3-DIMENSIONAL COMPRESSIBLE EULER EQUATIONS

Abstract. In this paper, we mainly give two conclusions. The first conclusion is self-similar solutions of the compressible Euler equations in three dimensions. We find a new system, which is simplified by using the plane wave transform and self-similar transform. Next,we give the exact solution by using the Cardan formula. The second conclusions is that we find this equations have limit behavior.

1. Introduction

Now, we are discussion about the following 3-dimensional compressible isentropic Euler equations

$$\rho_t + \nabla\rho \cdot v + \rho\nabla \cdot v = 0, \tag{1.1}$$

$$(\rho v)_t + \sum_{i=1}^{i=3} (\rho v_i v)_{x_i} + \nabla(p(\rho)) = 0, \tag{1.2}$$

where ∇ denotes the gradient respect to the space coordinates $x = (x_1, x_2, x_3)$, $\rho = \rho(t, x)$ denotes the density of the gas, vector $v = (v_1, v_2, v_3) = v(t, x)$ is the velocity of the gas, and $p(\rho)$ denotes pressure.

In this article we only considering the equations under the polytropic pressure laws ($\theta -$ laws) with $\theta \geq 1$:

$$p(\rho) = \frac{c_0^2 \rho_0}{\theta} (\frac{\rho}{\rho_0})^\theta, \tag{1.3}$$

here c_0 is the sound speed at density ρ_0. Many subsequent results extend with little or no change to $\theta < 1$ or to general pressure laws.

The compressible Euler equations have drawn great interest since the vital physical importance and many mathematical challenges(see Lions [1]). Yuen [2] obtained the analytically self-similar solutions with elliptic symmetry and drift phenomenon for the compressible Euler and Navier-Stokes equations in $\mathbb{R}^n$ ($n \geq 2$) by the separation method.

Therefore its solutions are very meaningful in mathematical physics. Sideris [3] found that the smooth solutions to the three-dimensional Euler equations for a polytropic idea fluid must blow up in a finite time under some assumptions on the initial data. Godin [4] derived the asymptotic behavior of the lifespan of the smooth solution to three-dimensional spherically symmetric flows of ideal ploytropic

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gases with variable entropy, when the initial data is just perturbed from a constant state by smooth compactly supported functions. On the other hand, it is interesting that Grassin [5] showed that there exist global smooth solutions for ideal plolytropic fluids if the initial data can force the particles to spread out. In reference [6], the authors proved the global existence of the smooth solutions to the Cauchy problem for two-dimensional flow of Chaplygin gases under the assumption that the initial data is close to a constant stste and the vorticity of the initial velocity vanishes.

Recently, Li and Wang [7] studied the blow up phenomena of solutions for the multi-dimensional compressible Euler equations by constructing some special explicit solutions with spherical symmetry. Yuen [8] succeeded in constructing some non-spherically symmetric solutions for the 1-dimensions compressible Euler equations by perturbing the linear fluid velocity with a drifting term. By this perturbations, Yuen [9] derived a new class of blow up or global solutions with elementary functions to the 3-dimensional compressible or incompressible Euler and Navier-Stokes equations. Meanwhile Yeung and Yuen [10] constructed some self-similar blow-up solutions for the Navier-Stokes-Poisson equations with density-dependent viscosity and with pressure by the separation method. Most recently.

In this paper, we mainly give the proof of explicit exact solutions and limit behavior for the compressible Euler equations in three dimensions.This method is different from the study of above reference literature. Because the new system can be solved directly by using the plane wave transform and the Cardan formula.Finally,giving the proof of limit behavior.

The paper is organized as follows. In Section 2, we give some definitions and lemma. The Section 3 is devoted to simplify the system, and give the explicit self-similar solution of 3-dimensional Eluer equation. In Section 4, give a simple proof of the limit behavior.

2. PRELIMINARIES

Now, we first give same simpler definitions and lemma, which will be used in Section 3.

Definition 2.1. *(Plane wave) We say that a solution (u,ρ) of Eluer equations (1.1)-(1.2) in the $3+1$ variables $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $t\in\mathbb{R}^+$ having the form*

$$v(x,t)=f(y_1x-\sigma_1t),\ \ x=(x_1,x_2,x_3)\in\mathbb{R}^3, t\in\mathbb{R}^+,$$

$$\rho(x,t)=g(y_2x-\sigma_2t),\ \ x=(x_1,x_2,x_3)\in\mathbb{R}^3, t\in\mathbb{R}^+,$$

is called a plane wave, where $y_i\in\mathbb{R}^3$, $i=1,2$.

Definition 2.2. *(Self-similar solution) We say that a solution (u,ρ) of Eluer equations (1.1)-(1.2) in the $3+1$ variables $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $t\in\mathbb{R}^+$ having the form*

$$v=\frac{1}{t^\beta}u(\frac{x}{t^\alpha})=\frac{1}{t^\beta}u(y),$$

$$\rho=w(\frac{x}{t^\alpha}\frac{1}{t^\gamma})=\frac{1}{t^\gamma}w(y),$$

is called a self-similar solution, where $y\in\mathbb{R}^3$, α,β are constants.

Lemma 2.3. *(The Cardan formula) The general cubic equation over the field of complex numbers*

$$x^3+px+q=0.$$

Any cubic equation can be reduced to the above form, the roots of the equation has the form:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

3. MAIN RESULTS

In this part, we firstly get an equivalent system by using self-similar transform, and also find an explicit solutions of the new system.

Definition 3.1. *We define a C^∞ function π as follows*

$$\pi(\rho) = c_0^2 \cdot \begin{cases} \frac{\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} - 1}{\gamma - 1}, \gamma > 1, \\ \log\left(\frac{\rho}{\rho_0}\right), \gamma = 1, \end{cases}$$

where $\rho \in (0, \infty)$, $\gamma \in [1, \infty)$.

Utilize the self-similar transform to Euler equations (1.1)-(1.2), we have a new system as following.

Theorem 3.2. *Let $\beta = 0, \alpha = 1$ and arbitrary γ. Then the Euler equations (1.1)-(1.2) can be simplified to the self-similar form*

$$\gamma w + y \cdot \nabla w - u \cdot \nabla w - w\text{div}u = 0, \tag{3.1}$$

$$(y \cdot \nabla)u - u \cdot \nabla^T u - \pi_w \nabla w = 0, \tag{3.2}$$

where $y \in \mathbb{R}^3$.

Proof. We seek the self-similar solutions by lemma 2.2, we can get

$$-\frac{\gamma}{t^{\gamma+1}} w - \frac{\alpha}{t^{\alpha+\gamma+1}} \sum_{i=1}^{3} w_{y_i} x_i + \frac{1}{t^{\alpha+\beta+\gamma}} \sum_{i=1}^{3} w_{y_i} u_i + \frac{1}{t^{\alpha+\beta+\gamma}} w\text{div}u = 0$$

That is

$$-\gamma w - \alpha y \cdot \nabla w + \frac{1}{t^{\alpha+\beta-1}} \nabla w \cdot u + \frac{1}{t^{\alpha+\beta-1}} w\text{div}u = 0. \tag{3.3}$$

Suppose $\alpha + \beta - 1 = 0$, that is to say

$$\alpha + \beta = 1, \tag{3.4}$$

we have

$$\gamma w + \alpha y \cdot \nabla w - u \cdot \nabla w - w\text{div}u = 0. \tag{3.5}$$

Similarly, we have

$$-\frac{\beta}{t^{\beta+1}} u - \frac{\alpha}{t^{\alpha+\beta+1}} \sum_{i=1}^{3} u_{y_i} x_i + \frac{1}{t^{\alpha+2\beta}} u \cdot \nabla^T u + \frac{\pi_\rho}{t^{\alpha+\gamma}} \nabla w = 0.$$

According to the definition

$$\pi(\rho) = \pi(w),$$

we have

$$\frac{\beta}{t^{\beta+1}} u + \frac{\alpha}{t^{\beta+1}} (y \cdot \nabla)u - \frac{1}{t^{\alpha+2\beta}} u \cdot \nabla^T u - \frac{1}{t^{\alpha+\gamma}} \pi_w t^\gamma \nabla w = 0.$$

That is

$$\beta u + \alpha(y \cdot \nabla)u - \frac{1}{t^{\alpha+\beta-1}} u \cdot (\nabla)^T u - \frac{1}{t^{\alpha-\beta-1}} \pi_w \nabla w = 0. \tag{3.6}$$

Next we let $\alpha+\beta-1=0$ and $\alpha-\beta-1=0$. Then

$$\beta=0, \alpha=1. \tag{3.7}$$

Substituting (3.7) into (3.3) and (3.6) respectively, and (3.1)-(3.2) follows. □

Next, we will solve the new system (3.1)-(3.2) by using the plane wave transform and the Cardan formula.

Theorem 3.3. *Let $\gamma=1$, $\theta=2$. Then the new system (3.1)-(3.2) has the following exact solution*

$$w=\sqrt[3]{-\left(\frac{z^3}{27}+\frac{\sum_{i=1}^3 a_i^2 M}{4}\right)\pm\sqrt{\left(\frac{z^3}{27}+\frac{\sum_{i=1}^3 a_i^2 M}{4}\right)^2-\frac{4z^6}{729}}}, \tag{3.8}$$

$$u=\frac{-2M}{(N-z)^2}(a_1,a_2,a_3), \tag{3.9}$$

where $M=c\cdot c_0^2\cdot Q_0^{-1}$ with constant c, $z=\sum_{i=1}^3 a_i y_i$ with constant a_i, and $N=w+\frac{z^2}{9w}+\frac{2z}{3}$.

Proof. We seek the plane wave of (3.1)-(3.2) with the following forms

$$w=Q(z), \tag{3.10}$$

$$u=v(z), \tag{3.11}$$

where $z=a_1y_1+a_2y_2+a_3y_3$, $y=(y_1,y_2,y_3)$. Then

$$\nabla w=w_y=(w_{(y_1)},w_{(y_2)},w_{(y_3)})=w_z(a_1,a_2,a_3). \tag{3.12}$$

Substituting (3.10)-(3.12) into (3.1)-(3.2), we have

$$\begin{aligned}&\gamma Q+(a_1y_1+a_2y_2+a_3y_3)Q_z-(a_1v_1+a_2v_2+a_3v_3)Q_z\\&-Q\cdot(a_1v_{1z}+a_2v_{2z}+a_3v_{3z})=0\end{aligned}$$

and

$$\begin{aligned}&v_z\cdot(a_1y_1+a_2y_2+a_3y_3)-(a_1v_1+a_2v_2+a_3v_3)\cdot v_2\\&-c_0^2\cdot w_0^{1-\theta}\cdot Q^{\theta-2}\cdot(a_1,a_2,a_3)=0,\end{aligned}$$

where

$$\pi_w=c_0^2(\frac{w}{w_0})^{\theta-2}\frac{1}{w_0}=c_0^2w_0^{1-\theta}w^{\theta-2}=c_0^2Q_0^{1-\theta}Q^{\theta-2}.$$

That is

$$\begin{aligned}&\gamma\cdot Q+z\cdot Q_z-(a_1v_1+a_2v_2+a_3v_3)\cdot Q_z\\&-Q\cdot(a_1v_{1z}+a_2v_{2z}+a_3v_{3z})=0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&z\cdot v_z-(a_1v_1+a_2v_2+a_3v_3)\cdot v_z\\&-c_0^2\cdot Q_0^{1-\theta}\cdot Q^{\theta-2}\cdot Q_z(a_1,a_2,a_3)=0.\end{aligned} \tag{3.14}$$

Let $\theta=2$, we have

$$[z-(a_1v_1+a_2v_2+a_3v_3)]Q_z+[\gamma-(a_1v_{1z}+a_2v_{2z}+a_3v_{3z})]Q=0,$$

$$[z-(a_1v_1+a_2v_2+a_3v_3)]v_z-c_0^2\cdot Q_0^{-1}\cdot Q_z(a_1,a_2,a_3)=0.$$

It follows that

$$Q_z - \frac{\gamma - (a_1 v_{1z} + a_2 v_{2z} + a_3 v_{3z})}{(a_1 v_1 + a_2 v_2 + a_3 v_3) - z} Q = 0, \tag{3.15}$$

$$v_z + \frac{c_0^2 \cdot Q_0^{-1} \cdot (a_1, a_2, a_3)}{(a_1 v_1 + a_2 v_2 + a_3 v_3) - z} Q_z = 0. \tag{3.16}$$

According to (3.15), we get

$$Q = C \cdot e^{-\int \frac{(a_1 v_{1z} + a_2 v_{2z} + a_3 v_{3z}) - \gamma}{(a_1 v_1 + a_2 v_2 + a_3 v_3) - z} dz}.$$

Let $\gamma = 1$, we have

$$Q = c \cdot \frac{1}{[(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]}. \tag{3.17}$$

According to (3.16), we know that

$$v_z + \frac{c_0^2 \cdot Q_0^{-1} \cdot (a_1, a_2, a_3)}{(a_1 v_1 + a_2 v_2 + a_3 v_3) - z} Q_z = 0,$$

$$v_z + \frac{c_0^2 \cdot Q_0^{-1} \cdot (a_1, a_2, a_3)}{(a_1 v_1 + a_2 v_2 + a_3 v_3) - z} \cdot \frac{1 - (a_1 v_{1z} + a_2 v_{2z} + a_3 v_{3z})}{(a_1 v_1 + a_2 v_2 + a_3 v_3) - z} Q = 0,$$

$$v_z = \frac{c \cdot c_0^2 \cdot Q_0^{-1} \cdot (a_1, a_2, a_3)[(a_1 v_{1z} + a_2 v_{2z} + a_3 v_{3z}) - 1]}{[(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^3}.$$

Thus

$$v = c \cdot c_0^2 \cdot Q_0^{-1} \cdot (a_1, a_2, a_3) \cdot \frac{1}{-2 \cdot [(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^2}. \tag{3.18}$$

That is

$$v_1 = c \cdot c_0^2 \cdot Q_0^{-1} \cdot a_1 \cdot \frac{1}{-2 \cdot [(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^2},$$

$$v_2 = c \cdot c_0^2 \cdot Q_0^{-1} \cdot a_2 \cdot \frac{1}{-2 \cdot [(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^2},$$

$$v_3 = c \cdot c_0^2 \cdot Q_0^{-1} \cdot a_3 \cdot \frac{1}{-2 \cdot [(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^2}.$$

So we have

$$v_1 \cdot a_1 = c \cdot c_0^2 \cdot Q_0^{-1} \cdot a_1^2 \cdot \frac{1}{-2 \cdot [(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^2},$$

$$v_2 \cdot a_2 = c \cdot c_0^2 \cdot Q_0^{-1} \cdot a_2^2 \cdot \frac{1}{-2 \cdot [(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^2},$$

$$v_3 \cdot a_3 = c \cdot c_0^2 \cdot Q_0^{-1} \cdot a_3^2 \cdot \frac{1}{-2 \cdot [(a_1 v_1 + a_2 v_2 + a_3 v_3) - z]^2}.$$

Now, we assume that $M = c \cdot c_0^2 \cdot Q_0^{-1}$ and $\tilde{u} = a_1 v_1 + a_2 v_2 + a_3 v_3$, we have

$$\tilde{u} = -\frac{M(a_1^2 + a_2^2 + a_3^2)}{2(\tilde{u} - z)^2}.$$

That is to say

$$2\tilde{u}^3 - 4z\tilde{u}^2 + 2z^2\tilde{u} + M(a_1^2 + a_2^2 + a_3^2) = 0. \tag{3.19}$$

According to the idea of the Cardan formula, we suppose

$$\tilde{u} = t + \frac{2z}{3} \tag{3.20}$$

and substitute (3.20) into (3.19), we have

$$t^3 - \frac{z^2}{3}t + \frac{2z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{2} = 0. \tag{3.21}$$

According to the the idea of Cardan formula again, we suppose

$$t = w + \frac{z^2}{9w} \tag{3.22}$$

and substitute (3.22) into (3.21), we have

$$w^3 + \frac{z^6}{729w^3} + \frac{2z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{2} = 0. \tag{3.23}$$

That is

$$(w^3)^2 + [\frac{2z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{2}]w^3 + \frac{z^6}{729} = 0.$$

Thus

$$w^3 = -\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right) \pm \sqrt{(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4})^2 - \frac{4z^6}{729}}.$$

In view of (3.22), we get

$$\begin{aligned} t =& \sqrt[3]{-\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right) \pm \sqrt{(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4})^2 - \frac{4z^6}{729}}} \\ &+ \frac{z^2}{9\sqrt[3]{-\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right) \pm \sqrt{(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4})^2 - \frac{4z^6}{729}}}}. \end{aligned}$$

Due to (3.20), we get

$$\begin{aligned} \tilde{u} =& \sqrt[3]{-\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right) \pm \sqrt{\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right)^2 - \frac{4z^6}{729}}} \\ &+ \frac{z^2}{9\sqrt[3]{-\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right) \pm \sqrt{\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right)^2 - \frac{4z^6}{729}}}} + \frac{2z}{3}. \end{aligned}$$

we can get

$$\begin{aligned} &a_1v_1 + a_2v_2 + a_3v_3 \\ =& \sqrt[3]{-\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right) \pm \sqrt{\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right)^2 - \frac{4z^6}{729}}} \\ &+ \frac{z^2}{9\sqrt[3]{-\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right) \pm \sqrt{\left(\frac{z^3}{27} + \frac{M(a_1^2+a_2^2+a_3^2)}{4}\right)^2 - \frac{4z^6}{729}}}} + \frac{2z}{3}, \end{aligned} \tag{3.24}$$

Substituting (3.24) into (3.18) and concludes the Theorem 3.3. □

Remark 3.4. *The solution (3.8)-(3.9) are explicit, in view of (1.1)-(1.2), we can get the explicit and exact self-similar solution of 3-dimensional Eluer equations.*

Corollary 3.5. *Let $\gamma = 1$, $\theta = 2$. Then the new system (3.1)-(3.2) has the following special exact solution*

$$w = \sqrt[3]{-(\frac{z^3}{729b^3} + \frac{M}{36b}) \pm \sqrt{(\frac{z^3}{729b^3} + \frac{M}{36b})^2 - \frac{4z^6}{81^3b^6}}},$$

$$u = \left(w + \frac{z^2}{81b^2w} + \frac{2z}{9b}\right)\hat{e},$$

where $\hat{e} = (1, 1, 1)$, $M = c \cdot c_0^2 \cdot Q_0^{-1}$ with constant c, $z = b(y_1 + y_2 + y_3)$ with constant b.

Proof. Now we substitute (3.24) into (3.18). Let $v_1 = v_2 = v_3 = u_0$, then we can find that

$$a_1 = a_2 = a_3.$$

Let $a_i = b$, $i = 1, 2, 3$. Then

$$Q = c \cdot \frac{1}{3bu_0 - z}, \tag{3.25}$$

$$u_0 = M \cdot b \cdot \frac{1}{-2(3bu_0 - z)^2}. \tag{3.26}$$

It follows (3.26) that

$$18b^2 \cdot u_0^3 - 12b \cdot z \cdot u_0^2 + 2 \cdot z^2 u_0 + Mb = 0. \tag{3.27}$$

According to the idea of the Cardan formula, we suppose

$$u_0 = t + \frac{2z}{9b}. \tag{3.28}$$

Substituting (3.28) into (3.27), we have

$$t^3 - \frac{z^2}{27b^2}t + \frac{2z^3}{729b^3} + \frac{M}{18b} = 0. \tag{3.29}$$

According to the idea of the Cardan formula again, we suppose

$$t = w + \frac{z^2}{81b^2w}, \tag{3.30}$$

and substitute it into (3.29), we have

$$w^3 + \frac{z^6}{81^3b^6w^3} + \frac{2z^3}{729b^3} + \frac{M}{18b} = 0,$$

that is to say

$$(w^3)^2 + (\frac{2z^3}{729b^3} + \frac{M}{18b})w^3 + \frac{z^6}{81^3b^6} = 0.$$

Thus

$$w^3 = -(\frac{z^3}{729b^3} + \frac{M}{36b}) \pm \sqrt{(\frac{z^3}{729b^3} + \frac{M}{36b})^2 - \frac{4z^6}{81^3b^6}}. \tag{3.31}$$

Because (3.30), we have

$$\begin{aligned} t = & \sqrt[3]{-(\frac{z^3}{729b^3} + \frac{M}{36b}) \pm \sqrt{(\frac{z^3}{729b^3} + \frac{M}{36b})^2 - \frac{4z^6}{81^3b^6}}} \\ & + \frac{z^2}{81b^2\sqrt[3]{-(\frac{z^3}{729b^3} + \frac{M}{36b}) \pm \sqrt{(\frac{z^3}{729b^3} + \frac{M}{36b})^2 - \frac{4z^6}{81^3b^6}}}}. \end{aligned} \tag{3.32}$$

In view of (3.32), we have

$$u_0 = \sqrt[3]{-(\frac{z^3}{729b^3} + \frac{M}{36b}) \pm \sqrt{(\frac{z^3}{729b^3} + \frac{M}{36b})^2 - \frac{4z^6}{81^3b^6}}} + \frac{z^2}{81b^2\sqrt[3]{-(\frac{z^3}{729b^3} + \frac{M}{36b}) \pm \sqrt{(\frac{z^3}{729b^3} + \frac{M}{36b})^2 - \frac{4z^6}{81^3b^6}}}} + \frac{2z}{9b}. \tag{3.33}$$

□

4. LIMIT BEHAVIOR

In this section, we mainly discuss the limit behavior of (1.1)-(1.2). In other words , we discuss whether the weak solution of (1.1)-(1.2) tend to the one of (4.3)-(4.4) when $\gamma \longrightarrow 1$.

Now, we lable equations (1.1)-(1.2) as follows

$$\rho^*_t + \nabla \cdot (\rho^* v^*) = 0 \tag{4.1}$$

$$(\rho^* v^*)_t + \sum_{i=1}^{d} (\rho^* v^*_i v^*)_{x_i} + \nabla(P(\rho^*)) = 0 \tag{4.2}$$

When $\gamma \longrightarrow 1$ *, the limit equation is*

$$\rho_t + \nabla \cdot (\rho v) = 0 \tag{4.3}$$

$$(\rho v)_t + \sum_{i=1}^{d} (\rho v_i v)_{x_i} + \nabla(c_0^2 \rho) = 0 \tag{4.4}$$

when equation(4.1), (4.3)and(4.2),(4.4)respectively to do bad,we can get

$$(\rho^*_t - \rho_t) + \nabla \cdot (\rho^* v^*) - \nabla \cdot (\rho v) = 0 \tag{4.5}$$

$$(\rho^* v^*)_t - (\rho v)_t + \sum_{i=1}^{d} (\rho^* v^*_i v^*)_{x_i} - \sum_{i=1}^{d} (\rho v_i v)_{x_i} + \nabla(P(\rho^*)) - \nabla(c_0^2 \rho) = 0 \tag{4.6}$$

Theorem 4.1. *Let* $\Omega_T = \Omega \times [0, T]$*,here* $0 \le T < +\infty$ *and* $\Omega \subset R^3$*.If* (v^*, ρ^*) *and* (v, ρ) *is the weak solution of (4.1)-(4.2)and(4.3)-(4.4), satisfy the same boundery conditions, respectively.Then when* $\gamma \to 1$

$$\| \rho^* - \rho \|_{L^2} (\Omega_\tau) + \| v^* - v \|_{L^2} (\Omega_\tau) \to 0$$

Proof. let $v^* - v = \tilde{v}$,$\rho^* - \rho = \tilde{\rho}$, it follows (4.5),(4.6)that

$$\tilde{\rho}_t + \nabla \cdot (\tilde{\rho} v) + \nabla \cdot (\rho^* \tilde{v}) = 0 \tag{4.7}$$

$$(\rho^* \tilde{v} + v\tilde{\rho})_t + \nabla \rho^* v^* \tilde{v} + \rho^* \nabla v^* \tilde{v} + \nabla \rho^* \tilde{v} v + \nabla \tilde{\rho} \cdot v^2 + \rho^* \nabla \tilde{v} v + v \cdot \tilde{\rho} \nabla v + (\tilde{\rho} v + \rho^* \tilde{v}) \nabla \cdot v^* + \rho v \nabla \cdot \tilde{v} + c_0^2 \rho_0^{1-\gamma} \rho^{*(\gamma-1)} \cdot \nabla \rho^* - c_0^2 \nabla \rho = 0 \tag{4.8}$$

Multiply (4.7) by $\tilde{\rho}$, we have

$$\tilde{\rho}\tilde{\rho}_t + \tilde{\rho}(\nabla \tilde{\rho} \cdot v + \tilde{\rho} \nabla \cdot v) + \tilde{\rho}(\nabla \rho^* \cdot \tilde{v} + \rho^* \nabla \cdot \tilde{v}) = 0 \tag{4.9}$$

integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|\tilde{\rho}|^2+\frac{1}{2}\int_{\Omega}\nabla|\tilde{\rho}|^2\cdot v+\int_{\Omega}\tilde{\rho}^2\nabla\cdot v+ \int_{\Omega}\tilde{\rho}\nabla\rho^*\cdot\tilde{v}+\int_{\Omega}\tilde{\rho}\rho^*\nabla\cdot\tilde{v}=0 \tag{4.10}$$

Similarly, multiplying (4.8)by $-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)$, and integrating over Ω, we can get

$$\begin{aligned}\int_{\Omega}(\rho^*\tilde{v}+v\tilde{\rho})_t[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\int_{\Omega}\nabla\rho^*v^*\tilde{v}[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\\ \int_{\Omega}\rho^*\nabla v^*\tilde{v}[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\int_{\Omega}\nabla\rho^*\tilde{v}v[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\\ \int_{\Omega}\nabla\tilde{\rho}\cdot v^2[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\int_{\Omega}\rho^*\nabla\tilde{v}v[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\\ \int_{\Omega}v\cdot\tilde{\rho}\nabla v[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\int_{\Omega}(\tilde{\rho}v+\rho^*\tilde{v})\nabla\cdot v^*[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\\ \int_{\Omega}\rho v\nabla\cdot\tilde{v}[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\int_{\Omega}c_0^2\rho_0^{1-\gamma}\rho^{*(\gamma-1)}\cdot\nabla\rho^*[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]\\ -\int_{\Omega}c_0^2\nabla\rho[-\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]=0\end{aligned} \tag{4.11}$$

Due to

$$\begin{aligned}-\int_{\Omega}(\rho^*\tilde{v}+v\tilde{\rho})_t[\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]=\\ \int_{\Omega}\nabla(\rho^*\tilde{v}+v\tilde{\rho})_t\cdot\nabla(\rho^*\tilde{v}+\tilde{\rho}v)-\int_{\partial\Omega}(\rho^*\tilde{v}+v\tilde{\rho})_t\cdot\nabla(\rho^*\tilde{v}+\tilde{\rho}v)\cdot n=\\ \frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla(\rho^*\tilde{v}+v\tilde{\rho})|^2-\int_{\partial\Omega}(\rho^*\tilde{v}+v\tilde{\rho})_t\cdot\nabla(\rho^*\tilde{v}+\tilde{\rho}v)\cdot n\end{aligned} \tag{4.12}$$

Integraling over$[0,\tau]$with respect to t,where $\tau\in[0,\tau]$,we can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^{\tau}\int_{\Omega_\tau}(|\nabla(\rho^*\tilde{v}+v\tilde{\rho})|^2+|\tilde{\rho}|^2)+c_0^2\int_{\Omega_\tau}\nabla\rho[\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\\ \frac{1}{2}\int_{\Omega_\tau}\nabla|\tilde{\rho}|^2\cdot v+\int_{\Omega_\tau}\tilde{\rho}\nabla\rho^*\cdot\tilde{v}+\int_{\Omega_\tau}\tilde{\rho}\rho\nabla\cdot\tilde{v}+\\ \frac{1}{2}\int_{\Omega_\tau}\nabla|(\tilde{\rho}v+\rho^*\tilde{v})|^2\cdot\triangle v^*\\ =-(\int_{\Omega_\tau}|\nabla(\tilde{\rho}v+\rho^*\tilde{v})|^2\nabla\cdot v^*+\int_{\Omega_\tau}|\tilde{\rho}|^2\nabla\cdot v^*)+\\ \int_{\Omega_\tau}\nabla(\rho^*v^*)\tilde{v}[\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\int_{\Omega_\tau}\nabla(\rho^*\tilde{v})v[\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]\\ \int_{\Omega_\tau}\nabla(\tilde{\rho}v)v[\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\int_{\Omega_\tau}\rho v\nabla\cdot\tilde{v}[\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\\ c_0^2\int_{\Omega_\tau}\rho^{*(\gamma-1)}\rho_0^{1-\gamma}\cdot\nabla\rho^*[\Delta(\rho^*\tilde{v}+\tilde{\rho}v)]+\\ (\int_{\partial\Omega_\tau}(\rho^*\tilde{v}+v\tilde{\rho})_t\cdot\nabla(\rho^*\tilde{v}+\tilde{\rho}v)\cdot n+\int_{\partial\Omega_\tau}(\tilde{\rho}v+\rho^*\tilde{v})\nabla\cdot v^*\cdot\nabla(\tilde{\rho}v+\rho^*\tilde{v})\cdot n)\end{aligned}$$

Suppose

$$G(\tau) = \int_0^\tau \int_\Omega |\nabla(\rho^*\tilde{v} + v\tilde{\rho})|^2 + |\tilde{\rho}|^2$$

because

$$\rho^{*(\gamma-1)}\rho_0^{1-\gamma} \leq \frac{\rho^{*\gamma(\gamma-1)}}{\gamma} + \frac{\gamma-1}{\gamma}\rho_0^{-\gamma}$$

(there p is γ, q is $\frac{\gamma}{\gamma-1}$, and $\gamma \geq 1$) we have

$$G(T) = \int_0^T \int_\Omega (|\nabla(\rho^*\tilde{v} + v\tilde{\rho})|^2 + |\tilde{\rho}|^2) \leq C\frac{\gamma-1}{\gamma}\varepsilon_0^{-\gamma}(e^{CT} - 1)$$

thus

$$\int_0^T \int_\Omega (|\tilde{v}|^2 + |\tilde{\rho}|^2) \leq \int_0^T \int_\Omega (|\nabla(\rho^*\tilde{v} + v\tilde{\rho})|^2 + |\tilde{\rho}|^2) \leq C\frac{\gamma-1}{\gamma}\varepsilon_0^{-\gamma}(e^{CT} - 1)$$

Therefore, we obtain that

$$\| \rho^* - \rho \|_{L^2(\Omega_T)} + \| v^* - v \|_{L^2(\Omega_T)} \to 0$$

□

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