Original research paper
Analytical Design Method for Widening the Intertrack Space

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ABSTRACT

In the paper a new approach is made to the problem of widening the intertrack space by presenting a thesis that it is necessary to form it using two joined curves of appropriate curvature distribution in length. It has been assumed that along the widening length there appear segments of variable curvature in extension zones of both the curves, whereas in the central zone the curvature is constant. To solve the problem use has been made of the analytical design technique, employing the identification of the curvature distribution by different equations. Particular attention has been given to the nonlinear distribution of curvature, regarded as the most advantageous undertaking. To obtain a specific final ordinate in the widening the intertrack space it is necessary to obey a sequential correction procedure of the adopted geometric parameters.

Keywords: Railway track, widening the intertrack space, curvature modelling, analysis of horizontal ordinates

1. INTRODUCTION

An enlargement of the track spacing commonly known as widening of the intertrack space is one of the basic geometric layouts of the railway track. It is most often based on shifting one of the tracks to a particular distance. The design principles for such systems are specified in the relevant regulations in force in the given country. In Poland this procedure is carried out in compliance with the standards established in the seventies of last century by prof. Henryk Baluch [1, 2]. He expressed disapproval of the method used then, consisting in employing two circular arcs (opposite ones) without cant, separated by a straight section of the track. He presented his own propositions relating to the solution of the problem:

- by the use of two circular arcs with cant and four transition curves with gradients due to cant,
- by means of four transition curves without cant.

The first solution was mainly connected with railway lines dominated by passenger traffic, whereas the second one was judged to be advantageous in the case of significant cargo traffic domination. Also appropriate calculation algorithms were prepared, where the cubic parabolas were applied as transition curves, and regarding the occurrence of cant in the circular arc – straight-lined gradients due to cant.

The described situation has estabished a routine in Poland for a long time and has continued this practice to this day, presenting a subject for both the didactic activities and also scientific investigations (it is possible to mention here paper [3]). In the meantime in Europe in 2010 there appeared also other propositions about the solution of the problem under consideration [4]:

- by the application of two opposite circular arcs of radii R < 4000 m with four transition curves (solution applied when the speeds do not exceed 120 km/h),
- by the use of two opposite circular arcs of radii $R \geq 4000$ m with no transition curves,
- by the use of four transition curves without circular arcs.

As can be seen the first and the last procedure mentioned above correspond to the proposition given in papers [1, 2], but the solution obtained by the use of circular arc of radii R < 4000 m a significant speed limit is imposed, as a rule, the radii of $1500 \leq R \leq 2500$ m are applicable. In solutions using circular arcs of radii $R \geq 4000$ m no cant along the arc is anticipated, that is also referring to gradients due to cant.

And yet it should be noted that it is possible to make another approach to the problem. The fact of the intertrack spacing suggests an analogy to the case of linking paralel tracks with each other using two railway turnouts. This takes place along the length of the diverging tracks, assuming the shape of opposite arcs. For the reason that in typical geometric shaping of the diverging track in a railway turnout that has been used since the beginning of the railway system, a circular arc is employed (with no transition curves). This explains the occurrence of areas of a turbulent changes of coordinates in the curvature diagram at the outset and the end of the turnout. That's why between the ends of the turnouts applied a straight track segment it is necessary to make used in the rail track connection.

However, in recent years in some countries attempts have been made to smooth the curvature diagram using the so-called „clothoid segments" on both sides of the circular arc where the curvature changes linearly very often not reaching the zero values at extension points [5-7]. In such case the straight track segment becomes useless and the end ordinate of the first turnout is equal to half the required track spacing. An identical second turnout is inserted in the paralel track. However, the ends of both the turnouts are connected with each other.

Taking into consideration the above it is possible to present a thesis on the intertrack space extension that should be shaped by making use of two connected curves of appropriate curvature distribution along their length provided with opposite curvature signs (i.e. opposite arcs), where their end ordinates must reach half the extension values. Of course, the requirements relating to the turnout crossing angle of 1 : n need not be fulfilled.

2. FUNDAMENTAL ASSUMPTIONS

In contrast to the solution by the use of circular arcs with cant and transition curves with gradients due to cant, it does not seem sensible to differentiate the height of the track rails in view of the maintenance problems (in widening the intertrack space the cant values cannot be large). However, the solution using four transition curves with no cant requires an application of the circular arc segments. Nevertheless, the fundamental question should become the proper configuration of the curvature along the length of the entire geometric layout. This is a significant factor responsible for the speed of the train.

In the proposed solution it is only necessary to take into consideration half of the geometric layout. Values of the abscissa for the second half are determined by taking advantage of the symmetry axis which appears at the point connecting both the curves, whereas the ordinates of that zone come from the difference between the spacing value and the ordinates of the first curve.

In the analysis of a half of the geometric layout the same conditions shoud be fulfilled as for the turnout diverging track. For the reason that the solution by using two circular arcs (opposite ones) with no cant is unimportant for us, an assumption is made that along the length of spacing there appear segments of variable curvature in the extreme areas and that in the central zone the curvature is constant (which means the circular arc is used). To obtain a correct solution it is necessary to satisfy some kinematic conditions and to determine the end ordinate being equal to half of the assumed extention value.

The curve length of the intertrack spacing has been divided into three zones:
- initial zone of length l_1 provided with a variable curvature,
- central zone of length l_2 provided with a constant curvature,
- end-zone of length l_3 provided with the variable curvature.

Of course, there are different variants of solutions connected with the curvature values and the lengths of some specific zones.

An assumption is made that the kinematic parameters determine the value of the circular arc radius and the length of segments of variable curvature for a given travelling speed of trains. The length of the arc part of a constant curvature is implemented using an iterative method until the assumed final ordinate is obtained. The analytical method of design [8-11] was used in the analyzes.

3. EXISTING POSSIBILITIES OF SOLVING THE PROBLEM

There are two basic variants in shaping the curvature in extreme areas of the geometric layout, linear (Fig. 1) and nonlinear (Fig. 2).

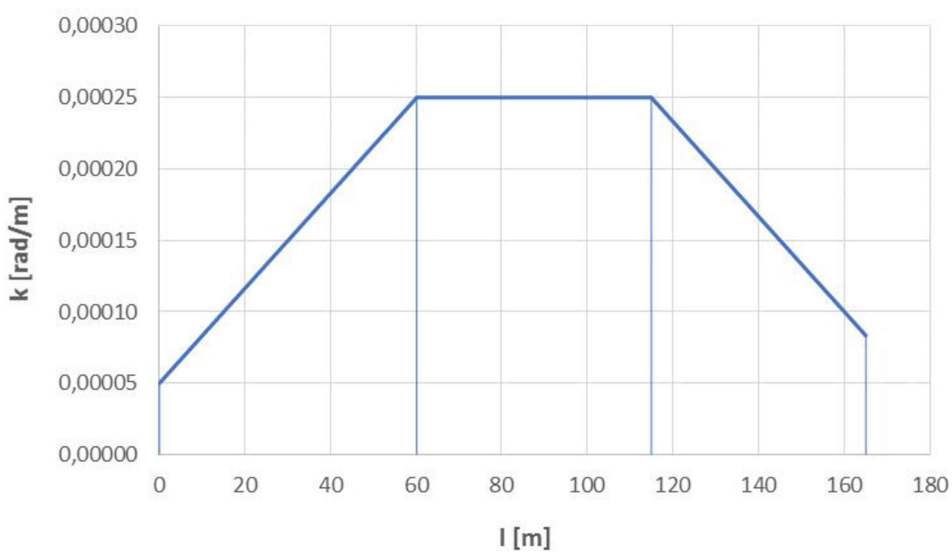

Fig. 1. An example of diagram with segments of linear curvature: k_1 = 1/20000 rad/m, l_1 = 60 m, k_2 = 1/4000 rad/m, l_2 = 55 m, l_3 = 50 m, k_3 = 1/12000 rad/m

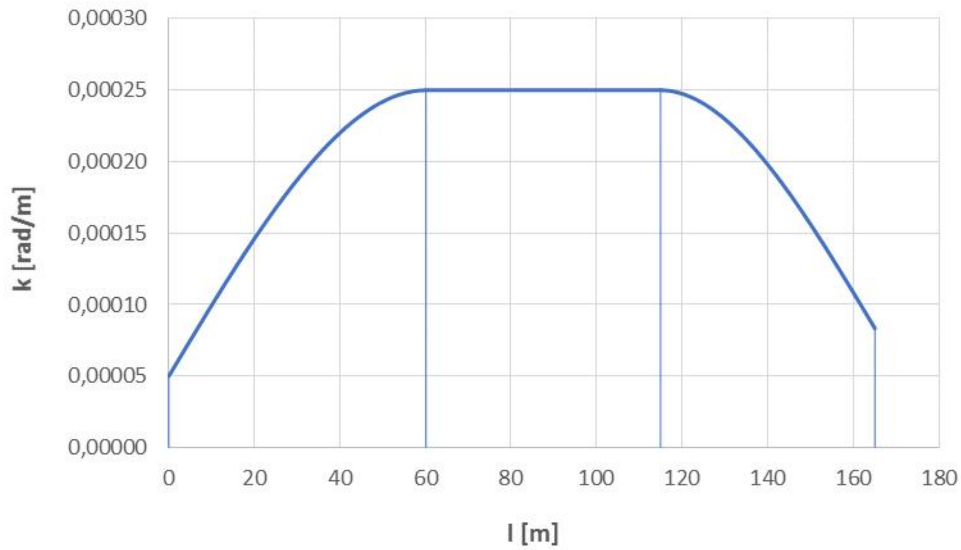

Fig. 2. An example of curvature diagram with nonlinear segments: k_1 = 1/20000 rad/m, l_1 = 60 m, k_2 = 1/4000 rad/m, l_2 = 55 m, l_3 = 50 m, k_3 = 1/12000 rad/m

As already mentioned the linear variant is now used in shaping the diverging tracks in railway turnouts. The nonlinear variant has been proposed in paper [12], which provides theory of both the analyzed variants, along with the determination of parametric equations in extreme curve zones of the Cartesian coordinate system. However, there arises a crucial question as to the choice of parameters, namely k_1, k_2 and k_3 and l_1, l_2 and l_3, that would be most advantageous in a given situation.

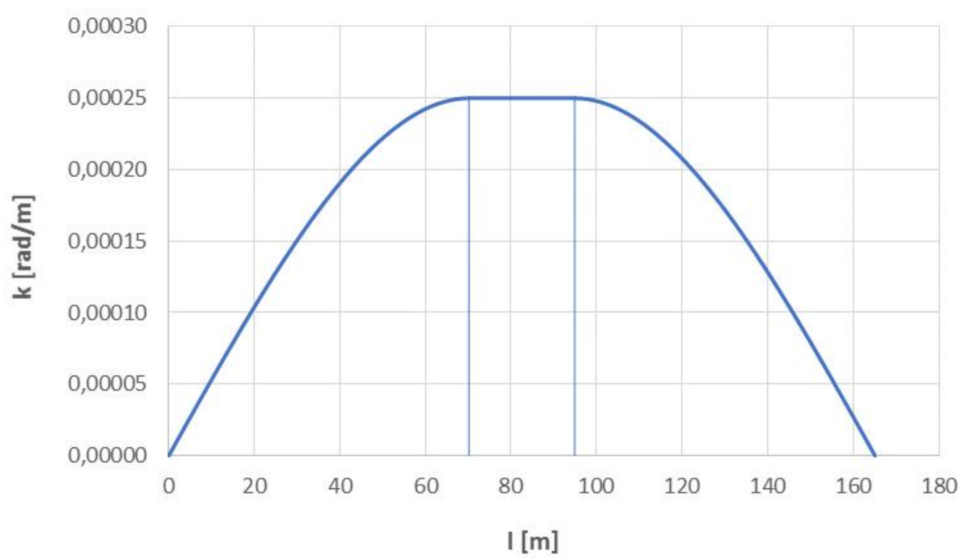

Fig. 3. The most advantageous diagram of nonlinear curvature segments: k_1 = 0, l_1 = 70 m, k_2 = 1/4000 rad/m, l_2 = 25 m, l_3 = 70 m, k_3 = 0

The number of variants applicable to practical use is largely limited by the dynamic analysis presented in paper [13]. A dozen instances of the application of linear and nonlinear curvature segments are given attention in the paper. Definitely the most advantageous solution characterized by the least values of dynamic interactions (accelerations) has appeared to be the case with nonlinear curvature segments of equal length and zero curvature at the outset and the end of the geometric layout. Figure 3 illustrates the curvature diagram in length for such a situation. Therefore further considerations are concentrated on the assumption that curvatures k_1 = k_3 = 0 and that its curvature diagram is symmetric in length (i.e. length l_1 = l_3). Under this circumstances it is worthwhile to introduce the denotation $k_2 = k$.

The modelling of the variable curvature segments along the length of the turnout diverging track makes it possible to make an analytical record in the form of function $k(l)$, where parameter l denotes the position of a given point along the length of the curve. The equations of the coordinates of the connection sought can be written in parametric form [9]:

$$x(l) = \int \cos \Theta(l) dl \tag{3-1}$$

$$y(l) = \int \sin \Theta(l) dl \tag{3-2}$$

The function of the tangent inclination angle $\Theta(l)$ is determined from the formula

$$\Theta(l) = \int k(l) dl \tag{3-3}$$

A method currently in use (among others, in commercial programmes in aid of design [14, 15]) to determine coordinates $x(l)$ and $y(l)$ is the numerical integration of functions $\cos \Theta(l)$ and $\sin \Theta(l)$. From the practical point of view it provides a sufficient accuracy. However, the above approach has a fundamental drawback – every geometric occurrence has to be analyzed individually and the use of any generalization is here very inconvenient. If a problem is to be viewed in a more general way, it is necessary to make use of the analytical technique, which in principle is general and complete. The paper offers universal analytical equations for particular zones of widening the intertrack space.

4. CHOICE OF THE APPLIED TYPE OF THE NONLINEAR CURVATURE

The application of the nonlinear curvature sections indicates that in the initial zone is bound by the following boundary conditions:

$$\begin{cases} k(0) = 0 & k(l_1) = k \\ k'(0) = C\dfrac{k}{l_1} & k'(l_1) = 0 \end{cases} \tag{4-1}$$

and the differential equation

$$k^{(4)}(l) = 0 \tag{4-2}$$

where $k = \dfrac{1}{R}$, and the numerical factor $C \geq 0$.

In consequence of solving the differential problem (4-1), (4-2) we have

$$k(l) = \frac{Ck}{l_1} l - \frac{(2C-3)k}{l_1^2} l^2 + \frac{(C-2)k}{l_1^3} \tag{4-3}$$

while the tangential slope angle function $\Theta(l)$ is described by the relation

$$\Theta(l) = \frac{Ck}{2l_1} l^2 - \frac{(2C-3)k}{3l_1^2} l^3 + \frac{(C-2)k}{4l_1^3} l^4 \tag{4-4}$$

At the end of initial zone, for $l = l_1$, the value of angle $\Theta(l_1) = \frac{6+C}{12} kl$.

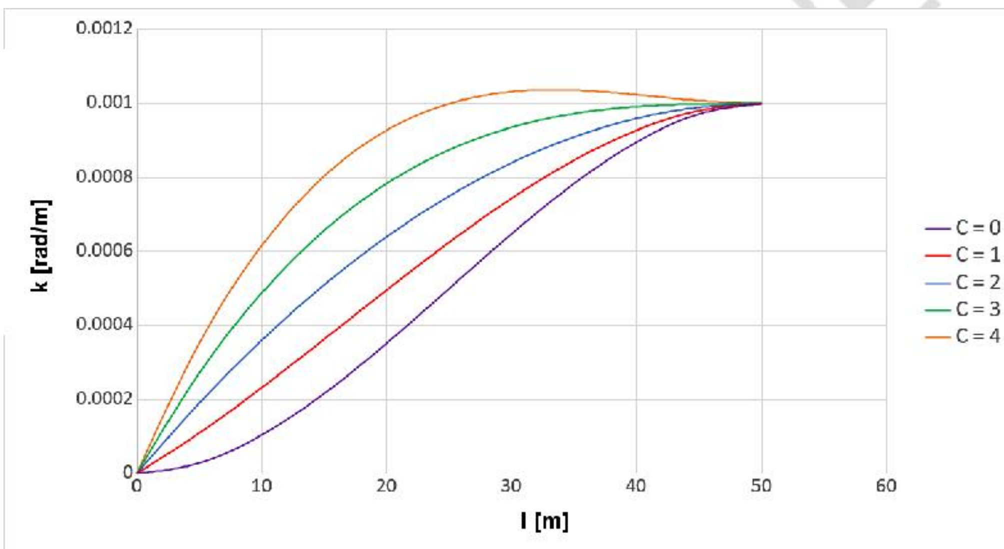

Fig. 4. Examples of curvature diagrams along the length of the initial zone for selected values of coefficient *C* (*k* = 1/1000 rad/m, l_1 = 50 m)

Figure 4 gives examples of curvature diagrams along length for some selected values of coefficient *C*. As can be seen the monotonic curvature is due to the curves for $C \in \langle 0;3 \rangle$. The curve for *C* = 0 is characterized by the most moderate feature, but as in the case of other curves, the accomplishment of satisfying the condition imposed on preserving the permissible value of speed of acceleration change makes it necessary to elongate it in relations to the adopted linear curvature.

While searching for the most advantageous curve out of the analyzed ones, one should primarily take into consideration the criterion of the least required length. The length is determined by the permissible value of the speed of acceleration change, which besides the trains speed, is directly connected with the curvature derivative.

$$k'(l) = \frac{Ck}{l_1} - \frac{2(2C-3)k}{l_1^2} l + \frac{3(C-2)k}{l_1^3} l^2 \tag{4-5}$$

Figure 5 illustrates diagrams of curvature derivative along the length of curves for which $C \in \langle 0; 2.5 \rangle$.

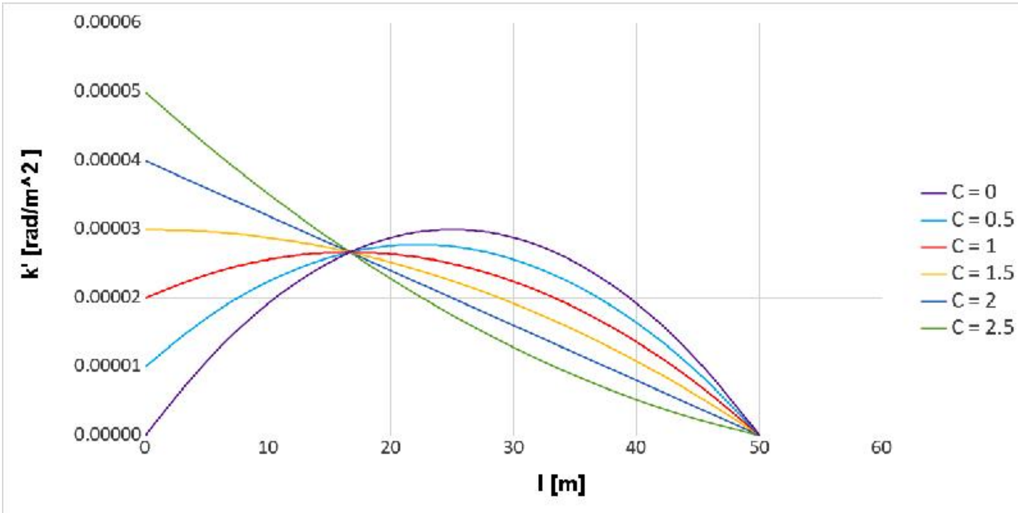

Fig. 5. Examples of diagrams of curvature derivative along the initial zone length for selected values of coefficient *C* (*k* = 1/1000 rad/m, l_1 = 50 m).

The most significant here is the maximum value of the derivative which can be obtained from the condition

$$k''(l) = -\frac{2(2C-3)k}{l_1^2} + \frac{6(C-2)k}{l_1^3} l_0 = 0 \tag{4-6}$$

From equation (4-6) it follows that the abscissa of the point presenting the maximum function *k'*(*l*) is

$$l_0 = \frac{2C-3}{3(C-2)} l \tag{4-7}$$

which used in Equation (4-5) determines the maximum function *k'*(*l*).

$$\max k'(l) = \left[C - \frac{(2C-3)^2}{3(C-2)} \right] \frac{k}{l_1} \tag{4-8}$$

As it appears by the use of this procedure one can find the max *k'*(*l*) when $C \le 1.5$. In other cases the largest value of function *k'*(*l*) appears at the initial point (Table 1).

Table 1. Effect of coefficient *C* on the maximum value of curvature derivative

Coefficient *C*	Arc radius *R* [m]	Length l_1 [m]	Abscissa l_0 [m]	max $k'(l)$ [rad/m^2]
0	1000	50	25.000	0.00003
0.5	1000	50	22.222	2.78E-0.5
1.0	1000	50	16.667	2.67E-0.5
1.5	1000	50	0	0.00003
2.0	1000	50	0	0.00004
2.5	1000	50	0	0.00005
3.0	1000	50	0	0.00006

From Equation (4-8) it follows that for C = 0, the $\max k'(l) = 3k/2l_1$, thus on account of the permissible value of the speed of acceleration change the length of the extreme segment is to be 50% bigger than the one of the linear curvature. While for C = 1 the $\max k'(l) = 4k/3l_1$, which means that the length of the extreme segment should be greater than for the linear curvature only by ⅓. Therefore, on the basis of the performed analysis it follows that the most advantageous solution is the use of coefficient C = 1. This leads to the following equations for $k(l)$ and $\Theta(l)$:

$$k(l) = \frac{k}{l_1} l + \frac{k}{l_1^2} l^2 - \frac{k}{l_1^3} l^3 \tag{4-9}$$

$$\Theta(l) = \frac{k}{2l_1} l^2 + \frac{k}{3l_1^2} l^3 - \frac{k}{4l_1^3} l^4 \tag{4-10}$$

At the end of the initial zone the tangential inclination angle $\Theta(l_1) = \frac{7}{12} k l_1$.

5. ANALYTICAL SOLUTION OF THE PROBLEM

5.1 Solution of the Problem for the Initial Zone

Equations $k(l)$ and $\Theta(l)$ relating to the initial zone have been determined at point 4. With respect to the adopted coefficient C = 1, relations (4-9) and (4-10) are in force. Function $\Theta(l)$ makes it possible to determine parametric equations $x(l)$ and $y(l)$ of this widening curve zone by taking advantage of relations (3-1) and (3-2). To expand functions $\cos\Theta(l)$ and $\sin\Theta(l)$ into Maclaurin series use has been made of the Maxima program [16] followed by integration of respective terms.

$$x(l) = l - \frac{k^2}{40l_1^2} l^5 - \frac{k^2}{36l_1^3} l^6 + \frac{5k^2}{504l_1^4} l^7 + \frac{k^2}{96l_1^5} l^8 + \left(\frac{k^4}{3456l_1^4} - \frac{3k^2}{864l_1^6} \right) l^9 \tag{5-1}$$

$$y(l) = \frac{k}{6l_1} l^3 + \frac{k}{12l_1^2} l^4 - \frac{k}{20l_1^3} l^5 - \frac{k^3}{336l_1^3} l^7 - \frac{k^3}{192l_1^4} l^8 + \frac{k^3}{2592l_1^5} l^9 \tag{5-2}$$

5.2 Solution of the Problem for the Central Zone

Within the zone of the circular arc, i.e. for $l \in \langle l_1, l_1 + l_2 \rangle$, there appears a constant curvature

$$k(l) = k \tag{5-3}$$

and function *Θ*(*l*) is described by the relationship

$$\Theta(l) = \frac{(C-6)k}{12} l_1 + kl \tag{5-4}$$

The value of angle *Θ*(*l*) at the end of the circular arc is $\Theta(l_1 + l_2) = \frac{6+C}{12} kl_1 + kl_2$; for coefficient *C* =1, value $\Theta(l_1 + l_2) = \frac{7}{12} kl_1 + kl_2$.

The circular arc equation can be noted in the form of explicit function $y(x)$. The methods of its presentation is here analogical to papers [8, 10,11]. The scheme presenting the position of the circular arc is given in Figure 6. It is assumed that the circular arc length is l_2(measured along the very arc). Its radius is R, whereas the tangential inclination at the initial point $s_1 = \tan \theta(l_1)$.

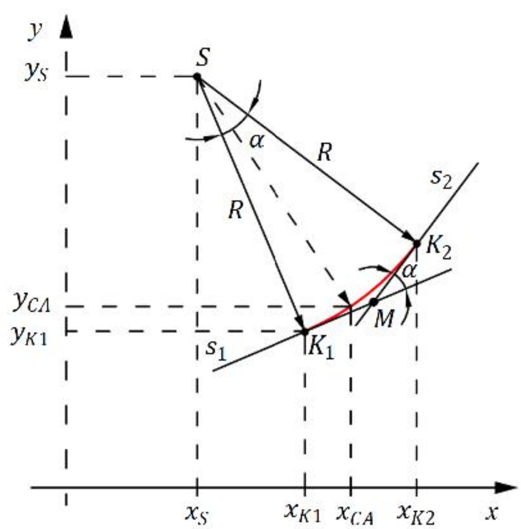

Fig. 6. Diagram illustrating the position of the circulat arc

The coordinates of the arc mid-point $S(x_S, y_S)$ are determined as follows

$$x_S = x(l_1) - \frac{s_1}{\sqrt{1+s_1^2}} R \tag{5-5}$$

$$y_S = y(l_1) + \frac{1}{\sqrt{1+s_1^2}} R \tag{5-6}$$

The circular arc equation is

$$y(x) = y_S - \sqrt{R^2 - (x_s - x)^2} \quad , \quad x \in \langle x(l_1), x(l_1 + l_2) \rangle \tag{5-7}$$

For the reason that the angle of return of tangents of the circular arc is α = l_2 / R, the formula for the tangential angle of indication to the arc at its end, i.e. for $x(l_1 + l_2)$, can also be denoted as $\Theta(l_1 + l_2) = \Theta(l_1) + \alpha$. Hence it follows that the value of the tangential inclination at this point is $s_2 = \tan[\Theta(l_1) + \alpha]$. In order to determine the circular arc end coordinates it is necessary to find out first the coordinates of point M (Fig. 6). Finally the following are obtained

$$x(l_1 + l_2) = x(l_1) + \tan\frac{\alpha}{2}\left(\frac{1}{\sqrt{1+s_1^2}} + \frac{1}{\sqrt{1+s_2^2}}\right)R \tag{5-8}$$

$$y(l_1 + l_2) = y(l_1) + \tan\frac{\alpha}{2}\left(\frac{s_1}{\sqrt{1+s_1^2}} + \frac{s_2}{\sqrt{1+s_2^2}}\right)R \tag{5-9}$$

5.3 Solution of the Problem for the Final Zone

An assumption is made for the following boundary conditions

$$\begin{cases} k(l_1 + l_2) = k & k(2l_1 + l_2) = 0 \\ k'(l_1 + l_2) = 0 & k'(2l_1 + l_2) = -C\dfrac{k}{l_1} \end{cases} \tag{5-10}$$

and the differential equation (4-2). The solution of the differential problem (4-2), (5-10) is as follows

$$k(l) = c_1 + c_2 l + c_3 l^2 + c_4 l^3 \tag{5-11}$$

where
$$c_1 = \left[1 - \frac{3-C}{l_1^2}(l_1 + l_2)^2 - \frac{2-C}{l_1^3}(l_1 + l_2)^3\right]k$$

$$c_2 = \left[\frac{2(3-C)}{l_1^2}(l_1 + l_2) - \frac{3(2-C)}{l_1^3}(l_1 + l_2)^2\right]k$$

$$c_3 = -\left[\frac{3-C}{l_1^2} + \frac{3(2-C)}{l_1^3}(l_1 + l_2)\right]k$$

$$c_4 = \frac{2-C}{l_1^3}k$$

The equation of the tangential inclination angle has the form

$$\Theta(l) = c_0 + c_1 l + \frac{1}{2} c_2 l^2 + \frac{1}{3} c_3 l^3 + \frac{1}{4} c_4 l^4 \qquad (5\text{-}12)$$

where $$c_0 = \left[\frac{C-6}{12} l_1 + \frac{3-C}{3l_1^2} (l_1 + l_2)^3 + \frac{2-C}{4l_1^3} (l_1 + l_2)^4 \right] k$$

At the end of the widening curve, the value of angle *Θ*(*l*) is $\Theta(2l_1 + l_2) = \frac{6+C}{6} k l_1 + k l_2$; with respect to coefficient *C* = 1 value $\Theta(2l_1 + l_2) = \frac{7}{6} k l_1 + k l_2$.

For *C* = 1 the values of the numerical coefficients are

$$c_0 = \left[-\frac{5}{12} l_1 + \frac{2}{3l_1^2} (l_1 + l_2)^3 + \frac{1}{4l_1^3} (l_1 + l_2)^4 \right] k$$

$$c_1 = \left[1 - \frac{2}{l_1^2} (l_1 + l_2)^2 - \frac{1}{l_1^3} (l_1 + l_2)^3 \right] k$$

$$c_2 = \left[\frac{4}{l_1^2} (l_1 + l_2) - \frac{3}{l_1^3} (l_1 + l_2)^2 \right] k$$

$$c_3 = -\left[\frac{2}{l_1^2} + \frac{3}{l_1^3} (l_1 + l_2) \right] k$$

$$c_4 = \frac{1}{l_1^3} k$$

On expanding functions $\cos\Theta(l)$ and $\sin\Theta(l)$ into Taylor series using the Maxima program [16] and after the integration of respective terms it is possible to obtain parametric equations:

$$x(l) = x(l_0) + \cos\Theta_0 (l - l_0) - \frac{k}{2} \sin\Theta_0 (l - l_0)^2 - \frac{k^2}{6} \cos\Theta_0 (l - l_0)^3 + \frac{k^3}{24} \sin\Theta_0 (l - l_0)^4 + \frac{k^4}{120} \cos\Theta_0 (l - l_0)^5 \qquad (5\text{-}13)$$

$$y(l) = y(l_0) + \sin\Theta_0 (l - l_0) + \frac{k}{2} \cos\Theta_0 (l - l_0)^2 - \frac{k^2}{6} \sin\Theta_0 (l - l_0)^3 - \frac{k^3}{24} \cos\Theta_0 (l - l_0)^4 + \frac{k^4}{120} \sin\Theta_0 (l - l_0)^5 \qquad (5\text{-}14)$$

where $l_0 = l_1 + l_2$, $\Theta_0 = \Theta(l_1 + l_2)$.

6. EXECUTION OF THE INTERTRACK SPACE WIDENING

Point 5 provides calculation equations relating to the first intertrack space widening curve which is a half of the designed geometric layout. The execution of the intertrack space widening makes it necessary to introduce a second curve equipped with opposite curvature. In equations describing functions $k(l)$ and $\Theta(l)$ relating to this curve, it will be more advantageous to make use of abscissa $\bar{l} = (4l_1 + 2l_2) - l$, which enables us to employ, after some modification, equations valid for the first curve.

Respective equations for the second curve are as follows:

– for $l \in \langle 2l_1 + l_2, 3l_1 + l_2 \rangle$, i.e. $\bar{l} \in \langle l_1 + l_2, 2l_1 + l_2 \rangle$

$$k(\bar{l}) = -\left[c_1 + c_2\bar{l} + c_3\bar{l}^2 + c_4\bar{l}^3\right] \tag{6-1}$$

$$\Theta(\bar{l}) = c_0 + c_1\bar{l} + \frac{1}{2}c_2\bar{l}^2 + \frac{1}{3}c_3\bar{l}^3 + \frac{1}{4}c_4\bar{l}^4 \tag{6-2}$$

when the same coefficients are obligatory as in equations (5-11) and (5-12).

– for $l \in \langle 3l_1 + l_2, 3l_1 + 2l_2 \rangle$, i.e. $\bar{l} \in \langle l_1, l_1 + l_2 \rangle$

$$k(\bar{l}) = k \tag{6-3}$$

$$\Theta(\bar{l}) = -\frac{5k}{12} + k\bar{l} \tag{6-4}$$

– for $l \in \langle 3l_1 + 2l_2, 4l_1 + 2l_2 \rangle$, i.e. $\bar{l} \in \langle 0, l_1 \rangle$

$$k(\bar{l}) = -\left(\frac{k}{l_1}\bar{l} + \frac{k}{l_1^2}\bar{l}^2 + \frac{k}{l_1^3}\bar{l}^3\right) \tag{6-5}$$

$$\Theta(\bar{l}) = \frac{k}{2l_1}\bar{l}^2 + \frac{k}{3l_1^2}\bar{l}^3 - \frac{k}{4l_1^3}\bar{l}^4 \tag{6-7}$$

The second curve of the intertrack space widening is generated in consequence of a double mirror reflection of the first curve: at first with respect to the perpendicular straight passing through the entire centre of the system (with abscissa $x(2\,l_1 + l_2)$). In the next step a paralel straight passing through the value of the ordinate in the centre of the system (i.e. through $y(2\,l_1 + l_2)$). Coordinates of the analyzed curve are determined by means of the following equations:

– for $x \in \langle x(2l_1 + l_2), x(3l_1 + l_2) \rangle$, i.e. $\bar{l} \in \langle l_1 + l_2, 2l_1 + l_2 \rangle$

$$x(\bar{l}) = 2x(2l_1 + l_2) - x(l_0) - \cos\Theta_0(\bar{l} - l_0) + \frac{k}{2}\sin\Theta_0(\bar{l} - l_0)^2 + \frac{k^2}{6}\cos\Theta_0(\bar{l} - l_0)^3$$
$$- \frac{k^3}{24}\sin\Theta_0(\bar{l} - l_0)^4 - \frac{k^4}{120}\cos\Theta_0(\bar{l} - l_0)^5 \tag{6-8}$$

$$y(\bar{l}) = 2y(2l_1 + l_2) - y(l_0) - \sin\Theta_0(\bar{l} - l_0) - \frac{k}{2}\cos\Theta_0(\bar{l} - l_0)^2 + \frac{k^2}{6}\sin\Theta_0(\bar{l} - l_0)^3$$

$$+\frac{k^3}{24}\cos\Theta_0\left(\bar{l}-l_0\right)^4-\frac{k^4}{120}\sin\Theta_0\left(\bar{l}-l_0\right)^5 \tag{6-9}$$

– for $x\in\left\langle x\left(3l_1+l_2\right),x\left(3l_1+2l_2\right)\right\rangle$

The tangent inclination at the initial point of the circular arc is $\overline{s_1}=-\tan\left[\Theta\left(l_1\right)+\alpha\right]$, while at the end point $\overline{s_2}=-\tan\Theta\left(l_1\right)$. The coordinates of the arc centre are determined for point $\overline{S}\left(\overline{x_S},\overline{y_S}\right)$.

$$\overline{x_S}=x\left(3l_1+l_2\right)-\frac{\overline{s_1}}{\sqrt{1+\overline{s_1^2}}}R \tag{6-10}$$

$$\overline{y_S}=y\left(3l_1+l_2\right)-\frac{1}{\sqrt{1+\overline{s_1^2}}}R \tag{6-11}$$

The equation of the circular arc is

$$y(x)=\overline{y_S}+\sqrt{R^2-\left(\overline{x_S}-x\right)^2}\quad,\quad x\in\left\langle x\left(3l_1+l_2\right),x\left(3l_1+2l_2\right)\right\rangle \tag{6-12}$$

and the coordinates of the circular arc end are as follows

$$x\left(3l_1+2l_2\right)=x\left(3l_1+l_2\right)+\tan\frac{\alpha}{2}\left(\frac{1}{\sqrt{1+\overline{s_1^2}}}+\frac{1}{\sqrt{1+\overline{s_2^2}}}\right)R \tag{6-13}$$

$$y\left(3l_1+2l_2\right)=y\left(3l_1+l_2\right)-\tan\frac{\alpha}{2}\left(\frac{\overline{s_1}}{\sqrt{1+\overline{s_1^2}}}+\frac{\overline{s_2}}{\sqrt{1+\overline{s_2^2}}}\right)R \tag{6-14}$$

– for $x\in\left\langle x\left(3l_1+2l_2\right),x\left(4l_1+2l_2\right)\right\rangle$, i.e. $\bar{l}\in\left\langle 0,l_1\right\rangle$

$$x\left(\bar{l}\right)=2x\left(2l_1+l_2\right)-\left[\bar{l}-\frac{k^2}{40l_1^2}\bar{l}^5-\frac{k^2}{36l_1^3}\bar{l}^6+\frac{5k^2}{504l_1^4}\bar{l}^7+\frac{k^2}{96l_1^5}\bar{l}^8+\left(\frac{k^4}{3456l_1^4}-\frac{3k^2}{864l_1^6}\right)\bar{l}^9\right] \tag{6-15}$$

$$y\left(\bar{l}\right)=2y\left(2l_1+l_2\right)-\left[\frac{k}{6l_1}\bar{l}^3+\frac{k}{12l_1^2}\bar{l}^4-\frac{k}{20l_1^3}\bar{l}^5-\frac{k^3}{336l_1^3}\bar{l}^7-\frac{k^3}{192l_1^4}\bar{l}^8+\frac{k^3}{2592l_1^5}\bar{l}^9\right] \tag{6-16}$$

7. EXAMPLE OF PRACTICAL APPLICATION

The intertrack space widening is assumed to reach the value of 12 m (e.g. due to an existing island platform) along the railway line designed for trains travelling at a speed of *V* = 140 km/h. The solution of the problem will be based on the application of two opposite curves provided with circular arc segments in the middle part, and segments of nonlinear curvature in extreme areas. The minimal radius of the circular arc in the mid-part is calculated by the use of the formula

$$R_{\min} = \left(\frac{V}{3.6}\right)^2 \frac{1}{a_{per}} \tag{7-1}$$

where a_{per} – permissible value of unbalanced acceleration.

On the assumption that is a_{per} = 0.85 m/s^2, the value of R_{min} is equal to 1779 m. Further calculations assume the circular arc radius R = 1800 m as the outset data.

Along the length of the nonlinear curvature segments (where coefficient C = 1) the transverse acceleration $a(l)$ is desribed by the following formula

$$a(l) = \left(\frac{V}{3.6}\right)^2 \frac{k}{l_1} l + \left(\frac{V}{3.6}\right)^2 \frac{k}{l_1^2} l^2 + \left(\frac{V}{3.6}\right)^2 \frac{k}{l_1^3} l^3$$

The speed of acceleration change $\psi = \frac{V}{3.6}\frac{d}{dl}a(l)$ is here variable along its length, and therefore it is necessary to satisfy the condition

$$\psi_{\max} = \max\left[\left(\frac{V}{3.6}\right)^3 \frac{k}{l_1} + \left(\frac{V}{3.6}\right)^3 \frac{2k}{l_1^2} l + \left(\frac{V}{3.6}\right)^3 \frac{3k}{l_1^3} l^2\right] \le \psi_{per}$$

where ψ_{per} – permissible value of the speed of acceleration change.

It follows from Equation (10) that the value ψ_{max} appears at point $l_0 = \frac{1}{3} l_1$. Thus, finally the value should be

$$\psi_{\max} = \left(\frac{V}{3.6}\right)^3 \frac{4}{3} \frac{k}{l_1} \le \psi_{per}$$

The formula for the minimum length of the nonlinear curvature segments is

$$l_1 \ge \left(\frac{V}{3.6}\right)^3 \frac{4}{3} \frac{k}{\psi_{per}} \tag{7-2}$$

Making an assumption that the permissible value of acceleration increment ψ_{per} = 0.3 m/s^3 (as for single transition curves of linear curvature) one can obtain the condition that $l_1 \ge$ 145.218 m. In the calculations made the lengths of the extreme segments $l_1 = l_3$ = 146 m are treated as the output ones. Table 2 presents the procedure of the calculations in which initial stages the length of the mid-segment is equal to the length of the extreme segments.

Table 2. Comparison of the characteristic values for successively generated variants

				$l = l_1$		$l = l_1 + l_2$		$l = 2\,l_1 + l_2$	
	R **[m]**	l_1 **[m]**	l_2 **[m]**	$x(l)$ **[m]**	$y(l)$ **[m]**	$x(l)$ **[m]**	$y(l)$ **[m]**	$x(l)$ **[m]**	$y(l)$ **[m]**
1	1800	146	146	145.966	2.368	291.362	15.177	435.243	39.724
2	2000	131	131	130.980	1.716	261.627	11.002	391.386	28.819
3	2200	119	119	118.987	1.287	237.769	8.256	356.000	21.636
4	2400	109	109	108.992	0.990	217.851	6.350	326.354	16.647
5	2600	101	101	100.995	0.785	201.899	5.034	302.562	13.198
6	2800	94	94	93.996	0.631	187.930	4.049	281.695	10.618
7	3000	88	88	87.997	0.516	175.950	3.312	263.782	8.686
8	3000	88	80	87.997	0.516	167.958	2.951	255.804	8.091
9	3000	88	70	87.997	0.516	157.967	2.530	245.830	7.378
10	3000	88	60	87.997	0.516	147.974	2.143	235.853	6.697
11	3000	88	50	87.997	0.516	137.980	1.788	225.874	6.050
12	3000	88	49	87.997	0.516	136.981	1.755	224.876	5.987
13	3000	88	49.3	87.997	0.516	137.281	1.765	225.175	6,006
14	3000	88	49.2	87.997	0.516	137.181	1.761	225.076	5.999
15	3000	88	49.22	87.997	0.516	137.201	1.762	225.096	6.00052
16	3000	88	49.21	87.997	0.516	137.191	1.762	225.086	5.99989
17	3000	88	49.213	87.997	0.516	137.194	1.762	225.089	6.00008
18	**3000**	**88**	**49.212**	**87.997**	**0.516**	**137.193**	**1.762**	**225.088**	**6.00001**
19	3000	88	49.211	87.997	0.516	137.192	1.762	225.087	5.99995

At the beginning of calculations the final ordinate $y(2\,l_1 + l_2)$ is determined for the adopted R = 1800 m, $l_1 = l_3$ = 146 m and l_2 = 146 m. It amounts to 39.724 m, which means that it significantly deviates from the required value of 6 m (that is, a half of the assumed intertrack space extension). It turns out that an essential procedure to reduce it, is to increase radius R. This is an advantageous situation causing simultaneously a possibility of reducing the lengths of extreme segments according to condition (7-2).

In an iterative way one can arrive at radius R = 3000 m and adequate lengths $l_1 = l_2 = l_3$ = 88 m for which the final ordinate is 8.686 m. Further reducing of the ordinate is obtained by reducing length l_2. By appropriate conditions, using the trial method it is possible to find out a solution for which the final ordinate is 6 m. In the situation analyzed the ordinate is for l_2 = 49.212 m. The diagram of the curvature along the entire length of the intertrack space widening is shown in Fig. 7, while the tangential inclination angle is given in Fig. 8. Fig. 9 illustrates the extension ordinates using the rectangular coordinate system.

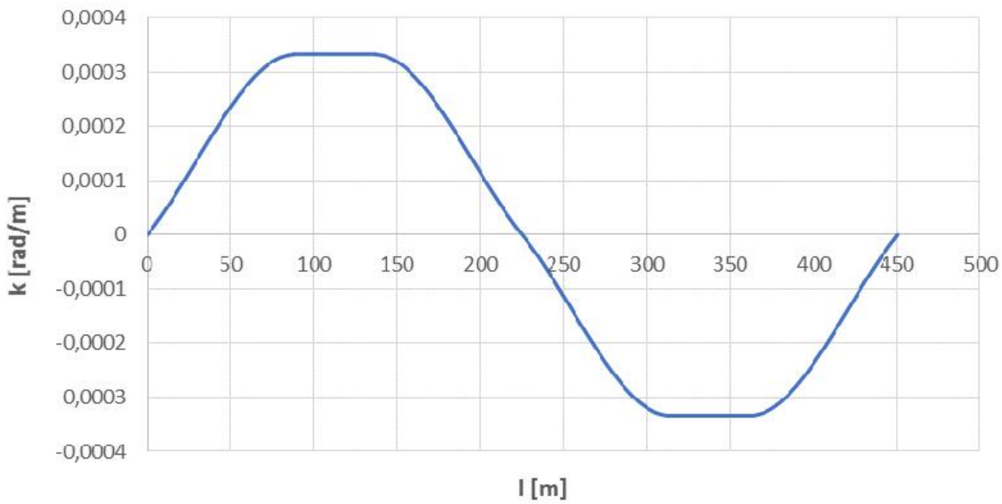

Fig. 7. Diagram of curvature along the determined widening intertrack space length (k = 1/3000 rad/m, l_1 = l_3 = 88 m, l_2 = 49.212 m)

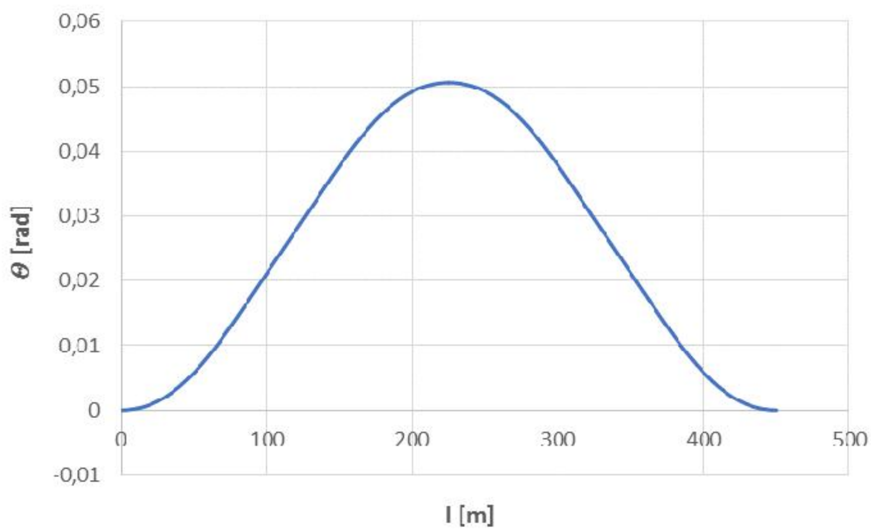

Fig. 8. Diagram of the tangent inclination angle along the length of the determined intertrack space widening (k = 1/3000 rad/m, l_1 = l_3 = 88 m, l_2 = 49.212 m)

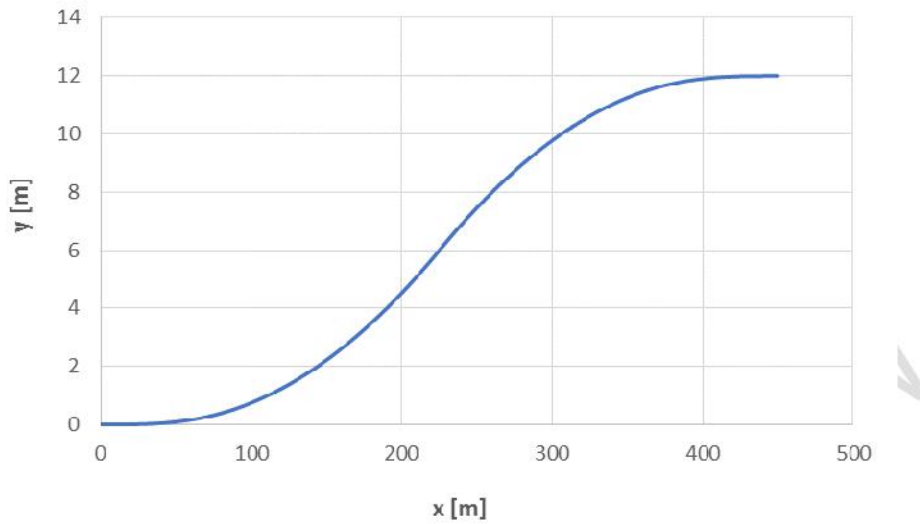

Fig. 9. Diagram of horizontal ordinates along the length of the determined intertrack space widening (k = 1/3000 rad/m, $l_1 = l_3$ = 88 m, l_2 = 49.212 m, in non-comparative scale)

8. CONCLUSION

Although the binding design principles for the intertrack space widening were already put into use a dozen years ago (and are still applicable) it is worthwhile making an attempt at a new approach to the problem in view of the tremendous progress that has taken place since that time in the calculation technique.

Taking the above into account a thesis has been submitted relating to the intertrack space extension which should be formed by the application of two connected curves of appropriate curvature distribution along this length with opposite curvature signs (that is, with contrary arcs). An assumption is made that along the length of the widening one can find nonlinear curvature segments in the extreme zones of both the used curves, whereas, the curvature in the mid-zone is constant (which means that there appears a circular arc). To obtain a correct solution it is necessary to satisfy kinematic conditions and to find out the final ordinate of the first curve equal to a half of the assumed extention value.

To solve the problem an advantage has been taken of the analytical design technique equipped with curvature distribution using the differential equations and a mathematical record of ordinates in various zones of the extention curves. The obtained solutions are of a universal type and make it possible to deal with any curvature values in the circular arc zone and any arbitrary lengths of respective zones. To obtain a required end ordinate of the intertrack space widening does not cause any special difficulties, but makes it necessary to correct the assumed geometric parameters.

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