ON PAIRWISE L-CLOSED SPACES

Abstract: In this paper we define pairwise L-closed spaces and study their properties, we obtain several results concerning pairwise L-closed spaces, and some product theorems. Some examples dealing with pairwise L-closed spaces are discussed.

Key words: pairwise L-closed space,pairwise P-space,p-Lindlöf space,s-Lindlöf space,p-continuous function, p-homeomorphism function.

1.Introduction

In mathematics, the notion of bitopological spaces are introduced and studied by J.C Kelly [1] in 1963, he defined pairwise Hausdorff, pairwise regular, pairwise normal spaces, and obtained generalizations of several standard results such as Urysohn's Lemma and Tietze's extension theorem.

Since then several mathematicians studied various concepts in bitopological spaces which turned to be an important field in general topology. We use $\mathbb{R}$ and $\mathbb{N}$ to denote the set of all real and natural numbers respectively, p- to denote pairwise and τcoc, τdis, τs, τu, τr to denote cocountable, discrete, Sorgenfrey, usual and right ray topologies on $\mathbb{R}$ or $\mathbb{N}$

Also the τ-closure of a set A is denoted by clA. Also we study the properties of pairwise L-closed spaces and their relations with other related concepts.

2.Preliminaries

Definition2.1: A bitopological space (X,τ1,τ2) is said to be pairwise L-closed space if each τ1-Lindlöf subset of X is τ2-closed and each τ2-Lindlöf subset of X is τ1-closed.

Definition2.2: [2] A family $\widetilde{F}$ of non empty τ1-closed or τ2-closed subsets of a bitopological space (X,τ1,τ2) is called p-closed if it contains at least one member F1 and at least one member F2 such that F1 is τ1-closed proper subset of X and F2 is τ2-closed proper subset of X. A family $\widetilde{F}$ of non empty subsets of X is τ1τ2-closed if every member of F is τ1-closed or τ2-closed.

Definition2.3: [3] A cover Ũ of a bitopological space (X,τ1,τ2) is called $\tau 1\tau 2$-open cover if Ũ⊆ τ1∪τ2, and it is called p-open cover for X if it contains at least one non empty member of τ1 and at least one non empty member of τ2.

Definition2.4: [6] A bitopological space (X,τ1,τ2) is said to be p-Lindlöf if every p-open cover for X has a countable subcover. Also X is called s-Lindlöf if every $\tau 1\tau 2$-open cover for X has a countable subcover.

Definition2.5: [6] (X,τ1,τ2) is τ1-*Lindlöf* with respect to τ2 if for each τ1-open cover for X there is a countable τ1-open subcover. Now if X is τ1-*Lindlöf* with respect to τ2 and it is τ2-Lindlöf with respect to τ1, then X is called B-*Lindlöf*.

Definition2.6: [7] A bitopological space (X,τ1,τ2) is called pairwise T1 if for each pair of distinct points x, y in X, there exists a τ1-neighbourhood U of x and a τ2-neighbourhood V of y such that x∈U, y∉U and y∈V, x∉V.

Definition2.7: [7] A bitopological space (X,τ1,τ2) is called p-Hausdorff if ∀x≠y in X, there exists a τ1-neighbourhood U of x and a τ2-neighbourhood V of y such that x∈U, y∈V, and U∩V=ϕ.

Definition2.8: [4] A bitopological space (X,τ1,τ2) is said to be a pairwise P-space if countable intersection of τ1-open subsets of X is a τ2-open subset of X and countable intersection of τ2-open subsets of X is a τ1-open subset of X. A point x∈X is called a P-point if the intersection of countably many τ1-neighborhoods of x is a τ2-neighborhood of x, and the intersection of countably many τ2-neighborhoods of x is a τ1-neighborhood of x.

Definition2.9: [6] A bitopological space (X,τ1,τ2) is called second countable if (X,τ1) is second countable and (X,τ2) is second countable.

Definition2.10: [6] A bitopological space (X,τ1,τ2) is called Lindlöf

(resp. compact) if it is τ1-Lindlöf (resp. τ1-compact) and τ2-Lindlöf (resp. τ1-compact).

Example2.11: Consider the bitopological space ($\mathbb{R}$,τu,τr) let A=[0,1], then A is a τu-closed subset of $\mathbb{R}$. Furthermore A is τu-Lindlöf because ($\mathbb{R}$,τu) is Lindlöf. Now A is neither closed nor open in (R,τr), hence (R,τu,τr) is not a pairwise L-closed space.

Proposition2.12: [8] In a bitopological space (X,τ1,τ2), if every countable subset of X is closed, then every countable subset is discrete and every compact subset is finite.

Proof: Let $F = \cup_{k=1}^{\infty}\{x_k\}$ where x_k ∈X k∈ $\mathbb{N}$, then F is a countable subset of X and closed by the assumption.

Each point of F is an isolated point, thus F is discrete. Now let A be a compact subset of X and Ũ={{x}:x∈X} be an open cover for A, then {x} is a clopen subset of X.

If A is a compact subset of X, then there exists a finite subset {x1, x2,. . . ,xm} such that A⊆ $\cup_{k=1}^{\infty}${xn} n,m∈ $\mathbb{N}$.Thus A is a finite subset of X.

Corollary2.13: [8] If (X,τ1,τ2) is a pairwise L-closed space, every countable subset of X is closed, discrete and every compact subset of X is finite.

Proposition2.14: Every subspace of a pairwise L-closed space is pairwise L-closed.

Proof: Suppose that (X,τ_1,τ_2) is a pairwise L-closed space and Y is a subspace of it, let F be a τ_1-Lindlöf subset of Y, then F is a τ_1-Lindlöf subset of X, hence F is a τ_2-closed subset of X because X is a pairwise L-closed space.

Similarly if we suppose that G is a τ_1-Lindlöf subset of Y, then G is τ_2-closed. Thus Y is a pairwise L-closed space.

Corollary2.15 : If (X,$\tau 1$,$\tau 2$) is a p-Hausdorff pairwise P-space, then X is a pairwise L-closed space.

Proof: Let F be a $\tau 1$-Lindlöf subset of X, let x$\in$X such that x$\notin$F. Since (X,$\tau 1$,$\tau 2$) is p-Hausedorff, $\exists$ a sequence (wk) of $\tau 1$-open subsets such that

x$\in \cup_{k=1}^{\infty}$(wk), also $\exists$ a sequence (vk) of $\tau 2$-open subsets such that

F$\subseteq \cup_{k=1}^{\infty}$(vk), and wk$\cap$vk=$\phi$ $\forall$k$\in \mathbb{N}$.

X is pairwise P-space, so $\cap_{k=1}^{\infty}$wk is $\tau 2$-open subset containing x and

$\cap_{k=1}^{\infty}$wk$\cap$F=ϕ, so F is a $\tau 2$-closed subset of X.

Similarly if we suppose that G is a $\tau 2$-Lindlöf subset of X, we will get that it is $\tau 1$-closed. Hence X is a pairwise L-closed space.

Proposition2.16: Every Lindlöf pairwise L-closed bitopological space is a pairwise P-space

Proof: Let (X,$\tau 1$,$\tau 2$) be a Lindlöf pairwise L-closed space, let A=$\cap_{k=1}^{\infty}$uk be a $\tau 1$-Gδ set,

then A is a $\tau 2$-open subset of X since X-A=X-$\cap_{k=1}^{\infty}$uk= $\cup_{k=1}^{\infty}$(X-uk) is a $\tau 1$-Fσ-set,

so X-A is a $\tau 1$-Lindlöf subset of X because X is Lindlöf, but X is a pairwise L-closed space,

so X-A is a $\tau 2$-closed subset of X. Hence A is a $\tau 2$-open subset of X.

Similarly, if we suppose that B is a $\tau 2$-Gδ-set, we will get that it is a $\tau 1$-open subset of X. Thus X is a pairwise P-space.

Corollary2.17: For a p-Hausdorff Lindlöf bitopological space (X,$\tau 1$,$\tau 2$), X is a pairwise L-closed space if and only if it is pairwise P-space.

The proof follows from 2.15 and 2.16.

Remark2.18: There is a Lindlöf pairwise L-closed space which is not pairwise P-space.

Example: [5] Consider a non empty set $X^{\omega_\circ}$ equipped with two discrete topologies

Let points of the Stone-Ce$\widetilde{c}$h compactification βX be τdis-ultrafilters of $X^{\omega_\circ}$

Take w to be a τdis-ultrafilter on $X^{\omega_\circ}$, each of whose members has power ω_1

i.e w$\in \beta X$ and w$\notin${cl$_{\beta X}$ A: A$\subseteq$X,|A|= ω_1}.

Let wX=$X^{\omega_\circ}\cup${x}, then wX is τdis-Lindlöf because if Ũ={u_α:$\alpha \in \Lambda$} is a τdis-open cover for wX and p$\in u_{\alpha\circ}$for some $\alpha\circ \in \Lambda$ where $u_{\alpha\circ}$is a τdis-open subset, then wX-$u_{\alpha\circ}$is a τdis-closed subset. But $X^{\omega_\circ}$ is τdis-Lindlöf, so wX-$u_{\alpha\circ}$is τdis-Lindlöf.

Since $X^{\omega_\circ}$ is a pairwise L-closed, wX-$u_{\alpha\circ}$ is τdis-closed, hence wX is pairwise L-closed.

Now w contains a countable subfamily with empty intersection because $|X^{\omega_\circ}|$

is non-measurable, so w is a τdis-G_δ-subset of wX, i.e wX is not pairwise P-space.

Definition2.19: [1] In a bitopological space (X,τ1,τ2), τ1 is regular with respect to τ2 if $\forall$x$\in$X and each τ1-closed set F such that x$\notin$F, there exists a τ1-open set U and a τ2-open set V such that x$\in$U and F$\in$V and U$\cap$V=ϕ.

Definition2.20: [1] A bitopological space (X,τ1,τ2) is called p-regular if τ1 is regular with respect to τ2 and τ2 is regular with respect to τ1.

Definition2.21: In a bitopological space (X,τ1,τ2), a point x$\in$X has a pairwise L-closed neighborhood U if each τ1-Lindlöf subset of U containing x is τ2-closed, and each τ2-Lindlöf subset of U containing x is τ1-closed.

Proposition2.22: Let (X,τ1,τ2) be a p-regular space. If every point in X has a pairwise

L-closed neighborhood, then (X,τ1,τ2) is pairwise L-closed.

Proof: Let F be a τ1-Lindlöf subset of X, let x$\in$X such that x$\notin$F.

If U is a τ1-open subset containing x, then U is L-closed neighborhood.

Since X is p-regular, $\exists$ a τ1-open set H such that x$\in$H$\subseteq$cl2H$\subseteq$U and cl2H$\cap$F is a τ1-Lindlöf subset of U, hence cl2H$\cap$F is a τ2-closed subset of U.

U–(cl2H$\cap$F) is a τ2-open neighborhood of x, so $\{$U–(cl_2H$\cap$F)$\}\cap$F=ϕ is a contradiction.

Hence x$\in$F and F is a τ2-closed subset of X. Similarly if we assume that G is a τ2-Lindlöf subset of X, by a similar argument we will get hat G is τ1-closed.

So (X,τ1,τ2) is a pairwise L-closed space.

Definition2.23: [1] A space (X,τ1,τ2) is said to be p-normal if for a τ1-closed set C and a τ1-closed set F such that C$\cap$F =ϕ, there exist a τ1-open set G, a τ2-open set V such that F$\subseteq$G, C$\subseteq$V and V$\cap$ G=ϕ.

Proposition2.24: A p-regular pairwise L-closed space is p-normal

Proof: Let (X,τ1,τ2) be a pairwise L-closed space, let A be a τ1-Lindlöf subset of X and B be a τ2-Lindlöf subset of X such that A$\cap$B=ϕ, then A is τ2-closed and B is τ1-closed because X is pairwise L-closed.

Since (X,τ1,τ2) is p-regular we have, $\forall$a$\in$A, $\exists$a τ1-closed subset Fa and a τ2-open subset Ga such that a$\in$Ga$\subseteq$Fa$\subseteq$X-B.

Now $\forall$b$\in$B, $\exists$a τ1-open subset Cb and τ2-closed subset Mb such that

b$\in$Cb$\subseteq$Mb$\subseteq$X-A. Let Ũ={Ga:a$\in$A} be a τ2-open cover for A and

$\widetilde{V}$={Cb:b$\in$B} be a τ1-open cover for B.

A and B are τ1-Lindlöf and τ2-Lindlöf respectively, so A$\subseteq$ $\cup_{k=1}^{\infty}$Gk and B$\subseteq$ $\cup_{k=1}^{\infty}$Ck .

Let V1=C1 and for each positive integer k>1, let Vk=Ck-$\cup_{j=1}^{k-1}$Fj .

For each positive integer k, let Hk=Gk-$\cup_{j=1}^{k}$Mj and V=$\cup_{k=1}^{\infty}$Vk , H=$\cup_{k=1}^{\infty}$Hk , then V is a τ1-open subset of X and H is a τ2-open subset of X.

Also A$\subseteq$V, B$\subseteq$H. Furthermore, if x$\in$H$\cap$V, then x$\in$Hi$\cap$Vl for some i,l$\in$ $\mathbb{N}$,

and so x$\in$(Gi-$\cup_{j=1}^{i}$Mj)$\cap$(Cl-$\cup_{j=1}^{l-1}$Fj).

Consedering separately the cases $i > l$ and $i \leq l$ yields a contradiction and so H$\cap$V=ϕ.

Thus X is p-normal.

Definition2.25 : A bitopological space (X,τ1,τ2) is said to be pairwise hereditarily Lindlöf if every τ1-subspace of X is Lindlöf and τ2-subspace of X is Lindlöf.

Corollary2.26: For a pairwise hereditary Lindlöf bitopological space (X,τ1,τ2), the following are equivalent:

a. X is a pairwise L-closed space.

b. X is a countable discrete space.

Proposition2.27: Every p-regular space which can be represented as a countable union of subspaces each of which has the pairwise L-closedness property has itself the pairwise L-closedness property.

Proof: Suppose that X=$\cup_{k=1}^{\infty}$Xk , Xk is pairwise L-closed subspace.

Let A be a τi-Lindlöf subset of Xk for some k$\in \mathbb{N}$, then A is a τi-Lindlöf subset of X where X is p-regular.

But A is a τj-closed subset of Xk because Xk is a pairwise L-closed subspace of X, hence A is a τj-closed subset of X $\forall$i,j=1,2 i$\neq$j.

Thus X is pairwise L-closed.

Proposition2.28: In the bitopological space (X,τ1,τ2), the sum $\oplus_{\alpha\in\Lambda}X\alpha$ where Xα for some $\alpha \in \Lambda$ has a pairwise L-closedness property if and only if all spaces Xα have a pairwise L-closedness property.

Proof: $\Longrightarrow$) Suppose that the sum $\oplus_{\alpha\in\Lambda}X\alpha$ where X$\alpha \neq \phi$ for some $\alpha \in \Lambda$ is pairwise

L-closed space, then Xα is a pairwise L-closed subset of $\oplus_{\alpha\in\Lambda}X\alpha$ because Xα is a closed subspaces of $\oplus_{\alpha\in\Lambda}X\alpha$

$\Longleftarrow$) Suppose that F is a τi-Lindlöf subset of $\oplus_{\alpha\in\Lambda}X\alpha$, then F$\cap$Xα is a τi-Lindlöf subset of Xα.

But Xα is pairwise L-closed, hence F is a τj-closed subset of X$\alpha\forall\alpha \in \Lambda$,so F is a τj-closed subset of $\oplus_{\alpha\in\Lambda}X\alpha$ $\forall\alpha \in \Lambda$,i,j=1,2 i$\neq$j. Thus $\oplus_{\alpha\in\Lambda}X\alpha$ is pairwise L-closed.

Definition2.29: [4] A bitopological space (X,τ1,τ2) is pairwise almost Lindlöf if every τi-open cover $\tilde{U}$=$\{u_\alpha$:$\alpha \in \Lambda\}$of X has a countable subcollection

$\tilde{U}'$=$\{u_\alpha$:$\alpha \in \Lambda_1 \subseteq \Lambda\}$ of Λsuch that X=$\cup_{\alpha\in\Lambda}$clj u_α $\forall$i,j=1,2 i$\neq$j.

Definition2.30: A bitopological space (X,τ1,τ2) is called pairwise hereditarily almost Lindlöf if every subspace of X is pairwise almost Lindlöf.

Proposition2.31: If (X,τ1,τ2) is a pairwise L-closed space, then the following are equivalent:

a. X is pairwise hereditarily almost Lindlöf.

b. X is pairwise hereditarily Lindlöf.

c. X is countable discrete.

Proof: c$\rightarrow$a) Suppose that X is a countable discrete space such that X=$\cup_{k\in\mathbb{N}}$Fk , Fk is a τi-Lindlöf subset of X. If $\tilde{U}$=$\{u_\alpha$:$\alpha \in \Lambda\}$ is a τi-open cover for Fk and $\tilde{U}'$=$\{u_\alpha$:$\alpha \in$ Λ1$\subseteq \Lambda\}$is a countable subcollection of $\Lambda$where u_α is a τi-open subset of X, then Fk$\subseteq \cup_{\alpha\in\Lambda_1} u_\alpha$. But X is pairwise L-closed,hence Fk is τj-closed, and X=$\cup_{k\in\mathbb{N}}$Fk $\subseteq \cup_{\alpha\in\Lambda_1} u_\alpha$ $\forall$i,j=1,2 i$\neq$j.Thus X is pairwise hereditarily almost Lindlöf.

Proposition2.32: Every τ1τ2-open cover for a p-regular pairwise L-closed space (X,τ1,τ2) has locally countable open refinement.

Proof: Suppose that $\tilde{U}$={u_x: x∈X} is a $\tau 1\tau 2$-open cover for X.Since X is p-regular, ∀x∈X, there exists a $\tau 1$-open set u_x such that x∈v_x ⊆cl2v_x ⊆ u_x for some $\tau 1$-open set v_x.

Let {u_{x_k} :k∈ ℕ} be a countable subcover for $\tilde{U}$. The sets Hk= u_{x_k}-(cl2v_{x_1}∪ cl2v_{x_2}∪. . .) are $\tau 1$-open or $\tau 2$-open that constitude a $\tau 1\tau 2$-open cover for X.

∀x∈X we have x∈H$k_{(x)}$ where $k_{(x)}$ is the smallest integer such that x∈ u_{x_k}.

{Hk: k∈ ℕ} refines $\tilde{U}$ and it is locally countable because v_{x_k}∩ Hj =ϕ ∀j>k.

3.Product properties of pairwise L-closed spaces

Definition3.1: [6] If (X,$\tau 1$,$\tau 2$) and (Y,σ_1,σ_2) are two bitopological spaces, a function f:(X,$\tau 1$,$\tau 2$)→(Y,σ_1,σ_2) is said to be p-continuous if f:(X,$\tau 1$)→(Y,σ_1) is continuous and f:(X,$\tau 2$)→(Y,σ_2) is continuous.

Proposition3.2 : Let (X,$\tau 1$,$\tau 2$) and (Y,σ_1,σ_2) be two bitoplogical spaces such that (Y,σ_1,$\sigma 2$) is a pairwise L-closed space, if f: (X,$\tau 1$,$\tau 2$)→(Y,σ_1,σ_2) is a p-continuous one to one function, then (X,$\tau 1$,$\tau 2$) is a pairwise L-closed space.

Proof: Suppose that f:(X,$\tau 1$,$\tau 2$)→(Y,σ_1,σ_2) is a p-continuous one to one function.

Let (Y,σ_1,σ_2) be a pairwise L-closed space, let F be a $\tau 1$-Lindlöf subset of X, then f(F) is $\sigma 2$-Lindlöf because f is a p-continuous function, but (Y,σ_1,σ_2) is a pairwise L-closed space, so f(F) is a $\sigma 2$-closed subset of Y.

Now F=f^{-1}(f(F)) is a $\tau 2$-closed subset of X since f is one to one.

Similarly if we suppose that G is a $\tau 2$-Lindlöf subset of X, we will get that it is $\tau 2$-closed. Hence (X,$\tau 1$,$\tau 2$) is pairwise L-closed.

Definition 3.3: [6] Let (X,$\tau 1$,$\tau 2$) and (Y,σ_1,σ_2) be two bitopological spaces, a function f:(X,$\tau 1$,$\tau 2$)→(Y,$\sigma 1$,$\sigma 2$) is called p-homeomorphism if f is bijection, p-continuous and $f^{-1}$is p-continuous. (X,$\tau 1$,$\tau 2$) and (Y,σ_1,σ_2) are called p-homeomorphic.

Definition3.4 : A function f: (X,$\tau 1$,$\tau 2$)→(Y,σ_1,σ_2) is called pairwise open if the induced functions f:(X,$\tau 1$)→(Y,σ_1) and f:(X,$\tau 2$)→(Y,σ_2) are both open. A function f:(X,τ)→(Y,σ) is closed if it sends closed sets onto closed sets. The function f: (X,$\tau 1$,$\tau 2$)→(Y,σ_1,σ_2) is called pairwise closed if the induced functions f:(X,$\tau 1$)→(Y,σ_1) and f:(X,$\tau 2$)→(Y,σ_2) are both closed.

Proposition3.5: Let (X,$\tau 1$,$\tau 2$) be a p-Lindlöf bitopological space, (Y,σ_1,σ_2) is a pairwise L-closed space, if f: (X,$\tau 1$,$\tau 2$)→(Y,σ_1,σ_2) is a bijection p-continuous function, then f is p-homeomorphism.

Proof: It suffices to show that f is a pairwise closed function. Let C be a $\tau 1$-closed proper subset of X, since X is p-Lindlöf, C is a $\tau 2$-Lindlöf subset of X ([6] Corollary2.29), hence f(C) is σ_2-Lindlöf because f is p-continuous.

But (Y,σ_1,σ_2) is a pairwise L-closed space,so f(C) is σ_1-closed.

Similarly if we suppose that F is a $\tau 2$-closed proper subset of X, we will get that f(F) is a σ_2-closed subset of Y. Hence f is a pairwise closed function and f is p-homeomorphism.

Corollary3.6: If a p-continuous function from a p-Hausdorff Lindlöf bitopological space to a pairwise L-closed space is pairwise closed,then every p-continuous bijective function is p-homeomorphism.

Proposition3.7: To be pairwise L-closed space is a bitopological property.

Proof: Let (X,τ1,τ2) be a pairwise L-closed space and (Y,σ_1,σ_2) be any bitopological space.

Suppose that h:(X,τ1,τ2)$\rightarrow$(Y,σ_1,σ_2) is p-homeomorphism, let A be a τ1-Lindlöf subset of X, then h(A) is σ_1-Lindlöf because h is p-continuous.

Since X is a pairwise L-closed space, A is τ2-closed, hence h(A) is σ_2-closed because h is pairwise L-closed.

Similarly if we suppose that B is τ2-Lindlöf, we will get that h(B) is σ_1-closed. Thus Y is a pairwise L-closed space.

Remark3.8: The product of two Lindlöf topological spaces need not to be Lindlöf. In general the product of two Lindlöf bitopological spaces is not necessarly Lindlöf as the following example shows [6].

Let X=$\mathbb{R}$×I where I is an interval, let "<" be the lexicographical order in X

Let β_1={[x,y): x<y x,y$\in$ $\mathbb{R}$}be a base for the lower limit topology (or Sorgenfrey topology) τ1 on X and β_2={(x,y]: x<y x,y$\in$ $\mathbb{R}$} be a base for τ2 on X, so (X,τ1,τ2) is a Lindlöf bitopological space.

(X×X,τ1×τ1,τ2×τ2) is not (τ1×τ1)-Lindlöf because (τ1×τ1)-closed subpace L={(x,y):x=-y x,y$\in$ $\mathbb{R}$} is not a (τ1×τ1)-Lindlöf subspace, it is discrete.

Proposition3.9: If (X,τ1,τ2) and (Y,σ_1,σ_2) are pairwise L-closed bitopological spaces such that either X or Y is p-regular, then X×Y is a pairwise L-closed space

Proof: Suppose that (X,τ_1,τ_2) and (Y,σ1,σ_2) are pairwise L-closed spaces, let Y be p-regular, let F be a (τ_1xσ_1)-Lindlöf subset of X×Y. If ($x_\circ$,$y_\circ$)$\notin$F, so ($x_\circ$,$y_\circ$)$\notin$[({$x_\circ$}×Y}$\cap$F and ({$x_\circ$}×Y)$\cap$F is a τ_2-closed subset of X×Y because Y is pairwise L-closed.

Since Y is p-regular,$\exists$ a σ_1-open set H containing $y_\circ$ such that (X×cl2H)$\subseteq${X-({$x_\circ$}×Y)$\cap$F], so the projection function π_x ((X×cl2H)$\cap$(({$x_\circ$}×Y)$\cap$F) is a τ_2-closed subset of X because π_x is p-continuous.

X-[π_x(X×cl2H)$\cap$F)×Y$\cap$(X×H)] is τ_2-open neighborhood of ($x_\circ$,$y_\circ$) disjoint from F, hence F is (τ_2 ×σ_2)-closed subset of X×Y. Similarly if we suppose that G is a (τ_2 ×σ_2)-Lindlöf subset of X×Y, then it is a (τ_1 ×σ_1)-closed subset of X×Y.

Therefore X×Y is pairwise L-closed.

Proposition3.10: The product of two finite number of pairwise L-closed p-regular spaces is pairwise L-closed.

Proof: Let {Xk:k$\in$ $\mathbb{N}$} be a family of finitely many p-regular pairwise L-closed spaces.

Let X=$\prod_{k\in\mathbb{N}}$Xk , by induction on k, for k=2 the result is given by 3.9.

Suppose that the result is true for k=n $\forall$n$\in$ $\mathbb{N}$, we want to show that it is true for k=n+1.

Now $(X_1 \times X_2 \times \ldots \times X_n) \times X_{n+1}$ is p-homeomorphic to

$X_1 \times X_2 \times \ldots \times X_n \times X_{n+1}$, so by induction hypothesis we get that

$X_1 \times X_2 \times \ldots \times X_n \times X_{n+1}$ is pairwise L-closed. Hence X is a pairwise L-closed.

Definition3.11: [6] A surjective function f:(X,τ)$\rightarrow$(Y,σ) is a Lindlöf function if whenever K is a Lindlöf closed subset of Y, we have f^{-1}(K) is a Lindlöf subset of X. A surjective function f:(X,τ_1,τ_2)$\rightarrow$(Y,σ_1,σ_2) is called pairwise Lindlöf function if the induced function f:(X,τ_i)$\rightarrow$(Y,σ_i) is Lindlöf function $\forall$i=1,2.

Proposition3.12: Let (X,τ_1,τ_2) be a pairwise L-closed space, and (Y,σ_1,σ_2) be a Lindlöf space, then π_x:X×Y$\rightarrow$X is a pairwise Lindlöf function.

Proof: Let F be a τ_1-Lindlöf subset of X, then F is a τ_2-closed subset of X because X is pairwise L-closed.

The projection function $\pi_{x|F\times Y}$ is pairwise-closed such that $(\pi_{x|F\times Y}\ )^{-1}$(x) is τ_1-Lindlöf because π_x is p-continuous. Similarly if we suppose that G is τ_2-Lindlöf, we will get that $(\pi_{x|F\times Y}\ )^{-1}$(x) is τ_2-Lindlöf.

Hence Lindlöf is a pairwise Lindlöf function.

Proposition3.13: Let (X,τ_1,τ_2) be a pairwise L-closed space and (Y,σ_1,σ_2) be any bitopological space. If f:(X,τ_1,τ_2)$\rightarrow$(Y,σ_1,σ_2) is any pairwise function and {(x,f(x)):x$\in$X} is p-Lindlöf, then f is p-continuous.

Proof: Let π_x and π_y be two projection functions, then X and f(X) are two Lindlöf sets as images of Lindlöf sets under π_x and π_y. Let π'_x=$\pi_{x|f}$, then π'_x is a pairwise closed projection function...(1) and this is because if A$\subseteq$f(X) is τi-closed subset, then A is τj-Lindlöf where f(X) is Lindlöf $\forall$i,j=1,2 i$\neq$j.

So π'_x(A) is p-Lindlöf p-closed because X is a pairwise L-closed space.

Since f is defined on X, π'_x is a bijection function. . . (2).

From (1) and (2) we get $\forall\tau_i$-open set v$\subseteq$f we have π'_x(v) is τ_j-open in X. Hence f =$\pi_y\circ(\pi'_x)^{-1}$ is p-continuous.

Proposition3.14: If (X,τ_1,τ_2) and (Y,σ_1,σ_2) are p-Hausdorff pairwise L-closed spaces, then (X×Y,$\tau_1\times\sigma_1$,$\tau_1\times\sigma_2$) is a $\tau_i\times\sigma_i$-L-closed space $\forall$i=1,2.

Proof: Let F be a $\tau i\times\sigma i$-Lindlöf subset of X×Y such that ($x_\circ$,$y_\circ$)$\notin$F.

Let $\tilde{U}$={u_n: n$\in\mathbb{N}$} be a countable collection of $\tau i\times\sigma i$-open subsets of X×Y,

then $u_n$is the union of $\tau i\times\sigma i$–basic open sets of the form Gn×Hn where Gn and Hn are τi-open subset and σi-open subset of X and Y respectively $\forall$n$\in\mathbb{N}$. Now ($x_\circ$,$y_\circ$)$\notin$ cl$_i$Gn×cljHn $\forall$i,j=1,2 i$\neq$j. F$\subseteq\cup${Gn×Hn: n$\in\mathbb{N}$}. Let K1={n$\in\mathbb{N}$: $x_\circ\notin$cliGn}, K2={n$\in\mathbb{N}$: $y_\circ\notin$cljHn}. Let F=F1$\cup$F2 where

F1={F$\cap$cliGn×cljHn:n$\in$K1} and F2={F$\cap$cliGn×cljHn:n$\in$K2}.

$x_\circ\notin\pi_x$(F1), so there exists a τ1-open subset U of X such that $x_\circ\in$U and U$\cap\pi_x$(F1)=ϕ. Also $y_\circ\notin\pi_x$(F2), so there exists a τ_2-open subset V of X such that $y_\circ\in$V and V$\cap\pi_x$(F2)=ϕ. Claim: (U×V)$\cap$F=ϕ

Let $(x,y)\in$U×V, then x$\notin\pi_x$(F1), so $(x,y)\notin$F1, also y$\notin\pi_x$(F2), so (x,y) $\notin$F2.

Hence F$\subseteq$ X×Y-(U×V), i.e F a is τj-closed subset of (U×V). Thus X×Y is pairwise L-closed.

Definition3.15: Let (X,τ) be a topological space and A$\subset$X. If for every neighborhood Ux of x$\in$X we have |Ux$\cap$A|=|A|, then x is called a complete accumulation point of A.

Proposition3.16: If (X,τ_1,τ_2) is a pairwise L-closed space and A is a τi-Lindlöf subset of X such that |A|=$\omega_1\forall$i=1,2, if x is an accumulation point of A, then x is a complete accumulation point.

Proof: Let A be a τ_i-Lindlöf subset of X such that|A|=ω_1, then A is τ_j-Lindlöf-closed $\forall$i,j=1,2 i$\neq$j because X is pairwise L-closed. Let x be an accumulation point of A, hence x$\in$cljA=A

Let Ox be a τj-Lindlöf-neighborhood of x, if we take the identity function I:A∩Ox→A , then I is a p-continuous function. Hence |Ox∩A|=|A|.

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