*Original Research Article*

# COBHAM'S THESIS AS THE SORITES PARADOX


**Abstract**

According to Cobham's thesis, computational problems can be practically (or, in other words, feasibly) computed on some computational device only if they can be computed in polynomial time. Despite the presence of many objections to this claim, they are all not decisive. Then again, there is one not explored yet critical objection to the claim. Namely, Cobham's thesis is susceptible to paradoxical reasoning emerging as a result of the indeterminacy surrounding limits of application of the vague predicate "is practical" ("is feasible"). What is more, as it is demonstrated in the present paper, any attempt to defuse such reasoning and make Cobham's thesis nonparadoxical causes it to become of no purpose at all.



**Keywords**: Complexity theory; Cobham's thesis; Sorties paradox; Truth-value assignment; Supervaluationism; Many-valued logic.




## 1. Introduction

As it is known, the celebrated Alan Cobham's paper entitled "The intrinsic computational difficulty of functions" [1] makes an explicit claim about *practicality* (in other words, feasibility) of an algorithm used for solving a problem. Consistent with the major thesis of this paper (*Cobham's thesis*, henceforth), the P complexity class (explicitly, the class of problems that can be solved in "polynomial time" [2]) is a good way to describe the set of *practically* (feasibly) *computable* problems. Along these lines, any problem that cannot be contained in P is expected to be *impractical*. Under Cobham's thesis, "P" means "easy, fast, and practical," while "not in P" means "hard, slow, and impractical."

There are many objections to Cobham's thesis [3]. For the most part, they are general complaints about analysis of algorithms (e.g., Cobham's thesis ignores constant factors, the size of the exponent and the size of the input) and models of computation (e.g., Cobham's thesis ignores models of computation other than the Turing machine and random-access machines). However, these complaints cannot be considered crucial to Cobham's thesis.

On the other hand, there is another not explored yet critical objection to this thesis: As a matter of fact, at the heart of Cobham's thesis one may recognize the phenomenon of vagueness. Specifically, predicates like "is easy", "is fast" and "is practical" (as well as "is hard", "is slow" and "is impractical") are all vague. Since they lack sharp boundaries, their extensions lead to the indeterminacy. For this reason, Cobham's thesis is susceptible to *paradoxical reasoning* that completely undermines its meaningfulness.

Let us demonstrate this in the present paper.

## 2. The sorites paradox

Recall that *the sorites paradox* is the name given to a class of paradoxical arguments which arise as a result of the indeterminacy surrounding limits of application of the vague predicates involved [4].

Let $I$ denote an input length of a problem and let the function $f(I)$ determine the running time of an algorithm used for solving the problem. This algorithm is expected to finish on an order of seconds calculated as follows:

$$A = \frac{f(I)}{K} T \text{ (sec)} \quad , \tag{1}$$

where $K$ is the number of operations that a typical CPU can do during the time interval $T$ (given in seconds).

Now, take the predicate "is practical" and consider the following argument:

> Suppose that the running time $A$ is practical and so the algorithm is practical.
> If the running time $A$ is practical, then the time $A+1$ second is practical too.
> If the time $A+1$ second is practical, then the time $A+2$ seconds is practical too.
> …
> If the time $A+10^L-1$ seconds is practical, then the time $A+10^L$ seconds is practical too.
> ___
> The running time $A+10^L$ seconds for any $L \in \mathbb{N}$ is practical and so this algorithm is practical.

As one can see, this argument employs only *modus ponens* and *cut* (that is, the chaining together of each sub-argument resulting from a single application of modus ponens). Furthermore, the premises of all the sub-arguments seem true because they are arranged as weakly as possible. More to this point, they can always be combined such that the expression $n'' - n'$ would be negligible in order to make any apparent difference in the truth-values of the respective antecedent $A + n'$ and consequent $A + n''$. As a result, the whole argument seems valid. And yet its conclusion is false: An algorithm whose running time is unlimited cannot be considered practical at all.

Similarly, any vague predicate (i.e., "is easy", "is fast" and so on) assigning the truth-values to the propositions relating to the concept of feasibility of computational problems admits the same sorites paradox.

Let us state (reminiscent of [5]) the conditions under which any argument of the form presented above is soritical.

First, denote the soritical predicate (such as "is practical", "is easy", "is fast" and so on) by Φ. At that point, the proposition asserting that an algorithm is feasible can be represented as $\Phi(A)$, and accordingly $\Phi(A)$ reads "the algorithm is practical (is easy, is fast, and so on)".

Next, the sorites can be schematically represented by way of a series of conditionals as follows:

$$\begin{array}{l} \Phi(A_1) \\ \text{If } \Phi(A_1) \text{, then } \Phi(A_2) \\ \text{If } \Phi(A_2) \text{, then } \Phi(A_3) \\ \cdots \\ \underline{\text{If } \Phi(A_n) \text{, then } \Phi(A_{n+1})} \\ \Phi(A_{n \in N}) \text{, where } 1 \leq n \leq N \text{ and } N \in \mathbb{N} \text{ can be arbitrary large} \end{array}$$

or, in the compact form,

$$\begin{array}{l} \Phi(A_1) \\ \underline{\square n \in \mathbb{N}\big(\Phi(A_n) \Longrightarrow \Phi(A_{n+1})\big)} \\ \square n \in \mathbb{N}\big(\Phi(A_N)\big) \end{array}$$

Suppose that the set $\{A_1, A_2, \dots, A_N\}$ is ordered (say, different running times are put in ascending order). Then, the predicate $\Phi$ will be soritical on the set $\{A_1, A_2, \dots, A_N\}$ (and therefore any argument of the above form using Φ and $\{A_1, A_2, \dots, A_N\}$ will be soritical) if Φ satisfies the following three constraints:

i. Φ assumes the value of true for $A_1$, the first member of the set;
ii. Φ assumes the value of false for $A_N$, the last member of the set;
iii. each adjacent pair in the set, such as $A_n$ and $A_{n+1}$, must be sufficiently similar to each other so that to appear indiscriminate in respect to Φ; that is, both $\Phi(A_n)$ and $\Phi(A_{n+1})$ must assume the value of true or neither must do.

It follows then that the key feature of any soritical predicate is *the tolerance* emerging due to the vagueness of the predicate [6]. In more detail, predicates such as “is practical” or “is fast” come out tolerant to some small changes in the running times. Consequently, the difference between adjacent (or nearby) members of the set $\{A_1, A_2, \dots, A_N\}$, on which Φ is soritical, namely, $\Delta A_i = A_n - A_{n+i}$ where $i \sim 1$, would be too small to make any difference in the truth-value taken in by the predicate Φ. Then again, the difference $\Delta A_i$ where $i \gg 1$ must change the truth-value of the predicate Φ even though $\Delta A_{i \gg 1}$ can be presented as the sum of $\Delta A_{i \sim 1}$.

Let us consider whether it is possible to make Cobham’s thesis not susceptible to paradoxical reasoning.

## 3. Treatments of Cobham's thesis

There are many ways in which the sorites paradox can be resolved. However, all of them can be reduced to the following two approaches in which the soundness of the sorites paradox is denied due to

(1). rejection of some premise(s) of a soritical argument,
(2). disallowance of its validity.

### *3.1. Supervaluationism*

Let us start with the first approach to which *supervaluationism* is related.

Recall that *supervaluation semantics* retains the classical consequence relation and classical laws at the same time as admitting *truth-value gaps* [7, 8].

Thus, in supervaluational semantics it is true that any running time $A_n$ is either practical or not. Hence, for any $n \in \mathbb{N}$ the disjunction $\Phi(A_n) \vee \neg\Phi(A_n)$, where the predicates Φ and $\neg\Phi$ stand for “is practical” and “is impractical”, respectively, assumes the value of true. In symbols,

$$\square_{n \in \mathbb{N}}\left(\Phi(A_n) \vee \neg\Phi(A_n)\right) \longrightarrow \text{true} \quad . \tag{2}$$

However, the disjuncts $\Phi(A_n)$ and $\neg\Phi(A_n)$ have no truth-value given that the predicates $\Phi$ and $\neg\Phi$ fail to refer to the limits of their application; in symbols,

$$\square_{n \in \mathbb{N}}(\Phi(A_n)) \nrightarrow \{\text{true, false}\} \quad , \tag{3}$$

$$\square_{n \in \mathbb{N}}(\neg\Phi(A_n)) \nrightarrow \{\text{true, false}\} \quad . \tag{4}$$

At the same time, recall that according to supervaluationism, the universally quantified conditional is false, so,

$$\square_{n \in \mathbb{N}}(\Phi(A_n) \Longrightarrow \Phi(A_{n+1})) \longrightarrow \text{false} \quad . \tag{5}$$

One infers from this that

$$\square_{n \in \mathbb{N}}(\Phi(A_n) \wedge \neg\Phi(A_{n+1})) \longrightarrow \text{true} \quad . \tag{6}$$

The last statement means that even though it is true that there is some borderline which separates practical (feasible) running times from impractical (infeasible) ones, there is no running time $A_n$ for which it is true that it is the borderline.

After such supervaluational treatment, Cobham’s thesis becomes impervious to paradoxical reasoning but, at the same time, of no use.

To be sure, suppose that there are running times $A'$ and $A''$ determined by the functions $f'(I)$ and $f''(I)$ associated with two different algorithms used for solving problems on one and the same computer. But since in the supervaluationist interpretation of Cobham’s thesis, one must be ignorant about the existence of a sharp boundary between “practical” and “impractical” concepts, one cannot differentiate $f'(I)$ and $f''(I)$ basing on their practicality.

### *3.2. Many-valued logic*

In contrast to supervaluational semantics which is not truth-functional, *a many-valued logic approach* to Cobham's thesis can defuse its soritical paradoxical reasoning and keep the logical connectives truth-functional.

Take, for example, a three-valued logic and suppose that the soritical predicate "is practical" (denoted by Φ) assumes the truth-value $t$ from the three-element set of truth degrees, namely,

$$\forall_{n \in \mathbb{N}(\Phi(A_n))} \longrightarrow t \in \{\text{true, indeterminate, false}\} \ . \tag{7}$$

Then, the treatment of soritical paradoxical reasoning will depend on the adopted definition of *validity*, in other words, the definition of *the designated truth degrees* which act as substitutes for the bivalent truth value "true" [9, 10]. For example, if the designated truth degrees belong to the set {true, indeterminate}, the many-valued logic treatment will result in a type (2) approach (denying the validity of the sorites paradox).

However, be that as it may, one must admit that there is no ground to decide what particular truth degree should be assigned to each sentence $\Phi(A_n)$. For, had such ground existed, it could be apparently used to draw a borderline separating practical running times from impractical ones.

Hence, once again, treating soritical paradoxical reasoning of Cobham's thesis makes this thesis unusable.

## **4. Conclusion remarks**

Cobham's thesis, as it is made evident in the present paper, is susceptible to paradoxical reasoning emerging as a result of the indeterminacy surrounding limits of application of the vague predicate such as "is practical".

Undeniably, as long as *the concept of a practical (feasible) algorithm* has no sharp boundaries, no one second taken by an algorithm in order to achieve a solution can be identified as making the difference between being "a practical algorithm" and not being "a practical algorithm". So, if the algorithm, which takes one second to reach the solution on the given input, is considered practical, then it seems logical to assume that another algorithm that takes two seconds (three seconds, four seconds, and so on) to reach the same solution on the same input (and the same computational device) must also be regarded as practical. But then it would appear that no matter what number of seconds an algorithm takes it must be considered practical. Thus, one gets the sorites paradox, namely, from the apparently true premises and through uncontroversial reasoning one gets an obviously false conclusion.

Clearly, soritical paradoxical reasoning undermines Cobham's thesis. However, as it has been shown in the present paper, any attempt to resolve such reasoning (using known approaches to the sorites paradox such as supervaluationism and many-valued logic) makes Cobham's thesis serving no purpose at all.

A lesson one can learn from this is that an attempt at melding a robust mathematical definition (e.g., a definition of a class of computable functions) and a vague predicate made from English words (or words of any other language) leads to either a paradoxical argument or a claim without purpose.

It is quite certain that the idea underlying Cobham's thesis is to prioritize somehow complexity classes. But, instead of identifying the P class as "practical" ("feasible" or the like) and the others as "impractical", one may rather argue that P is preferable to another class because it is naturally chosen by processes of the physical world.

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