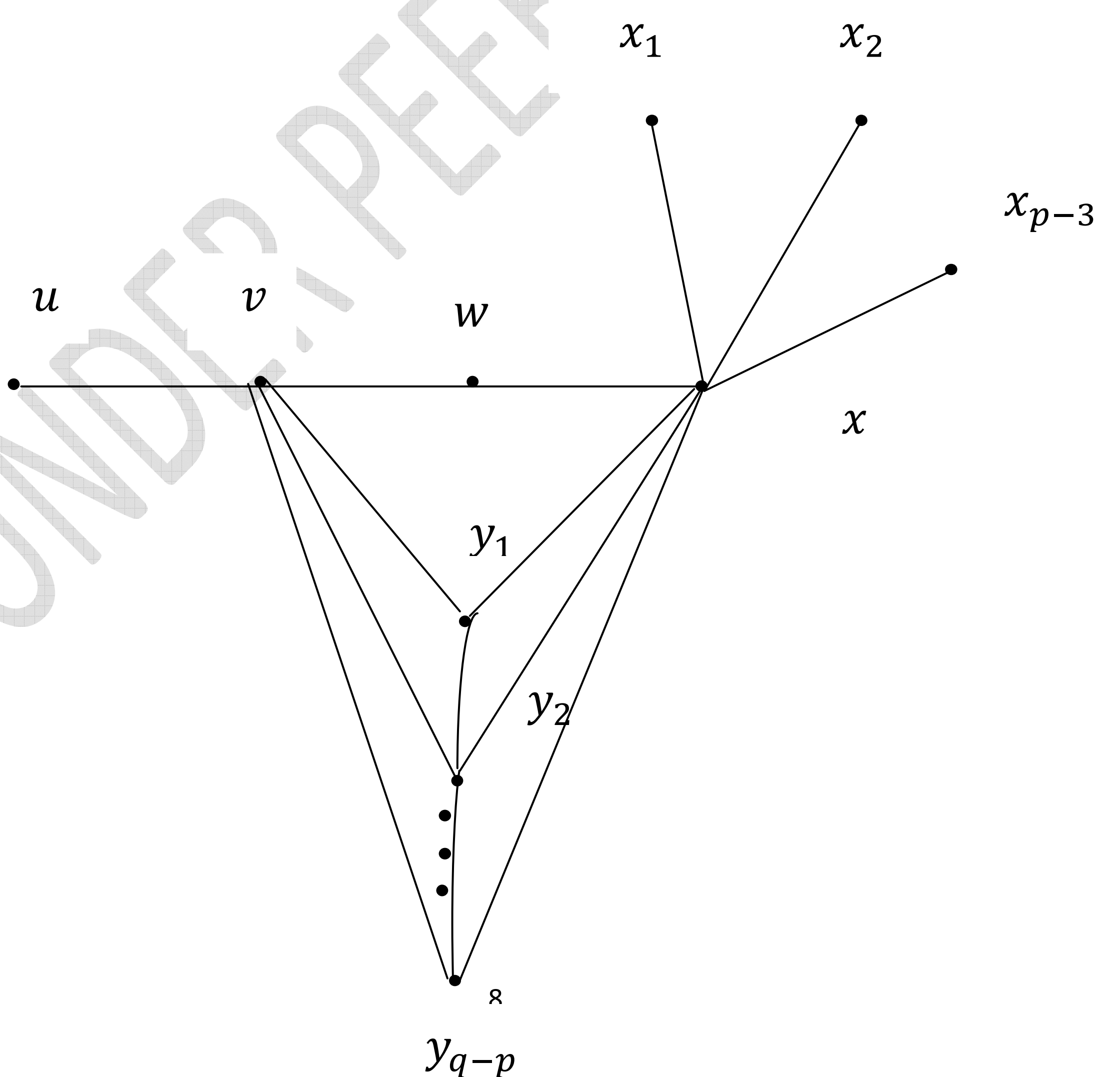

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Fig 2.3

Let $Z = \{u, x_1, x_2, \dots, x_{p-3}\}$. It is clear that Z is not a total monophonic dominating set of G. However, $M_1 = Z \cup \{v, x\}$ is the minimum total monophonic dominating set of G so that $\gamma_{mt}(G) = p$. Next, we show that $\gamma_{mt}^{oi}(G) = q$. It can be easily verified that M_1 is not a total outer independent monophonic dominating set of G. But $M_2 = M_1 \cup \{y_1, y_2, \dots, y_{q-p}\}$ is the total outer independent monophonic dominating set of G and so that $\gamma_{mt}^{oi}(G) = q$.

4 Crown Graph

Definition 4.1 The crown graph $H_{n,n}$ (Fig 2.4) is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

Theorem 4.2: Let $G = H_{n,n}$ be a crown graph of order $n \geq 4$, then

$$\gamma_{mt}^{oi}(H_{n,n}) = \begin{cases} 4 & for\ n = 3 \\ n+2 & for\ n \geq 3 \end{cases}$$

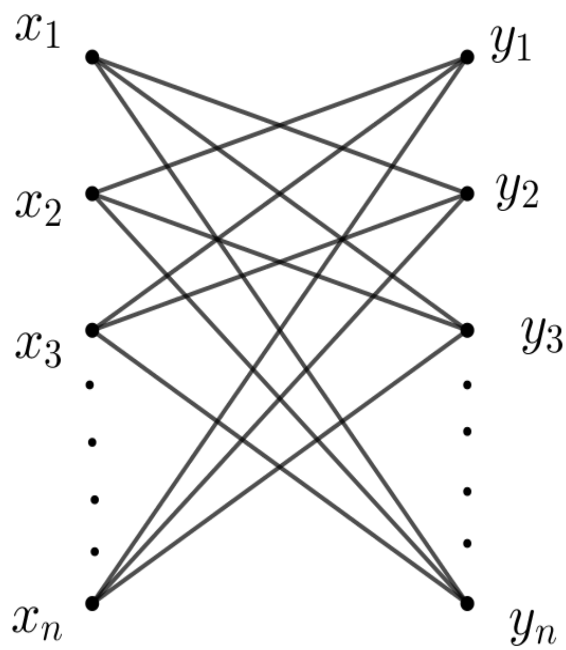

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Fig 2.4

Proof. Let $\{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$ be the vertices of the crown graph with $2n$ vertices.

Case(i) For $n = 3$, the set $S = \{v_1, v_n, v'_1, v'_n\} \subseteq V(G)$, forms the total monophonic domination set and so $\langle V - S \rangle$ is an independent set. Therefore, $\gamma_{mt}^{oi}(H_{n,n}) = 4$.

Case (ii) For $n \geq 3$, the set $S = \{v_i, v_j, v'_i, v'_j\} \subseteq V(G)$, where $1 \leq i.j \leq n$, be the minimum total monophonic dominating set and so $\langle V - S \rangle$ is not an independent set. Consider $B = \{v_1, v_2, \dots, v_{n-2}\} \subseteq N(v'_k), k = i \text{ or } j$. Also $A = S \cup B$ is the minimum total outer independent monophonic domination set of G and $\langle V - A \rangle$ is an independent set. Therefore, $|A| = |S| + |B| = 4 + n - 2 = n + 2 = \gamma_{mt}^{oi}(H_{n,n})$.

Theorem 4.3: Let $H_{n,n}$ be a crown graph of order $n \geq 3$, then $\gamma_{mt}(H_{n,n}) \leq \gamma_{mt}^{oi}(H_{n,n})$

Proof. Clearly, $S = \{v_1, v_2, v_1', v_2'\}$ forms the total monophonic domination set and so $\langle V - S \rangle$ is an independent set. Therefore it is also the total outer independent monophonic domination set of G and so $\gamma_{mt}(H_{n,n}) = \gamma_{mt}^{oi}(H_{n,n})$ for $n = 3$. For $n > 3$, let $S_1 = \{v_i, v_j, v_i', v_j'\} \subseteq V(G)$, where $1 \leq i, j \leq n$, be the minimum total monophonic dominating set and so $\langle V - S_1 \rangle$ is not an independent set. Consider, $S_2 = S_1 \cup \{v_1, v_2, \dots, v_{n-2}\}$ which is the minimum total outer independent monophonic domination set of G and $\langle V - S_2 \rangle$ is an independent set. Therefore, $\gamma_{mt}(H_{n,n}) \leq \gamma_{mt}^{oi}(H_{n,n})$.

5 CONCLUSION

We can extend the concept of total outer independent monophonic domination number to find the total outer independent edge monophonic domination number, connected total outer independent monophonic domination number of graphs and so on.

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