ASYMPTOTIC ANALYSIS OF THE STATIC AND DYNAMIC BUCKLING OF A COLUMN WITH CUBIC - QUINTIC NONLINEARITY STRESSED BY A STEP LOAD.

ABSTRACT

In this paper, the static and dynamic buckling loads of a viscously damped imperfect finite column lying on an elastic foundation with cubic – quintic nonlinearity but trapped by a step load (in the dynamic case) is analytically investigated. The main aim here is to analytically determine the static and dynamic buckling loads by means of perturbation and asymptotic procedures and relate both buckling loads in one single formula. The formulation contains small perturbations particularly in the viscous damping and imperfection amplitude. Multi – timing perturbation techniques and asymptotics are easily utilized in analyzing the problem. The results, which are nontrivially obtained, are implicit in nature and are valid as long as the magnitudes of the small perturbations become asymptotically small compared to unity.

Keywords: Non – linear Elastic foundations, Dynamic Buckling, Step load, Perturbation and Asymptotic Analyses

1. INTRODUCTION

Investigations concerning static and dynamic buckling of structures under prescribed loading histories, have received tremendous patronage in recent times. Such investigations include studies by Chitra and Priyardarsini [1], Ferri et al. [2], Kolakowski [3], Kowal-Michalska [4] and Mcshane et al. [5]. Priyardarsini et al. [6] investigated numerical and experimental study of advanced fibre composite cylinders under axial compression while Reda and Forbes [7] studied the dynamic effect of lateral buckling of high temperature / high pressure of off shore pipelines. Of special note is the investigation by Belyav et al. [8], who investigated the stability of transverse vibration of rod under longitudinal step – wise loading while Kripka and Martin [9] investigated cold – formed steel channel columns optimization with simulated annealing method. Similarly, Jatav and Datta [10] investigated shape optimization of damaged columns subjected to conservative and non – conservative forces while Artem and Aydin [11] studied exact solution and dynamic buckling analysis of beam – column loading.

Worthy of mention is also the following recent research works on the buckling of elastic structures and the effects of imperfection on the structures. Patil et al. [12] reviewed the buckling analyses of various structures like plates and shells while Hu and Burgueño [13] studied the elastic post buckling response of axially – loaded cylindrical shells with seeded geometric imperfection design. In the same way, Ziolkowski and Imieowski [14] discussed the buckling and post buckling behavior of prismatic aluminium column submitted to a series of compressive loads while Adman and Saidani [15] discussed the elastic buckling of columns with end restraint effects. Similarly, Avcar [16] studied the elastic buckling of steel columns under axial compression while Kriegesmann et al. [17] studied sample size dependent probabilistic design of axially compressed cylindrical shells.

2. GOVERNING EQUATION OF MOTION

The governing differential equation satisfied by the normal displacement W(X, T) of the viscously damped column trapped by an arbitrary load P(T) is

$$mW_{,TT} + c_0 W_{,T} + EIW_{,XXXX} + 2PW_{,XX} + k_1 W + \alpha k_2 W^3 - \beta_1 k_3 W^5 = -2P(T)\frac{d^2\bar{W}}{dX^2}, \quad T > 0, \quad (1)$$

$$0 < X < \pi, \quad (2)$$

$$W(X,0) = W_{,T}(X,0) = 0, \quad 0 < X < \pi, \qquad c_0 > 0 \quad (3)$$

$$W = W_{,XX} = 0 \; at \; X = 0, \pi, \;\; T \geq 0 \quad (4)$$

where, m is the mass per unit length of the finite column, c_0 is the positive but light viscous damping coefficient, EI is the bending stiffness, where E and I are the Young's modulus and the moment of inertia respectively, $\alpha \; and \; \beta_1$ are exponents of imperfection sensitivity parameters which are to be chosen such that the structure is imperfection sensitive, $\bar{W}$ is the stress – free time independent but twice – differentiable imperfection while X and T are spatial coordinate and time variable respectively. The cubic – quintic nonlinear elastic foundation exerts a force per unit length given by $k_1 W + \alpha k_2 W^3 - \beta_1 k_3 W^5$ on the column. All nonlinearities higher than quintic are neglected while all nonlinear derivatives are similarly neglected. Here, a subscript after a comma indicates partial differentiation.

3. NONDIMENSIONALIZATION OF THE GOVERNING EQUATIONS

We now let

$$X = \left(\frac{EI}{k_1}\right)^{\frac{1}{4}} x, \quad W = \left(\frac{k_1}{k_2}\right)^{\frac{1}{2}} w, \quad \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}}, \quad \bar{W} = \epsilon\left(\frac{k_1}{k_2}\right)^{\frac{1}{2}} \varpi$$

$$t = \left(\frac{k_1}{m}\right)^{\frac{1}{2}} T, \quad 2\epsilon^2 = \frac{c_0}{(mk_1)^{\frac{1}{2}}}, \quad \beta = \left(\frac{\beta_1 k_1 k_3}{k_2^2}\right)^{\frac{3}{2}} \bar{W}$$

On substituting these non-dimensional quantities into the governing equations, the resultant equations are

$$w_{,tt} + 2\epsilon^2 w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon f(t)\frac{d^2\varpi}{dx^2}, \quad t > 0 \quad (5)$$

$$0 < x < \pi \quad (6)$$

$$w(x,0) = w_{,t}(x,0) = 0, \quad 0 < x < \pi \tag{7}$$

$$w = w_{,xx}(x,0) = 0 \; at \; x = 0, \pi, \quad t \geq 0 \tag{8}$$

Here, we have assumed simply -supported boundary conditions, while ϵ and λ are small parameters satisfying the inequalities $0 < \epsilon \ll 1$, and $0 < \lambda < 1$. Physically, ϵ denotes the amplitude of imperfection while λ is that of the applied load and $f(t)$ is the actual time dependent load function, which, in this investigation, is the step load given by

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{9}$$

4. SOLUTION OF THE ASSOCIATED STATIC PROBLEM

The governing equation in this case is

$$\frac{d^4w}{dx^4} + 2\lambda\frac{d^2w}{dx^2} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon\frac{d^2\varpi}{dx^2}, \quad 0 < x < \pi \tag{10a}$$

$$w = \frac{d^2w}{dx^2} = 0 \; at \; x = 0, \pi \tag{10b}$$

Let $w = \sum_{i=1}^{\infty} V^{(i)}(x)\epsilon^i$ (11)

By equating coefficients of powers of ϵ, the following equations are easily obtained.

$$\boldsymbol{O}(\epsilon): \frac{d^4V^{(1)}}{dx^4} + 2\lambda\frac{d^2V^{(1)}}{dx^2} + V^{(1)} = -2\lambda\epsilon\frac{d^2\varpi}{dx^2} \tag{12}$$

$$\boldsymbol{O}(\epsilon^2): \frac{d^4V^{(2)}}{dx^4} + 2\lambda\frac{d^2V^{(2)}}{dx^2} + V^{(2)} = 0 \tag{13}$$

$$\boldsymbol{O}(\epsilon^3): \frac{d^4V^{(3)}}{dx^4} + 2\lambda\frac{d^2V^{(3)}}{dx^2} + V^{(3)} = -\alpha\left(V^{(1)}\right)^3 \tag{14}$$

$$\boldsymbol{O}(\epsilon^4): \frac{d^4V^{(4)}}{dx^4} + 2\lambda\frac{d^2V^{(4)}}{dx^2} + V^{(4)} = -3\alpha\left(V^{(1)}\right)^2V^{(2)} \tag{15}$$

$$\boldsymbol{O}(\epsilon^5): \frac{d^4V^{(5)}}{dx^4} + 2\lambda\frac{d^2V^{(5)}}{dx^2} + V^{(5)} = -3\alpha\left({V^{(1)}}^2V^{(3)} + V^{(1)}{V^{(2)}}^2\right) + \beta {V^{(1)}}^5 \tag{16}$$

etc.

Let

$$\varpi = \bar{a}_m \sin mx, \quad V^{(i)}(x) = \sum_{n=1}^{\infty} V_n^{(i)} sinnx \tag{17}$$

The substitution of (17) into (12) yields

$$\sum_{n=1}^{\infty} (n^4 - 2n^2\lambda + 1)\, V_n^{(1)} sinnx = 2\lambda m^2 \bar{a}_m \sin mx \tag{18}$$

Multiplying (18) by, gives

$$V_m^{(1)} = \frac{2\lambda m^2 \bar{a}_m}{m^4 - 2m^2\lambda + 1} = B \tag{19a}$$

The solution of (13) easily yields

$$V^{(3)} = 0 \tag{19b}$$

Equation (14) now takes the form

$$\frac{d^4 V^{(3)}}{dx^4} + 2\lambda \frac{d^2 V^{(3)}}{dx^2} + V^{(3)} = -\frac{\alpha B^3}{4}(3 \sin mx - \sin 3mx) \tag{20}$$

On substituting (17) in (20), it is observed that, when $n = m$, the result is

$$V_m^{(3)} = \frac{-3\alpha B^3}{4\theta^2}, \quad \theta^2 = (m^4 - 2m^2\lambda + 1) > 0 \tag{21a}$$

However, for $n = 3m$, the result is

$$V_{3m}^{(3)} = \frac{-B^3 \alpha}{4\theta\omega^2}, \quad \omega^2 = (81m^4 - 18m^2\lambda + 1) > 0 \;\forall\, m \tag{21}$$

Thus, for $V^{(3)}$, we get

$$V^{(3)} = V_m^{(3)} \sin mx + V_{3m}^{(3)} \sin 3mx \tag{22}$$

On substituting in (15), using (19b), we have

$$V^4 \equiv 0$$

Substitution is next made into (16), using (19b) and (22) to get

$$\frac{d^4V^{(5)}}{dx^4} + 2\lambda\frac{d^2V^{(5)}}{dx^2} + V^{(5)}$$

$$= -3\alpha\left[\frac{1}{2}V^{(1)^2}\left(\frac{3V_m^{(3)}}{2} - V_{3m}^{(3)}\right)\sin mx\right.$$

$$\left. + \frac{1}{2}\left({V_m}^{(1)^2}V_{3m}^{(3)} - \frac{1}{4}{V_m}^{(1)^2}V_m^{(3)}\right)\sin 3mx + \frac{1}{4}V^{(1)^2}V_{3m}^{(3)}\sin 5mx\right]$$

$$+ \frac{\beta B^5}{16}(11\sin mx - 5sin3mx + \sin 5mx) \qquad (23)$$

Using (17) for $n = m$, we get

$$V_m^{(5)} = \left[\frac{-3\alpha}{\theta^2}\left\{\frac{1}{2}V^{(1)^2}\left(\frac{3}{2}V_m^{(3)} - V_{3m}^{(3)}\right)\right\} + \frac{11\beta}{16\theta^2}\right]\sin mx \qquad (24)$$

However, for $n = 3m$, the result is

$$V_{3m}^{(5)} = \left[\frac{5\beta B^5}{16\omega^2} - \frac{-3\alpha}{\omega^2}\left\{\frac{1}{2}{V_m}^{(1)^2}V_{3m}^{(3)} - \frac{1}{4}V^{(1)^2}V_{3m}^{(3)}\right\}\right]\sin 3mx \qquad (25a)$$

Now, when $n = 5m$ in (23), we get

$$V_{5m}^{(5)} = \frac{1}{\varphi^2}\left[\frac{{V_m^{(1)}}^5}{16} - \frac{3\alpha {V_m^{(1)}}^2}{4}V_{3m}^{(3)}\right]\sin 5mx \qquad (25b)$$

$$\varphi^2 = (625m^4 - 50m^2\lambda + 1) > 0 \ \forall\, m \qquad (25c)$$

It follows that

$$V^{(5)} = V_m^{(5)}\sin mx + V_{3m}^{(5)}\sin 3mx + V_{5m}^{(5)}\sin 5mx \qquad (26)$$

Thus, the displacement at static loading is

$$w(x) = \epsilon V_m^{(1)}\sin mx + \epsilon^3\left(V_m^{(3)}\sin mx + V_{3m}^{(3)}\sin 3mx\right)$$

$$+ \epsilon^3\left(V_m^{(5)}\sin mx + V_{3m}^{(5)}\sin 3mx + V_{3m}^{(5)}\sin 5mx\right) + \cdots \qquad (27)$$

5. STATIC BUCKLING LOAD

As in Amazigo [18], the static buckling load λ_S is obtained from the maximization

$$\frac{d\lambda}{dw} = 0 \tag{28}$$

We remark that $V_m^{(i)}$ depends on the load parameter λ through B. The static buckling load will here be given in two separate approximations, first, by taking the displacement strictly in the shape of the imperfection and next, by admitting the buckling mode in the combined shape of sin 3mx alongside the mode in the shape of imperfection.

5.1. STATIC BUCKLING LOAD USING MODES IN THE SHAPE OF IMPERFECTION

Here, we have

$$w(x) = \left(\epsilon V_m^{(1)} + \epsilon^3 V_m^{(3)} + \epsilon^5 V_m^{(5)}\right) \sin mx + \cdots \tag{29}$$

where,

$$V_m^{(1)} = B, \qquad V_m^{(3)} = \frac{-3\alpha B^3}{4\theta^2}, \qquad V_m^{(5)} = \frac{B^5 Q_{46}}{\theta^2} \tag{30}$$

$$Q_{46} = \frac{11\beta}{16} - 3\alpha^2 \left\{\frac{1}{8}\left(\frac{9}{2\theta^2} + \frac{1}{\omega^2}\right)\right\} \tag{31}$$

We shall now determine (29) at a convenient point, namely at $x = \frac{\pi}{2m}$, where $\frac{dw}{dx} = 0$. This gives

$$w(x) = \epsilon V_m^{(1)} + \epsilon^3 V_m^{(3)} + \epsilon^5 V_m^{(5)} + \cdots \tag{32}$$

Let $V_m^{(1)} = c_1, \qquad V_m^{(3)} = c_2, \qquad V_m^{(3)} = c_3.$

Then, we have

$$w = \epsilon c_1 + \epsilon^3 c_2 + \epsilon^5 c_3 + \cdots \tag{33}$$

We note that the choice of $x = \frac{\pi}{2m}$ follows from the fact that the accompanying dynamic problem will eventually be determined at the same point. As in Amazigo [18], the series (33) does not converge when $w > w_a$, where w_a is the displacement at buckling. The difficulty is overcome by reversing (33) in the form

$$\epsilon = d_1 w + d_2 w^3 + d_3 w^5 + \cdots \tag{34}$$

By substituting for w from (33) in (34) and equating the coefficients of powers of ϵ, we get

$$d_1 = \frac{1}{c_1}, \qquad d_2 = \frac{-c_2}{c_1^4}, \qquad d_3 = \frac{3c_2^2 - c_1 c_3}{c_1^7} \tag{35a}$$

i.e. $\quad d_1 = \frac{1}{B}, \qquad d_2 = \frac{-3\alpha}{4B\theta^2}, \qquad d_3 = \frac{-11Q_{46}}{16B}\left(\frac{\beta}{\theta^2}\right)Q_{47}$

$$Q_{47} = \left[\frac{27}{11Q_{46}}\left(\frac{\alpha^2}{\beta}\right) - 1\right] \tag{35b}$$

The maximization (28) is now accomplished through (34) to yield

$$d_1 + 3d_2 w_a^2 + 5d_3 w_a^4 = 0 \tag{36}$$

This yields

$$w_a^2 = \frac{-3d_2 \pm \sqrt{9d_2^2 - 20d_1 d_3}}{10d_3} \tag{37}$$

By taking the positive square root sign, we get

$$w_a^2 = \frac{18\alpha}{55\beta Q_{46}Q_{47}}\left[\sqrt{1 + \frac{220Q_{46}Q_{47}}{99}} - 1\right] \tag{38}$$

Therefore, we finally get

$$w_a = \frac{3\sqrt{\frac{2}{55}}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}}{\sqrt{Q_{46}Q_{47}}}\left[\sqrt{1 + \frac{220Q_{46}Q_{47}}{99}} - 1\right]^{\frac{1}{2}} \tag{39}$$

where, (38) and (39) are determined at $\lambda = \lambda_S$. The static buckling load in the case of buckling modes strictly in the shape of imperfection is determined by evaluating (34) at static buckling condition, where $w = w_a$ and $\lambda = \lambda_S$. This gives

$$\epsilon = w_a[d_1 + d_2 w_a^2 + d_3 w_a^4] \tag{40}$$

We next multiply (40) by 5 and after, substitute for $5d_3 w_a^4$ from (36) and simplify to get

$$5\epsilon = \frac{4w_a(m^4 - 2m^2\lambda_S + 1)}{m^2 \bar{a}_m \lambda_S}\left[1 - \frac{3\alpha w_a^2(\lambda_S)}{8\theta^2}\right] \tag{41}$$

Here, the static buckling load λ_S is given implicitly through B, Q_{46} and Q_{47}.

5.2. STATIC BUCKLING LOAD λ_S IN THE CASE OF MODES IN THE COMBINED SHAPES OF $\sin mx$ AND $\sin 3mx$

Here, the displacement (27) becomes

$$w = \epsilon V_m^{(1)} \sin mx + \epsilon^3 \left(V_m^{(3)} \sin mx + V_{3m}^{(3)} \sin 3mx\right) + \epsilon^5 \left(V_m^{(5)} \sin mx + V_{3m}^{(5)} \sin 3mx\right) + \cdots \quad (42)$$

On determining (42) at $x = \frac{\pi}{2m}$, we get

$$w = \epsilon V_m^{(1)} + \epsilon^3 \left(V_m^{(3)} - V_{3m}^{(3)}\right) + \epsilon^5 \left(V_m^{(5)} - V_{3m}^{(5)}\right) + \cdots \quad (43)$$

where,

$$V_m^{(3)} = \frac{-\alpha B^2}{4\theta^2}, \qquad V_{3m}^{(5)} = \frac{B^5 Q_{49}}{16\omega^2} \quad (44)$$

$$Q_{49} = (3\alpha^2 - 5\beta) \quad (45)$$

Thus, we get from (43)

$$w = \epsilon e_1 + \epsilon^3 e_2 + \epsilon^5 e_3 + \cdots \quad (46a)$$

where,

$$e_1 = B, \qquad e_2 = \frac{3\alpha B^3}{4\theta^2} Q_{51}, \qquad Q_{51} = \left[4\left(\frac{\theta}{\omega}\right)^2 - 1\right] \quad (46b)$$

$$e_3 = B^5 Q_{46} Q_{52}, \qquad Q_{52} = 1 - \frac{Q_{49}}{16 Q_{46} \omega^2} \quad (46c)$$

The series (46a) is now reversed in the form

$$\epsilon = f_1 w + f_2 w^3 + f_3 w^5 + \cdots \quad (46d)$$

where,

$$f_1 = \frac{1}{e_1}, \qquad f_2 = \frac{-e_2}{e_1^4}, \qquad f_3 = \frac{3e_2^2 - e_1 e_3}{e_1^7}$$

After simplification, we have

$$f_1 = \frac{1}{B}, \qquad f_2 = \frac{3\alpha Q_{51}}{4\theta^2 B}, \qquad f_3 = \frac{Q_{46}Q_{52}Q_{53}}{B}$$

where,

$$Q_{53} = \left(\frac{27\alpha^2 Q_{51}^2}{16\theta^4 Q_{46}Q_{52}} - 1\right) \tag{46e}$$

The maximization (28) is next invoked to yield the static buckling load λ_S (through (46d)) and through the equation

$$f_1 + 3f_2 w_{ac}^2 + 5f_3 w_{ac}^4 = 0 \tag{47a}$$

Where w_{ac} is the value of w for the displacement to have a maximum in the case of modes in the combined shapes of sinmx and sin3mx.

Thus, we have

$$w_{ac}^2 = \frac{-3f_2 \pm \sqrt{9f_2^2 - 20f_1 f_3}}{10f_3}$$

$$= \frac{3\alpha Q_{51}}{40\theta^2 Q_{46}Q_{52}Q_{53}}\left[-1 - \left\{1 - \frac{320Q_{46}Q_{52}Q_{53}B\theta^4}{81\alpha^2 Q_{51}^2}\right\}^{\frac{1}{2}}\right] \tag{47b}$$

where, we have taken the negative square root sign in (47b). Therefore, we get

$$w_{ac} = \left[\frac{3\alpha Q_{51}}{40\theta^2 Q_{46}Q_{52}Q_{53}}\left[-1 - \left\{1 - \frac{320Q_{46}Q_{52}Q_{53}B\theta^4}{81\alpha^2 Q_{51}^2}\right\}^{\frac{1}{2}}\right]\right]^{\frac{1}{2}} \tag{47c}$$

To get the static buckling load in this case, we determine (46d) at static buckling to get

$$\epsilon = w_{ac}[f_1 + w_{ac}^2(f_2 + w_{ac}^2 f_3)] \tag{48a}$$

On multiplying (48a) by 5 and substituting for $5f_3 w_{ac}^4$ from (47a) and simplifying, we

$$5m^2\bar{a}_m\epsilon\lambda_S = 4w_{ac}(m^4 - 2m^2\lambda_S + 1)\left[1 + \frac{3\alpha w_{ac}^2 Q_{51}}{8\theta^2}\right] \tag{48b}$$

The result (48b) is implicit in the load parameter λ_S through Q_{46}, Q_{51}, Q_{53} and θ.

6. SOLUTION OF THE DYNAMIC PROBLEM (5) – (8)

The full equations are here recast as

$$w_{,tt} + 2\epsilon^2 w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon f(t)\frac{d^2\varpi}{dx^2}, \quad t > 0, \qquad (49a)$$

$$0 < x < \pi \qquad (49b)$$

$$w(x,0) = w_{,t}(x,0) = 0, \quad 0 < x < \pi \qquad (49c)$$

$$w = w_{,xx} = 0 \; at \; x = 0, \pi, \;\; t \geq 0 \qquad (49d)$$

Let

$$\tau = \epsilon^2 t, \qquad w(x,t) = U(x,t,\tau,\epsilon) \qquad (50a)$$

$$\therefore \;\; w_{,t} = U_{,t} + \epsilon^2 U_{,\tau}; \qquad w_{,tt} = U_{,tt} + 2\epsilon^2 U_{,t\tau} + \epsilon^4 U_{,\tau\tau} \qquad (50b)$$

Substituting these in the governing equation of motion, yields

$$\left(U_{,tt} + 2\epsilon^2 U_{,t\tau} + \epsilon^4 U_{,\tau\tau}\right) + 2\epsilon^2\left(U_{,t} + \epsilon^2 U_{,\tau}\right) + U_{,xxxx} + 2\lambda U_{,xx} + U + \alpha U^3 - \beta U^5$$
$$= -2\lambda\epsilon\frac{d^2\overline{\omega}}{dx^2} \qquad (51)$$

Let

$$U(x,t,\tau,\epsilon) = \sum_{i=1}^{\infty} U^{(i)}\epsilon^i \qquad (52)$$

The following equations are obtained

$$\boldsymbol{O}(\epsilon): LU^{(1)} \equiv U_{,tt}^{(1)} + U_{,xxxx}^{(1)} + 2\lambda U_{,xx}^{(1)} + U^{(1)} = -2\lambda\epsilon\frac{d^2\varpi}{dx^2} \qquad (53)$$

$$\boldsymbol{O}(\epsilon^2): \; LU^{(2)} = 0 \qquad (54)$$

$$\boldsymbol{O}(\epsilon^3): \; LU^{(3)} = -2\left(U_{,t\tau}^{(1)} + U_{,t}^{(1)}\right) - \alpha\left(U^{(1)}\right)^3 \qquad (55)$$

$$\boldsymbol{O}(\epsilon^4): \; LU^{(4)} = -3\alpha\left(U^{(1)}\right)^2 U^{(2)} - 2\left(U_{,t\tau}^{(2)} + U_{,t}^{(2)}\right) \qquad (56)$$

$$\boldsymbol{O}(\epsilon^5): \; LU^{(5)} = -3\alpha\left[\left(U^{(1)}\right)^2 U^{(3)} + U^{(1)}\left(U^{(2)}\right)^2\right] + \beta\left(U^{(1)}\right)^5 - 2\left(U_{,t\tau}^{(3)} + U_{,t}^{(3)}\right) \qquad (57)$$

etc.

The initial conditions evaluated at $t = \tau = 0$ are

$$U^{(1)}(x,0,0) = 0, i = 1,2,3,\ldots \qquad (58a)$$

$$\boldsymbol{O}(\epsilon): U_{,t}^{(1)} = 0 \qquad (58b)$$

$$\boldsymbol{O}(\epsilon^2): \; U_{,t}^{(2)} = 0 \qquad (58c)$$

$$\boldsymbol{O}(\epsilon^3): \; U_{,t}^{(3)} + U_{,\tau}^{(1)} = 0 \qquad (58d)$$

$$\boldsymbol{O}(\epsilon^4): \; U_{,t}^{(4)} + U_{,\tau}^{(2)} = 0 \qquad (58e)$$

$$\boldsymbol{O}(\epsilon^5): \; U_{,t}^{(5)} + U_{,\tau}^{(3)} = 0 \qquad (58f)$$

$$U^{(i)} = \; U_{,xx}^{(i)} = 0 \; at \; x = 0, \pi \; ; \quad i = 1,2,3,\ldots \qquad (58g)$$

Let

$\bar{\omega} = \bar{a}_m sin\, mx, m = 1,2,3,\ldots$ and let

$$U^{(i)}(x,t,\tau) = \sum_{n=1}^{\infty} \; U_n^{(i)}(t,\tau)\sin mx \qquad (59)$$

On substituting into (53), the result is

$$\sum_{n=1}^{\infty} \; U_{n,tt}^{(1)} + \; (n^4 - 2n^2\lambda + 1) \;\; U_n^{(1)} = 2\bar{a}_m\lambda m^2 sinmx$$

For $n = m$, the result gives

$$U_{m,tt}^{(1)} + \; \theta^2 \, U_m^{(1)} = 2\bar{a}_m\lambda m^2 \qquad (60a)$$

$$U_m^{(1)}(0,0) = \;\; U_{m,t}^{(1)}(0,0) = 0 \qquad (60b)$$

The solution of (60a, b) is

$$U_m^{(1)}(t,\tau) = \; \alpha_1(\tau)\cos\theta t + \; \gamma_1(\tau)\sin\theta t + B \qquad (60c)$$

$$B = \left(\frac{2\bar{a}_m \lambda m^2}{m^4 - 2m^2\lambda + 1}\right) = \frac{2\bar{a}_m \lambda m^2}{\theta^2} \qquad (60d)$$

where, as in (21a), we have taken

$$\theta^2 = (m^4 - 2m^2\lambda + 1) > 0 \;\; \forall\, m. \qquad (60e)$$

$$\therefore \;\; \alpha_1(0) = -B, \qquad \gamma_1(0) = 0 \qquad (60f)$$

Substituting (59) into (54) (for $i = 2$) and for $n = m$, we have (using (58a, c))

$$U_m^{(2)}(t,\tau) = \alpha_2(\tau)\cos\theta t + \gamma_2(\tau)\sin\theta t \qquad (61a)$$

$$\therefore \;\; \alpha_2(0) = \gamma_2(0) = 0 \qquad (61b)$$

The next substitution into (55) requires the simplification

$$\begin{aligned} {U_m^{(1)}}^3 &= (\alpha_1(\tau)\cos\theta t + \gamma_1(\tau)\sin\theta t + B)^3 \\ &= \left(3B\alpha_1^2 + B^3 + \frac{3B\gamma_1^2}{2}\right) \\ &+ \left\{\frac{3\alpha_1^3}{4} + 3\alpha_1\left(B^2 + \frac{\gamma_1^2}{2}\right) - \frac{3\alpha_1\gamma_1^2}{4}\right\}\cos\theta t \\ &+ \left\{\frac{3\alpha_1^2\gamma_1}{4} + 3\left(B^2\gamma_1 + \frac{\gamma_1^3}{4}\right)\right\}\sin\theta t + 3\left(B\alpha_1^2 - \frac{B\gamma_1^2}{2}\right)\cos 2\theta t + 3\alpha_1\gamma_1 B\sin 2\theta t \\ &+ \left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\gamma_1^2}{4}\right)\cos 3\theta t + \left(\frac{3\alpha_1^2\gamma_1}{2} - \frac{\gamma_1^3}{4}\right)\sin 3\theta t \end{aligned} \qquad (62)$$

Substituting (59) into (55) and for $n = m$, we get

$$U_{m,tt}^{(3)} + \theta^2 U_m^{(3)} = -2\left(U_{m,t\tau}^{(3)} + \theta^2 U_{m,t}^{(3)}\right) - 3\alpha {U_m^{(1)}}^3 \qquad (63a)$$

$$U_m^{(3)}(0,0) = U_{m,t}^{(3)}(0,0) + U_{m,\tau}^{(1)}(0,0) = 0 \qquad (63b)$$

However, for $n = 3m$ in the substitution into (55), the resultant equation is

$$U_{3m,tt}^{(3)} + \omega^2 U_{3m}^{(3)} = \alpha {U_m^{(1)}}^3 \qquad (64a)$$

$$U_{3m}^{(3)}(0,0) = U_{3m,t}^{(3)}(0,0) = 0 \qquad (64b)$$

$$\omega^2 = (81m^4 - 18m^2\lambda + 1) > 0 \;\; \forall\, m. \qquad (64c)$$

Now, substituting from (60c, d) and (62) into (63a) and ensuring a uniformly valid solution in t by equating separately, on the right hand side, the coefficients of $\cos\theta t$ and $\sin\theta t$ to zero, we get, for coefficient of $\cos\theta t$:

$$-2\theta(\gamma_1' + \gamma_1) - 3\alpha\left\{\frac{3\alpha_1^3}{4} + 3\alpha_1\left(B^2 + \frac{\gamma_1^2}{2}\right)\right\} + \frac{9\alpha_1\gamma_1^2}{4} = 0 \qquad (65a)$$

and for coefficient of $sin\,\theta t$, we get

$$2\theta(\alpha_1' + \alpha_1) - 3\alpha\left\{\frac{3\alpha_1^2\gamma_1}{4} + 3\left(B^2\gamma_1 + \frac{\gamma_1^3}{4}\right)\right\} = 0 \qquad (65b)$$

Simplifying (65a, b) further, we have

$$2\theta(\gamma_1' + \gamma_1) + 9\alpha\alpha_1 K = 0 \qquad (65c)$$

$$2\theta(\alpha_1' + \alpha_1) - 9\alpha\gamma_1 K = 0 \qquad (65d)$$

where,

$$K = \frac{\alpha_1^2}{4} + B^2 + \frac{\gamma_1^2}{4} \qquad (65e)$$

Multiplying (65c) by γ_1 and (65d) by α_1 and adding, followed by simplification, yields

$$\frac{\frac{1}{2}\frac{d}{d\tau}(\gamma_1^2 + \alpha_1^2)}{\gamma_1^2 + \alpha_1^2} + 1 = 0 \qquad (65f)$$

The solution of (65f), using (60f), yields

$$(\gamma_1^2 + \alpha_1^2) = B^2 e^{-2\tau} \qquad (65g)$$

We are not however going to solve for α_1 and $\gamma_1$explicitly because every information needed later about $\alpha_1(\tau)$ and $\gamma_1(\tau)$ can be easily obtained from (65c – g). For example, we shall need

$$\alpha_1'(0) = -\alpha_1(0) = B, \qquad \alpha_1''(0) = \frac{2025\alpha^2B^5}{65\theta^2} - B \qquad (65h)$$

$$\gamma_1'(0) = \frac{45\alpha B^3}{8\theta}, \qquad \gamma_1''(0) = \frac{-27\alpha B^3}{2\theta} \qquad (65i)$$

The remaining equation in the substitution into (63a) is

$$U_{m,tt}^{(3)} + \theta^2 U_m^{(3)} = -3\alpha[r_0 + r_1 \cos 2\theta t + r_2 \sin 2\theta t + r_3 \cos 3\theta t + r_4 \sin 3\theta t] \qquad (66a)$$

$$U_m^{(3)}(0,0) = 0, \qquad U_{m,t}^{(3)}(0,0) + U_{m,\tau}^{(1)}(0,0) = 0$$

where,

$$r_0 = \left(3B\alpha_1^2 + B^3 + \frac{3B\gamma_1^2}{2}\right), \qquad r_0(0) = 4B^3 \qquad (66b)$$

$$r_1 = 3\left(B\alpha_1^2 - \frac{B\gamma_1^2}{2}\right), \qquad r_1(0) = 3B^3 \qquad (66c)$$

$$r_2 = 3\alpha_1\gamma_1 B, \qquad r_2(0) = 0; \qquad r_3 = \frac{\alpha_1^3}{4} - \frac{3\alpha_1\gamma_1^2}{4}, \qquad r_3(0) = \frac{-B^3}{4} \qquad (66d)$$

$$r_4 = \gamma_1\left(\frac{3\alpha_1^2}{2} - \frac{\gamma_1^2}{4}\right), \qquad r_4(0) = 0 \qquad (66e)$$

The solution of (66a – e) is

$$U_m^{(3)} = \alpha_{12}(\tau)\cos\theta t + \beta_{12}(\tau)\sin\theta t - 3\alpha\left[\frac{r_0}{\theta^2} - \frac{r_1\cos 2\theta t}{3\theta^2} - \frac{r_2 \sin 2\theta t}{3\theta^2} - \frac{1}{8\theta^2}(r_3\cos 3\theta t + r_4 \sin 3\theta t)\right] \qquad (67a)$$

where,

$$\alpha_{12}(0) = 3\alpha\left[\frac{r_0}{\theta^2} - \frac{r_1}{3\theta^2} - \frac{r_3}{8\theta^2}\right]\Big|_{\tau=0} = \frac{291\alpha B^3}{32\theta^2}; \qquad \beta_{12}(0) = \frac{-B}{\theta} \qquad (67b)$$

The following terms will be useful later:

$$r_1'(0) = -6B^3, \qquad r_2'(0) = \frac{-135\alpha B^5}{8\theta}, \qquad r_3'(0) = \frac{3B^3}{4}, \qquad r_4'(0) = \frac{135\alpha B^5}{16\theta}$$

Now, going back to the substitution in (55) to the case $n = 3m$, the resultant equation is

$$U_{3m,tt}^{(3)} + \omega^2 U_{3m}^{(3)} = \alpha[r_0 + r_5 cos\theta t + r_6 sin\theta t + r_1\cos 2\theta t + r_2 \sin 2\theta t + r_3\cos 3\theta t + r_4 \sin 3\theta t\,] \qquad (68a)$$

$$U_{3m}^{(3)}(0,0) = U_{3m,t}^{(3)}(0,0) = 0 \qquad (68b)$$

where,

$$r_5 = \left[\frac{\alpha_1^3}{4} + 3\alpha_1\left(B^2 + \frac{\gamma_1^2}{4}\right) - \frac{3\alpha_1\gamma_1^2}{4}\right], \qquad r_5(0) = \frac{-B^3}{4}$$

$$r_6 = \left[\frac{3\alpha_1^2\gamma_1}{4} + 3\left(B^2\gamma_1 + \frac{\gamma_1^3}{4}\right)\right], \qquad r_6(0) = 0$$

$$\omega^2 = (81m^4 - 18m^2\lambda + 1) > 0 \ \forall\, m. \tag{68c}$$

It also follows that

$$r_5'(0) = \frac{21B^3}{4}, \qquad r_6'(0) = \frac{6755\alpha B^5}{32\theta}$$

The solution of (68a, b) is

$$U_{3m}^{(3)} = \alpha_3 \cos\omega t + \beta_3 \sin\omega t + \alpha\left[\frac{r_0}{\omega^2} + \left(\frac{r_5 cos\theta t + r_6 sin\theta t}{\omega^2 - \theta^2}\right) + \left(\frac{r_1\cos 2\theta t + r_2 \sin 2\theta t}{\omega^2 - 4\theta^2}\right) + \left(\frac{r_3\cos 3\theta t + r_4\sin 3\theta t}{\omega^2 - 9\theta^2}\right)\right] \tag{69a}$$

$$\alpha_3(0) = -\alpha B^3 \Omega_1, \qquad \Omega_1 = \left[\frac{4}{\omega^2} - \frac{15}{4(\omega^2 - \theta^2)} + \frac{3}{\omega^2 - 4\theta^2} + \frac{1}{4(\omega^2 - 9\theta^2)}\right] \tag{69b}$$

$$\beta_3(0) = 0 \tag{69c}$$

The next substitution into (56) requires the following simplification

$$U_m^{(3)^2} U_m^{(2)} = [r_{18} + r_{19}cos\theta t + r_{20}sin\theta t + r_{21}\cos 2\theta t + r_{22}\sin 2\theta t + r_{23}\cos 3\theta t + r_{24}\sin 3\theta t\,] \tag{70a}$$

where,

$$r_{18} = B(\alpha_1\alpha_2 + \gamma_1\gamma_2), \qquad r_{18}(0) = 0 \tag{70b}$$

$$r_{19} = \left[\left(\frac{\alpha_1^2}{2} + \frac{\gamma_1^2}{2} + B\right)\alpha_2 + \frac{\alpha_1\gamma_1\gamma_2}{2} + \frac{\gamma_2}{4}(\alpha_1^2 - \gamma_1^2)\right], \qquad r_{19}(0) = 0 \tag{70c}$$

$$r_{20} = \left[\left(\frac{\alpha_1^2}{2} + \frac{\gamma_1^2}{2} + B^2\right)\gamma_2 + \frac{\alpha_1\alpha_2\gamma_1}{2} + \frac{\gamma_2}{4}(\alpha_1^2 - \gamma_1^2)\right], \qquad r_{20}(0) = 0 \tag{70d}$$

$$r_{21} = B(\alpha_1\alpha_2 - \gamma_1\gamma_2), \qquad r_{21}(0) = 0 \tag{70e}$$

$$r_{22} = B(\alpha_1\gamma_2 + \alpha_2\gamma_1), \qquad r_{22}(0) = 0 \tag{70f}$$

$$r_{23} = \left[\frac{(\alpha_1^2 - \gamma_1^2)\alpha_2}{4} - \frac{\alpha_1\gamma_1\gamma_2}{4}\right], \qquad r_{23}(0) = 0 \tag{70g}$$

$$r_{24} = \left[\frac{\alpha_1\alpha_2\gamma_1}{2} + \frac{\gamma_2}{4}(\alpha_1^2 - \gamma_1^2)\right], \qquad r_{24}(0) = 0 \tag{70h}$$

Now substituting into (56), yields

$$U^{(4)}_{,tt} + U^{(4)}_{,xxxx} + 2\lambda U^{(4)}_{,xx} + U^{(4)}$$
$$= -2\left(U^{(2)}_{m,t\tau} + U^{(2)}_{m,t}\right)\sin mx - \frac{3\alpha}{4}{U^{(1)}_m}^2 U^{(2)}_m(3\sin mx - \sin 3mx) \tag{71}$$

After substituting (59) into (71), for $i = 4$, and for $n = m$, we get

$$U^{(4)}_{m,tt} + \theta^2 U^{(4)}_m = -2\left(U^{(2)}_{m,t\tau} + U^{(2)}_{m,t}\right) - 9\alpha {U^{(1)}_m}^2 U^{(2)}_m \tag{72a}$$

$$U^{(4)}_m(0,0) = 0, \qquad U^{(4)}_{m,t}(0,0) + U^{(2)}_{m,\tau}(0,0) = 0 \tag{72b}$$

However, for $n = 3m$, we have

$$U^{(4)}_{3m,tt} + \omega^2 U^{(4)}_{3m} = \frac{3\alpha}{4}{U^{(1)}_m}^2 U^{(2)}_m \tag{72c}$$

$$U^{(4)}_{3m}(0,0) = U^{(4)}_{3m,t}(0,0) = 0 \tag{72d}$$

Now, on substituting into (72a) for $U^{(2)}_m$ from (61a) as well as for ${U^{(1)}_m}^2 U^{(2)}_m$ from (70a) and ensuring a uniformly valid solution in t by separately equating to zero the coefficients of $\cos\theta t$ and $\sin\theta t$, we have, respectively

$$\gamma_2' + \gamma_2 = \frac{-9\alpha r_{19}}{8\theta} \quad and \qquad \alpha_2' + \alpha_2 = \frac{-9\alpha r_{20}}{4} \tag{73a}$$

Solving (73a) gives

$$\gamma_2 = -e^{-\tau}\int \frac{9\alpha r_{19}}{8\theta}d\tau, \qquad \alpha_2 = e^{-\tau}\int \frac{9\alpha r_{20}}{8\theta}d\tau \tag{73b}$$

Observation shows that

$$\gamma_2'(0) = \alpha_2' = \gamma_2''(0) = \alpha_2''(0) = 0$$

Infact, all derivatives of γ_2 and α_2 evaluated at $\tau = 0$ vanish. Thus, without loss of generality, we can set

$$\gamma_2(\tau) = \alpha_2(\tau) \equiv 0$$

Hence, we get

$$U^{(2)} = U_m^{(2)}(t,\tau) = 0 \tag{73c}$$

The remaining equation in (72a) is

$$U_{m,tt}^{(4)} + \theta^2 U_m^{(4)} = -\frac{9\alpha}{4}[\, r_{21}\cos 2\theta t + r_{22}\sin 2\theta t + r_{23}\cos 3\theta t + r_{24}\sin 3\theta t] \tag{74a}$$

which is now solved together with (72b) to yield

$$\begin{aligned} U_m^{(4)} &= q_1(\tau)\cos\theta t + q_2(\tau)\sin\theta t \\ &\quad - \frac{9\alpha}{4}\left[-\frac{1}{3\theta^2}(r_{21}\cos 2\theta t + r_{22}\sin 2\theta t) - \frac{1}{8\theta^2}(r_{23}\cos 3\theta t + r_{24}\sin 3\theta t)\right] \end{aligned} \tag{74b}$$

$$q_1(0) = 0 = q_2(0) \tag{74c}$$

On substituting from (70a) into (72c) and solving, we get

$$\begin{aligned} U_{3m}^{(4)} &= \alpha_4(\tau)\cos\omega t + \gamma_4(\tau)\sin\omega t \\ &\quad + \frac{3\alpha}{4}\left[\frac{r_{18}}{\omega^2} + \left(\frac{r_{19}cos\theta t + r_{20}sin\theta t}{\omega^2 - \theta^2}\right) + \left(\frac{r_{21}\cos 2\theta t + r_{22}\sin 2\theta t}{\omega^2 - 4\theta^2}\right)\right. \\ &\quad \left. + \left(\frac{r_{23}\cos 3\theta t + r_{24}\sin 3\theta t}{\omega^2 - 9\theta^2}\right)\right] \end{aligned} \tag{75a}$$

$$\alpha_4(0) = \gamma_4(0) = 0 \tag{75b}$$

Thus,

$$U^{(4)} = U_m^{(4)}\sin mx + U_{3m}^{(4)}\sin 3mx \tag{75c}$$

The next substitution into (57), using (73c), needs the following simplications:

$$U_m^{(1)^5} = (\alpha_1 \cos\theta t + \gamma_1 \sin\theta t + B)^5 = r_7 + r_8 cos\theta t + r_9 sin\theta t + r_{10}\cos 2\theta t + r_{11}\sin 2\theta t + r_{12}\cos 3\theta t + r_{13}\sin 3\theta t + r_{14}\cos 4\theta t + r_{15}\sin 4\theta t + r_{16}\cos 5\theta t + r_{17}\sin 5\theta t \qquad (75d)$$

where,

$$r_7 = \frac{15\alpha_1^4 B}{8} + 5\alpha_1^2\left(B^3 + \frac{3\gamma_1^2 B}{4}\right) - \frac{15\alpha_1\gamma_1^2}{16} + \left(\frac{\gamma_1^2}{2} + B^2\right)\left(B^3 + \frac{\gamma_1^2 B}{2}\right) + 3B\gamma_1\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) + \frac{3\gamma_1^4 B}{8} \qquad (75e)$$

$$r_8 = \frac{5\alpha_1^5}{8} + \frac{10\alpha_1^3 B}{4}\left(\frac{\gamma_1^2}{2} + 3B^2\right) + 5\alpha_1\left\{\left(\frac{3\gamma_1^4}{8} + 3B^2\gamma_1 + B^4\right) - \frac{1}{2}\left(\frac{3\gamma_1^4}{8} + 3\gamma_1^2 B^2\right) - \frac{B\gamma_1^4}{4}\right\} \qquad (75f)$$

$$r_9 = \frac{-5\alpha_1^4\gamma_1}{16} - \frac{10\alpha_1^3\gamma_1 B}{4} + 5\alpha_1^2\left(\frac{\gamma_1^3}{2} - \frac{5B^2\gamma_1}{8}\right) + 3\left(\frac{\gamma_1^2}{2} + B^2\right)\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) + 2\gamma_1 B\left(B^3 + \frac{\gamma_1^2 B}{2}\right) + \frac{B^2\gamma_1^3}{2} + \frac{3}{4}\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) + \frac{\gamma_1^5}{16} \qquad (75g)$$

$$r_{10} = \frac{5\alpha_1^4 B}{4} + 5\alpha_1^2 B^3 - \frac{3}{2}\left(\frac{\gamma_1^2}{2} + B^2\right)\gamma_1^2 B - 3B\gamma_1\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right) - \left(B^3 + \frac{3\gamma_1^2 B}{2}\right)\frac{\gamma_1^2}{2} \qquad (75h)$$

$$r_{11} = \frac{15\alpha_1^3\gamma_1 B}{4} + 5\alpha_1\left\{\left(\frac{11B^2\gamma_1}{4} + \frac{13B\gamma_1^3}{8}\right)\right\} \qquad (75i)$$

$$r_{12} = \frac{5\alpha_1^5}{16} + \frac{5\alpha_1^3}{2}\left(3B^2 - \frac{\gamma_1^3}{4}\right) + 5\alpha_1\left\{-\frac{1}{2}\left(\frac{3\gamma_1^4}{8} + 3\gamma_1^2 B^2\right) + \frac{B\gamma_1^4}{4}\right\} \qquad (75j)$$

$$r_{13} = \frac{15\alpha_1^4\gamma_1}{8} + \frac{5\alpha_1^3\gamma_1 B}{2} + 5\alpha_1^2\left(\frac{3\gamma_1^3}{8} + \frac{5}{4}B^2\gamma_1\right) - \frac{1}{4}\left(\frac{\gamma_1^2}{2} + B^2\right)\gamma_1^3 - \frac{B^3\gamma_1^2}{2} - \frac{3}{4}\left(\frac{\gamma_1^3}{4} + B^2\gamma_1\right)\gamma_1^2 \qquad (75k)$$

$$r_{14} = \frac{5\alpha_1^4 B}{4} - \frac{15\alpha_1^2\gamma_1^2 B}{4} + \frac{3\gamma_1^4 B}{8} \qquad (75l)$$

$$r_{15} = \frac{-5\alpha_1\gamma_1^3 B}{2} \qquad (75m)$$

$$r_{16} = \frac{\alpha_1^5}{8} - \frac{5\alpha_1^3\gamma_1^2}{8} + \frac{15\alpha_1\gamma_1^4}{16} \qquad (75n)$$

$$r_{17} = \frac{5\alpha_1^4\gamma_1}{8} - \frac{15\alpha_1^2\gamma_1^2 B}{8} + \frac{\gamma_1^5}{16} \qquad (75o)$$

where,

$$r_7(0) = \frac{63B^5}{8}, \quad r_8(0) = \frac{-105B^5}{8}, \quad r_9(0) = 0, \quad r_{10}(0) = \frac{25B^5}{4}, \quad r_{11}(0) = 0, \quad r_{12}(0) = \frac{-125B^5}{16}, \quad r_{13}(0) = 0, \quad r_{14}(0) = \frac{5B^5}{4}, \quad r_{15}(0) = 0, \quad r_{16}(0) = \frac{-B^5}{8}, \quad r_{17}(0) = 0$$

We shall also need

$$U_m^{(1)^2} U_m^{(3)} = r_{25} + r_{26} cos\theta t + r_{27} sin\theta t + r_{28}\cos 2\theta t + r_{29}\sin 2\theta t + r_{30}\cos 3\theta t + r_{31}\sin 3\theta t + r_{32}\cos 4\theta t + r_{33}\sin 4\theta t + r_{34}\cos 5\theta t + r_{35}\sin 5\theta t \qquad (76a)$$

where,

$$r_{25} = \left\{\frac{1}{2}(\alpha_1^2 + \gamma_1^2) + B^2\right\}\left(\frac{-3\alpha r_0}{\theta^2}\right) + B\alpha_1\alpha_{12} + B\gamma_1\beta_{12} + \alpha\left(\frac{\alpha_1^2 - \gamma_1^2}{2\theta^2}\right)\gamma_1 \qquad (76b)$$

$$r_{26} = \left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\}\alpha_{12} - \frac{6\alpha r_0 B\alpha_1}{\theta^2} + \frac{B\alpha_1\alpha r_1}{\theta^2} + \frac{B\alpha_2\alpha\gamma_{12}}{\theta^2} + \frac{1}{4}(\alpha_1^2 - \gamma_1^2)\alpha_{12} + \frac{3\alpha(\alpha_1^2 - \gamma_1^2)r_3}{32\theta^2} \qquad (76c)$$

$$r_{27} = \left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\}\beta_{12} + \frac{B\alpha_1\alpha r_2}{\theta^2} - \frac{6\alpha r_0 B\gamma_1}{\theta^2} - \frac{B\alpha_1\alpha\gamma_1}{\theta^2} - \frac{3B\alpha\gamma_1 r_3}{8\theta^2} - \frac{1}{4}(\alpha_1^2 - \gamma_1^2)\beta_{12} + \frac{3\alpha(\alpha_1^2 - \gamma_1^2)r_4}{32\theta^2} \qquad (76d)$$

In the same vein, we shall similarly need

$$U_m^{(1)^2} U_{3m}^{(3)} = r_{36} + r_{37} cos\theta t + r_{38} sin\theta t + r_{39}\cos 2\theta t + r_{40}\sin 2\theta t + r_{41}\cos 3\theta t + r_{42}\sin 3\theta t + r_{43}\cos 4\theta t + r_{44}\sin 4\theta t + r_{45}\cos 5\theta t + r_{46}\sin 5\theta t + r_{47} cos(\theta + \omega)t + r_{48} sin(\theta + \omega)t + r_{49}\cos(\theta - \omega)t + r_{50}\sin(\theta - \omega)t + r_{51}\cos(2\theta + \omega)t + r_{52}\sin(2\theta + \omega)t + r_{53}\cos(\omega - 2\theta)t + r_{54}\sin(\omega - 2\theta)t + r_{55}\cos\omega t + r_{56}\sin\omega t \qquad (77a)$$

where,

$$r_{36} = \left(\frac{\alpha r_0}{\omega^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{2B\alpha\alpha_1 r_0}{\omega^2} + \frac{B\alpha\alpha_1 r_5}{\omega^2 - \theta^2} + \frac{B\alpha\gamma_1 r_6}{\omega^2 - \theta^2} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_1}{4(\omega^2 - \theta^2)} \qquad (77b)$$

$$r_{37} = \left(\frac{\alpha r_5}{\omega^2 - \theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{2B\alpha\alpha_1 r_1}{\omega^2 - 4\theta^2} + \frac{B\alpha\gamma_1 r_2}{\omega^2 - 4\theta^2} + \frac{1}{4}\left(\frac{\alpha r_5}{\omega^2 - \theta^2}\right)(\alpha_1^2 - \gamma_1^2) - \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_3}{4(\omega^2 - 9\theta^2)} \qquad (77c)$$

$$r_{38} = \left(\frac{\alpha r_6}{\omega^2 - \theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\gamma_1 r_0}{\omega^2} - \frac{B\alpha\gamma_1 r_1}{\omega^2 - 4\theta^2} - \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_6}{4(\omega^2 - \theta^2)} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_4}{4(\omega^2 - 9\theta^2)} \qquad (77d)$$

$$r_{39} = \left(\frac{\alpha r_1}{\omega^2 - 4\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\alpha_1 r_5}{\omega^2 - \theta^2} + \frac{B\alpha\alpha_1 r_3}{\omega^2 - 9\theta^2} - \frac{B\alpha\gamma_1 r_6}{\omega^2 - \theta^2} + \frac{B\alpha\gamma_1 r_4}{\omega^2 - 9\theta^2} + \frac{1}{4}(\alpha_1^2 - \gamma_1^2)\left(\frac{\alpha r_0}{\omega^2}\right) \qquad (77e)$$

$$r_{40} = \left(\frac{\alpha r_2}{\omega^2 - 4\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\alpha_1 r_6}{\omega^2 - \theta^2} + \frac{B\alpha\alpha_1 r_2}{\omega^2 - 4\theta^2} + \frac{B\alpha\alpha_1 r_4}{\omega^2 - 9\theta^2} + \frac{B\alpha\alpha_5\gamma_1}{\omega^2 - \theta^2} - \frac{B\alpha\gamma_1 r_3}{\omega^2 - 9\theta^2} \qquad (77f)$$

$$r_{41} = \left(\frac{\alpha r_3}{\omega^2 - 9\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\gamma_1\alpha_1}{\omega^2 - 4\theta^2} - \frac{B\alpha\gamma_1 r_4}{\omega^2 - 9\theta^2} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_5}{4(\omega^2 - 4\theta^2)} - \frac{B\alpha\gamma_1 r_2}{\omega^2 - 4\theta^2} \qquad (77g)$$

$$r_{42} = \left(\frac{\alpha r_4}{\omega^2 - 9\theta^2}\right)\left\{\frac{1}{2}(\alpha_1^2 - \gamma_1^2) + B^2\right\} + \frac{B\alpha\gamma_1 r_1}{\omega^2 - 4\theta^2} + \frac{1}{4}(\alpha_1^2 - \gamma_1^2)\left(\frac{\alpha r_6}{\omega^2 - \theta^2}\right) \qquad (77h)$$

$$r_{43} = \frac{B\alpha\alpha_1 r_3}{\omega^2 - 9\theta^2} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_1}{4(\omega^2 - 9\theta^2)} \qquad (77i)$$

$$r_{44} = \frac{B\alpha\alpha_1 r_4}{\omega^2 - 9\theta^2} + \frac{B\alpha\gamma_1 r_3}{\omega^2 - 9\theta^2} + \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_2}{4(\omega^2 - 9\theta^2)} \qquad (77j)$$

$$r_{45} = \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_3}{4(\omega^2 - 9\theta^2)}, \qquad r_{46} = \frac{\alpha(\alpha_1^2 - \gamma_1^2)r_4}{4(\omega^2 - 9\theta^2)} \qquad (77k)$$

$$r_{47} = -B\gamma_1\beta_3, \qquad r_{48} = B\alpha_1\beta_3 + B\gamma_1\gamma_3 \qquad (77l)$$

$$r_{49} = (B\alpha_1\alpha_3 + B\gamma_1\beta_3), \qquad r_{50} = B\alpha_1\beta_3 + B\gamma_1\gamma_3 \tag{77m}$$

$$r_{51} = \frac{\alpha_3(\alpha_1^2 - \gamma_1^2)}{4}, \qquad r_{52} = \frac{\beta_3(\alpha_1^2 - \gamma_1^2)}{4} \tag{77n}$$

$$r_{53} = \frac{\alpha_3(\alpha_1^2 - \gamma_1^2)}{4}, \qquad r_{54} = \frac{\beta_3(\alpha_1^2 - \gamma_1^2)}{4} \tag{77o}$$

$$r_{55} = \alpha_3\left\{\frac{1}{2}(\alpha_1^2 + \gamma_1^2) + B^2\right\}, \qquad r_{56} = \beta_3\left\{\frac{1}{2}(\alpha_1^2 + \gamma_1^2) + B^2\right\} \tag{77p}$$

where,

$$r_{36}(0) = \alpha B^5\Omega_2, \quad \Omega_2 = \left(\frac{9}{2(\omega^2 - \theta^2)} - \frac{2}{\omega^2}\right)$$

$$r_{37}(0) = \alpha B^5\Omega_3, \quad \Omega_3 = \left(\frac{9}{8(\omega^2 - \theta^2)} + \frac{1}{16(\omega^2 - \theta^2)} - \frac{3}{\omega^2 - 4\theta^2}\right)$$

$$r_{38}(0) = 0, \qquad r_{39}(0) = \alpha B^5\Omega_4, \quad \Omega_4 = \left(\frac{9}{2(\omega^2 - \theta^2)} + \frac{15}{4(\omega^2 - \theta^2)} + \frac{1}{4(\omega^2 - 9\theta^2)} + \frac{1}{\omega^2}\right)$$

$$r_{40}(0) = 0, \qquad r_{41}(0) = \alpha B^5\Omega_4^{(1)}, \quad \Omega_4^{(1)} = \left(\frac{3}{8(\omega^2 - 9\theta^2)} - \frac{111}{32(\omega^2 - 4\theta^2)}\right)$$

$$r_{42}(0) = 0, \qquad r_{43}(0) = \alpha B^5\Omega_5, \quad \Omega_5 = \left(\frac{3}{4(\omega^2 - 4\theta^2)} + \frac{1}{4(\omega^2 - 9\theta^2)}\right)$$

$$r_{44}(0) = 0, \qquad r_{45}(0) = -\alpha B^5\Omega_6, \quad \Omega_6 = -\frac{1}{16(\omega^2 - 9\theta^2)}$$

$$r_{46}(0) = 0, \qquad r_{47}(0) = 0, \qquad r_{48}(0) = 0$$

$$r_{49}(0) = \alpha B^5\Omega_1, \quad r_{50}(0) = 0, \qquad r_{51}(0) = \frac{-\alpha B^5\Omega_1}{4}$$

$$r_{52}(0) = 0, \qquad r_{53}(0) = \frac{-\alpha B^5\Omega_1}{4}, \quad r_{54}(0) = 0, \quad r_{55}(0) = \frac{-3\alpha B^5\Omega_1}{2}, \quad r_{56}(0) = 0.$$

Now, substituting the relevant terms into (57) yields

$$
\begin{aligned}
U_{,tt}^{(5)} + U_{,xxxx}^{(5)} + 2\lambda U_{,xx}^{(5)} + U^{(5)} &= -2\left(U_{m,t\tau}^{(3)} + U_{m,t}^{(3)}\right)\sin mx - \left(U_{3m,t\tau}^{(3)} + U_{3m,t}^{(3)}\right) sin3mx \\
&- 3\alpha\left[\frac{1}{4}{U_m^{(1)}}^2 U_m^{(3)}(3\sin mx - \sin 3mx) + \frac{1}{4}U_m^{(1)}U_m^{(3)}(\sin 3mx - \sin mx)\right] \\
&+ \frac{\beta}{16}{U_m^{(1)}}^5[11\sin mx - 5\sin 3mx + \sin 5mx] \qquad (78a)
\end{aligned}
$$

Using (59) for $n = m$, we get

$$U_{m,tt}^{(5)} + \theta^2 U_m^{(5)} = -2\left(U_{m,t\tau}^{(3)} + U_{m,t}^{(3)}\right) - \frac{3\alpha}{4}\left[3{U_m^{(1)}}^2 U_m^{(3)} - {U_m^{(1)}}^2 U_{3m}^{(3)}\right] + \frac{11\beta}{16}{U_m^{(1)}}^5 \qquad (78b)$$

$$U_m^{(5)}(0,0) = 0, \qquad U_{m,t}^{(5)}(0,0) + U_{m,\tau}^{(3)}(0,0) = 0 \qquad (78c)$$

whereas for $n = 3m$, we get

$$U_{3m,tt}^{(5)} + \omega^2 U_{3m}^{(5)} = -2\left(U_{3m,t\tau}^{(3)} + U_{3m,t}^{(3)}\right) - \frac{5\beta}{16}{U_m^{(1)}}^5 \qquad (78d)$$

$$U_{3m}^{(5)}(0,0) = 0, \qquad U_{3m,t}^{(5)}(0,0) + U_{3m,\tau}^{(3)}(0,0) = 0 \qquad (78e)$$

For $n = 5m$, we get

$$U_{5m,tt}^{(5)} + \varphi^2 U_{5m}^{(5)} = \frac{\beta}{16}{U_m^{(1)}}^5 \qquad (78f)$$

$$U_{5m}^{(5)}(0,0) = 0, \qquad U_{5m,t}^{(5)}(0,0) = 0 \qquad (78g)$$

where,

$$\varphi^2 = (625m^4 - 50m^2\lambda + 1) > 0 \ \forall\, m. \qquad (78h)$$

After substituting into (78b), using (75a – o) and (76a – d), and ensuring a uniformly valid solution in t by equating to zero the coefficients of $\cos\theta t$ and $\sin\theta t$, we get, for the coefficient of $\cos\theta t$,

$$\beta_{12}' + \beta_{12} = \rho_1(\tau), \quad \rho_1 = \frac{1}{2\theta}\left[\alpha\left(\frac{3}{4}r_{37} - \frac{9r_{26}}{4}\right) + \frac{11\beta r_8}{16}\right] \qquad (79a)$$

and for $\sin\theta t$, we get

$$\alpha_{12}' + \alpha_{12} = \rho_2(\tau), \qquad \rho_2 = \frac{1}{2\theta}\left[\frac{\alpha}{4}(9r_{27} - 3r_{38}) - \frac{11\beta r_9}{16}\right] \qquad (79b)$$

The solutions of (79a, b) are

$$\beta_{12} = e^{-\tau}\left[\int_0^{\tau} e^{s}\,\rho_1(s)ds\right], \quad \alpha_{12} = e^{-\tau}\left[\int_0^{\tau} e^{s}\,\rho_2(s)ds + \alpha_{12}(0)\right] \qquad (79c)$$

Meanwhile, it follows from (79a – c) that

$$\beta_{12}'(0) = B^5\Omega_7, \qquad \alpha_{12}'(0) = -\frac{483\alpha B^3}{32\theta^2} \qquad (79d)$$

where,

$$\Omega_7 = \left[\alpha^2\left\{\frac{3\Omega_3}{4} - \frac{27{,}297}{256\theta^2}\right\} - \frac{1155\beta}{128}\right] \qquad (79e)$$

The remaining equation in the substitution into (78b) is

$$\begin{aligned} U_{m,tt}^{(5)} + \theta^2 U_m^{(5)} &= r_{57} + r_{58}cos2\theta t + r_{59}sin2\theta t \\ &+ r_{60}\cos 3\theta t + r_{61}\sin 3\theta t + r_{62}\cos 4\theta t + r_{63}\sin 4\theta t + r_{64}\cos 5\theta t + r_{65}\sin 5\theta t \\ &+ r_{66}cos(\theta+\omega)t + r_{67}sin(\theta+\omega)t \\ &+ r_{68}\cos(\theta-\omega)t + r_{69}\sin(\theta-\omega)t + r_{70}\cos(2\theta+\omega)t + r_{71}\sin(2\theta+\omega)t \\ &+ r_{72}\cos(\omega-2\theta)t + r_{73}\sin(\omega-2\theta)t + r_{74}\cos\omega t + r_{75}\sin\omega t \qquad (80a) \end{aligned}$$

where,

$$r_{57} = \left[\frac{-9\alpha r_{25}}{4} + \frac{3\alpha r_{36}}{4} + \frac{11\beta r_7}{16}\right] \qquad (80b)$$

$$r_{58} = \left[\frac{-9\alpha r_{28}}{4} + \frac{3\alpha r_{39}}{4} + \frac{11\beta r_{10}}{16}\right] \qquad (80c)$$

$$r_{59} = \left[\frac{4\alpha(r_1' + r_1)}{\theta} - \frac{9\alpha r_{29}}{4} + \frac{3\alpha r_{40}}{4} + \frac{11\beta r_{11}}{16}\right] \qquad (80d)$$

$$r_{60} = \left[\frac{-9\alpha(r_4' + r_4)}{8\theta} - \frac{9\alpha r_{30}}{4} + \frac{3\alpha r_{41}}{4} + \frac{11\beta r_{12}}{16}\right] \qquad (80e)$$

$$r_{61} = \left[\frac{-9\alpha(r_3' + r_3)}{8\theta} - \frac{9\alpha r_{31}}{4} + \frac{3\alpha r_{42}}{4} + \frac{11\beta r_{13}}{16}\right] \qquad (80f)$$

$$r_{62} = \left[\frac{-9\alpha r_{32}}{4} + \frac{3\alpha r_{43}}{4} + \frac{11\beta r_{14}}{16}\right] \qquad (80g)$$

$$r_{63} = \left[\frac{-9\alpha r_{33}}{4} + \frac{3\alpha r_{14}}{4} + \frac{11\beta r_{15}}{16}\right] \quad (80h)$$

$$r_{64} = \left[\frac{-9\alpha r_{34}}{4} + \frac{3\alpha r_{45}}{4} + \frac{11\beta r_{16}}{16}\right] \quad (80i)$$

$$r_{65} = \left[\frac{-9\alpha r_{35}}{4} + \frac{3\alpha r_{46}}{4} + \frac{11\beta r_{17}}{16}\right] \quad (80j)$$

$$r_{66} = \frac{3\alpha r_{47}}{4}, \quad r_{67} = \frac{3\alpha r_{48}}{4}, \quad r_{68} = \frac{3\alpha r_{49}}{4} \quad (80k)$$

$$r_{69} = \frac{3\alpha r_{51}}{4}, \quad r_{70} = \frac{3\alpha r_{51}}{4}, \quad r_{71} = \frac{3\alpha r_{52}}{4} \quad (80l)$$

$$r_{72} = \frac{3\alpha r_{47}}{4}, \quad r_{73} = \frac{3\alpha r_{48}}{4}, \quad r_{74} = \frac{3\alpha r_{49}}{4} \quad (80m)$$

$$r_{75} = \frac{3\alpha r_{56}}{4} \quad (80n)$$

$$r_{57}(0) = B^5\Omega_8, \quad \Omega_8 = \left[\alpha^2\left(\frac{28107}{384\theta^2} + \frac{3\Omega_2}{4}\right) + \frac{693\beta}{128\theta^2}\right] \quad (80o)$$

$$r_{58}(0) = B^5\Omega_9, \quad \Omega_9 = \left[\left(\frac{135}{7\theta^2} + \frac{475}{128\theta^2} + \frac{3\Omega_5}{4}\right) + \frac{693\beta}{128}\right] \quad (80p)$$

$$r_{59}(0) = \frac{-12\alpha B^5}{\theta}, \quad r_{60}(0) = B^5\Omega_{10}, \quad \Omega_{10} = \left[\alpha^2\left(\frac{3\Omega_4}{4} - \frac{1215}{128\theta^2}\right) - \frac{1375\beta}{256}\right] \quad (80q)$$

$$r_{61}(0) = \frac{9\alpha B^3}{16\theta}, \quad r_{62}(0) = B^5\Omega_{11}, \quad \Omega_{11} = \left(\frac{3\Omega_5}{4} - \frac{9}{62\theta^2} + \frac{11\beta}{64}\right) \quad (80r)$$

$$r_{63}(0) = 0, \quad r_{64}(0) = B^5\Omega_{12}, \quad \Omega_{12} = \left[\alpha^2\left(\frac{27}{518\theta^2} - \frac{3\Omega_6}{512\theta}\right) - \frac{11\beta}{128}\right] \quad (80s)$$

$$r_{65}(0) = 0, \quad r_{66}(0) = \frac{3\alpha^2 B^5\Omega_1}{4}, \quad r_{67}(0) = 0 \quad (80t)$$

$$r_{68}(0) = \frac{3\alpha^2 B^5\Omega_1}{4}, \quad r_{69}(0) = 0 \quad (80u)$$

$$r_{70}(0) = \frac{3\alpha^2 B^5\Omega_1}{16}, \quad r_{71}(0) = 0, \quad r_{72}(0) = \frac{3\alpha^2 B^5\Omega_1}{16} \quad (80v)$$

$$r_{73}(0) = 0, \quad r_{74}(0) = \frac{9\alpha^2 B^5\Omega_1}{4}, \quad r_{75}(0) = 0 \quad (80w)$$

The solution of (80a – z), using (78c) is

$$\begin{aligned} U_m^{(5)} = {} & \alpha_5(\tau)\cos\theta t + \gamma_5(\tau)\sin\theta t + \frac{r_{57}}{\theta^2} - \frac{1}{3\theta^2}\,(r_{58}cos2\theta t + r_{59}sin2\theta t) \\ & - \frac{1}{8\theta^2}(r_{60}\cos 3\theta t + r_{61}\sin 3\theta t) - \frac{1}{15\theta^2}(r_{62}\cos 4\theta t + r_{63}\sin 4\theta t) \\ & - \frac{1}{15\theta^2}(r_{64}\cos 5\theta t + r_{65}\sin 5\theta t) - \left(\frac{r_{66}cos(\theta+\omega)t + r_{67}sin(\theta+\omega)t}{\omega(\omega+2\theta)}\right) \\ & + \frac{(r_{68}\cos(\theta-\omega)t + r_{69}\sin(\theta-\omega)t)}{\omega(2\theta-\omega)} - \left(\frac{r_{70}\cos(2\theta+\omega)t + r_{71}\sin(2\theta+\omega)t}{(3\theta+\omega)(\theta+\omega)}\right) \\ & + \left(\frac{r_{72}\cos(\omega-2\theta)t + r_{73}\sin(\omega-2\theta)t}{(\omega-\theta)(3\theta-\omega)}\right) + \left(\frac{r_{74}\cos\omega t + r_{75}\sin\omega t}{(\theta+\omega)(\theta-\omega)}\right) \qquad (81a) \end{aligned}$$

where,

$$\alpha_5(0) = \frac{B^5\Omega_{13}}{\theta^2} \qquad (81b)$$

$$\begin{aligned} \Omega_{13} = {} & \left[\frac{\Omega_9}{3} - \Omega_8 + \frac{\Omega_{10}}{8} + \frac{\Omega_{11}}{15} + \frac{\Omega_{12}}{24} \right. \\ & + 3\alpha^2\left\{\frac{\Omega_1}{4\omega(\omega+2\theta)} - \frac{\Omega_1}{4\omega(2\theta-\omega)} + \frac{\Omega_1}{16(\theta+\omega)(3\theta+\omega)} - \frac{\Omega_1}{16(\omega-\theta)(3\theta-\omega)} \right. \\ & \left.\left. - \frac{\Omega_1}{2(\theta+\omega)(\theta-\omega)}\right\}\right] \qquad (81c) \end{aligned}$$

$$\begin{aligned} \gamma_5(0) = {} & \frac{1}{\theta}\left[\frac{2r_{59}}{3\theta} + \frac{2r_{61}}{8\theta} + \frac{2r_{63}}{15\theta} + \frac{5r_{65}}{15\theta} + \frac{(\theta+\omega)r_{67}}{\omega(2\theta+\omega)} - \frac{(\theta-\omega)r_{69}}{\omega(2\theta-\omega)} + \frac{(2\theta+\omega)r_{71}}{(3\theta+\omega)(\theta+\omega)} \right. \\ & \left. - \frac{(\omega-2\theta)r_{73}}{(\omega-\theta)(3\theta-\omega)} - \frac{\omega r_{75}}{(\theta+\omega)(\theta-\omega)}\right]\Big|_{\tau=0} \\ & + \frac{1}{\theta}\left[-\alpha_{12}'(0) + 3\alpha\left\{\frac{r_0'}{\theta^2} - \frac{r_1'}{3\theta^2} - \frac{r_3'}{8\theta^2}\right\}\right]\Big|_{\tau=0} \qquad (81d) \end{aligned}$$

We next substitute into (78d) and ensure a uniformly valid solution by equating to zero the coefficients of $\cos\omega t$ and $\sin\omega t$ and get respectively

$$\beta_3' + \beta_3 = -\frac{3\alpha r_{55}}{8\omega} \; and \; \alpha_3' + \alpha_3 = \frac{3\alpha r_{56}}{8\omega} \qquad (82a, b)$$

On solving (82a, b), the result gives

$$\beta_3(\tau) = e^{-\tau}\left[\frac{-3\alpha}{8\omega}\int_0^\tau r_{55}e^s ds + \beta_3(0)\right] \tag{82c}$$

$$\alpha_3(\tau) = e^{-\tau}\left[\frac{3\alpha}{8\omega}\int_0^\tau e^s r_{56} ds\right] \tag{82d}$$

The remaining equation in the substitution into (78d) is

$$U_{3m,tt}^{(5)} + \omega^2 U_{3m}^{(5)} = r_{76} + r_{79}cos\theta t + r_{80}sin\theta t + r_{81}\cos 2\theta t + r_{82}\sin 2\theta t + r_{83}\cos 3\theta t + r_{84}\sin 3\theta t + r_{85}\cos 4\theta t + r_{86}\sin 4\theta t + r_{87}\cos 5\theta t + r_{88}\sin 5\theta t + r_{89}cos(\theta+\omega)t + r_{90}sin(\theta + \omega)t + r_{91}\cos(\theta-\omega)t + r_{92}\sin(\theta-\omega)t + r_{93}\cos(2\theta+\omega)t + r_{94}\sin(2\theta+\omega)t + r_{95}\cos(\omega - 2\theta)t + r_{96}\sin(\omega-2\theta)t \tag{83a}$$

$$U_{3m}^{(5)}(0,0) = 0, \quad U_{3m,t}^{(5)}(0,0) + U_{3m,\tau}^{(5)}(0,0) = 0 \tag{83b}$$

where,

$$r_{76} = \frac{3\alpha r_{25}}{4} - r_{36} - \frac{5\beta r_7}{16} \tag{83c}$$

$$r_{79} = \left[\frac{-2\theta(r_6' + r_6)\alpha}{\omega^2-\theta^2} + \frac{3\alpha r_{26}}{4} - \frac{3\alpha r_{37}}{4} - \frac{15\beta r_8}{16}\right] \tag{83d}$$

$$r_{80} = \left[\frac{2\theta(r_5' + r_5)\alpha}{\omega^2-\theta^2} + \frac{3\alpha r_{27}}{4} - \frac{3\alpha r_{38}}{4} - \frac{5\beta r_9}{16}\right] \tag{83e}$$

$$r_{81} = \left[\frac{4\theta(r_2' + r_2)\alpha}{\omega^2-\theta^2} + \frac{3\alpha r_{28}}{4} - \frac{3\alpha r_{39}}{4} - \frac{5\beta r_{10}}{16}\right] \tag{83e}$$

$$r_{82} = \left[\frac{4\theta(r_1' + r_1)\alpha}{\omega^2-4\theta^2} + \frac{3\alpha r_{29}}{4} - \frac{3\alpha r_{40}}{4} - \frac{5\beta r_{11}}{16}\right] \tag{83g}$$

$$r_{83} = \left[\frac{6\theta(r_4' + r_4)\alpha}{\omega^2-9\theta^2} + \frac{3\alpha r_{30}}{4} - \frac{3\alpha r_{41}}{4} - \frac{5\beta r_{12}}{16}\right] \tag{83h}$$

$$r_{84} = \left[\frac{6\theta(r_3' + r_3)\alpha}{\omega^2-9\theta^2} + \frac{3\alpha r_{31}}{4} - \frac{3\alpha r_{42}}{4} - \frac{5\beta r_{13}}{16}\right] \tag{83i}$$

$$r_{85} = \left[\frac{3\alpha r_{32}}{4} - \frac{3\alpha r_{43}}{4} - \frac{5\beta r_{14}}{16}\right] \tag{83j}$$

$$r_{86} = \left[\frac{3\alpha r_{33}}{4} - \frac{3\alpha r_{44}}{4} - \frac{5\beta r_{15}}{16}\right] \tag{83k}$$

$$r_{87} = \left[\frac{3\alpha r_{34}}{4} - \frac{3\alpha r_{45}}{4} - \frac{5\beta r_{16}}{16}\right] \tag{83l}$$

$$r_{88} = \left[\frac{3\alpha r_{35}}{4} - \frac{3\alpha r_{46}}{4} - \frac{5\beta r_{17}}{16}\right] \tag{83m}$$

$$r_{89} = \left[\frac{-3\alpha r_{47}}{4}\right], \qquad r_{90} = \left[-\frac{3\alpha r_{48}}{4}\right] \tag{83o}$$

$$r_{91} = \left[\frac{-3\alpha r_{49}}{4}\right], \qquad r_{92} = \left[-\frac{3\alpha r_{50}}{4}\right] \tag{83p}$$

$$r_{93} = \frac{-3\alpha r_{51}}{4}, \quad r_{94} = -\frac{3\alpha r_{52}}{4}, \quad r_{95} = -\frac{3\alpha r_{53}}{4}, \qquad r_{96} = -\frac{3\alpha r_{54}}{4} \tag{83q}$$

where,

$$r_{76}(0) = B^5 \Omega_{14}, \quad \Omega_{14} = \left[\frac{3123\alpha}{128\theta^2} + \alpha\Omega_2 + \frac{315\beta}{128}\right] \tag{83r}$$

$$r_{79}(0) = B^5 \Omega_{19}, \quad \Omega_{19} = \left[\frac{18198\alpha^2}{512\theta^2} = \frac{675\alpha^2}{16(\omega^2 - \theta^2)} - \frac{3\Omega_3}{4}\right] \tag{83s}$$

$$r_{80}(0) = B^3 \Omega_{20}, \qquad \Omega_{20} = \frac{3\theta\alpha}{(\omega^2 - \theta^2)} \tag{83t}$$

$$r_{81}(0) = B^5 \Omega_{21}, \qquad \Omega_{21} = -\left[\alpha^2\left(\frac{135}{2(\omega^2 - 4\theta^2)} - \frac{99}{8\theta^2} + \frac{3\Omega_4}{4}\right) + \frac{125\beta}{64}\right] \tag{83u}$$

$$r_{82}(0) = B^3 \Omega_{22}, \qquad \Omega_{22} = -\left(\frac{12\theta\alpha}{\omega^2 - 4\theta^2}\right) \tag{83v}$$

$$r_{83}(0) = B^5 \Omega_{23}, \quad \Omega_{23} = -\left[\alpha^2\left(\frac{405}{8(\omega^2 - 9\theta^2)B^2} + \frac{243}{512\theta^2} - \frac{3\Omega_4}{4}\right) + \frac{625\beta}{256}\right] \tag{83w}$$

$$r_{84}(0) = B^3 \Omega_{24}, \qquad \Omega_{24} = -\left(\frac{3\theta\alpha}{\omega^2 - 9\theta^2}\right) \tag{83x}$$

$$r_{85}(0) = B^5 \Omega_{25}, \quad \Omega_{25} = \left[\alpha^2\left(\frac{63}{128\theta^2} - \frac{3\Omega_5}{4}\right) - \frac{25\beta}{64}\right] \tag{83y}$$

$$r_{86}(0) = 0, \quad r_{87}(0) = B^5\Omega_{26}, \quad \Omega_{26} = \left[\alpha^2\left(\frac{9}{512\theta^2} + \frac{3\Omega_6}{4}\right) + \frac{5\beta}{128}\right] \tag{83z}$$

$$r_{88}(0) = 0, \quad r_{89}(0) = B^5\Omega_{27}, \quad \Omega_{27} = \frac{-3\alpha^2\Omega_1}{4} \tag{84a}$$

$$r_{90}(0) = 0, \quad r_{91}(0) = B^5\Omega_{28}, \quad \Omega_{28} = \frac{-3\alpha^2\Omega_1}{4}, \qquad r_{92}(0) = 0 \tag{84b}$$

$$r_{93}(0) = B^5\Omega_{29}, \quad \Omega_{29} = \frac{3\alpha^2\Omega_1}{16}, \qquad r_{94}(0) = 0 \tag{84c}$$

$$r_{95}(0) = 0 \quad B^5\Omega_{30}, \quad \Omega_{30} = \frac{3\alpha^2\Omega_1}{16}, \qquad r_{96}(0) = 0 \tag{84d}$$

In evaluating the above, we have used the fact that

$$\alpha_3'(0) = -\alpha_3(0) = \alpha B^3\Omega_1; \quad \beta_3'(0) = \frac{9\alpha^2B^5\Omega_1}{16\omega} \tag{84e}$$

The solution of (83a, b) is

$$\begin{aligned} U_{3m}^{(5)} = {} & \alpha_6(\tau)\cos\omega t + \gamma_6(\tau)\sin\omega t + \left(\frac{1}{\omega^2-\theta^2}\right)(r_{79}cos\theta t + r_{80}sin\theta t) \\ & + \left(\frac{1}{\omega^2-4\theta^2}\right)(r_{83}\cos 3\theta t + r_{84}\sin 3\theta t) + \left(\frac{1}{\omega^2-16\theta^2}\right)(r_{85}\cos 4\theta t + r_{86}\sin 4\theta t) \\ & + \left(\frac{1}{\omega^2-25\theta^2}\right)(r_{87}\cos 5\theta t + r_{88}\sin 5\theta t) \\ & + \left(\frac{1}{\theta(2\omega+\theta)}\right)(r_{89}cos(\theta+\omega)t + r_{90}sin(\theta+\omega)t) \\ & + \left(\frac{1}{\theta(2\omega+\theta)}\right)(r_{91}\cos(\theta-\omega)t + r_{92}\sin(\theta-\omega)t\ ) \\ & - \left(\frac{1}{4\theta(\theta+\omega)}\right)(r_{93}\cos(2\theta+\omega)t + r_{94}\sin(2\theta+\omega)t) \\ & + \left(\frac{1}{4\theta(\omega-\theta)}\right)(r_{95}\cos(\omega-2\theta)t + r_{96}\sin(\omega-2\theta)t) \end{aligned} \tag{85a}$$

where,

$$\alpha_6(0) = B^5\Omega_{31} \tag{85b}$$

$$\Omega_{31} = -\left[\frac{\Omega_{31}}{\omega^2-\theta^2} + \frac{\Omega_{21}}{\omega^2-4\theta^2} + \frac{\Omega_{23}}{\omega^2-9\theta^2} + \frac{\Omega_{25}}{\omega^2-16\theta^2} + \frac{\Omega_{26}}{\omega^2-25\theta^2} - \frac{\Omega_{27}}{\theta(2\omega+\theta)} + \frac{\Omega_{28}}{\theta(2\omega-\theta)} - \frac{\Omega_{29}}{4\theta(\theta+\omega)} + \frac{\Omega_{30}}{4\theta(\omega-\theta)}\right] \quad (85c)$$

The determination of $\gamma_6(0)$ follows from using

$$U^{(5)}_{3m,t}(0,0) + U^{(3)}_{3m,\tau}(0,0) = 0 \quad (85d)$$

Here,

$$U^{(3)}_{3m,\tau}(0,0) = B^3\Omega_{32} \quad (85e)$$

$$\Omega_{32} = \left[\Omega_1 + \left\{-\frac{6}{\omega^2} + \frac{21}{4(\omega^2-\theta^2)} - \frac{6}{\omega^2-4\theta^2} + \frac{3}{4(\omega^2-9\theta^2)}\right\}\right] \quad (85f)$$

$$\therefore \quad \gamma_6(0) = B^3\Omega_{33} \quad (85g)$$

$$\Omega_{33} = -\frac{1}{\omega}\left[\frac{\theta\Omega_{20}}{\omega^2-\theta^2} + \frac{2\theta}{\omega^2-4\theta^2} + \frac{3\theta}{\omega^2-9\theta^2} + \Omega_{32}\right] \quad (85h)$$

Now, simplifying (78f), we have

$$U^{(5)}_{5m,tt} + \varphi^2 U^{(5)}_{5m} = \frac{\beta}{16}\left(U^{(1)}_m\right)^5$$

$$= \frac{\beta}{16}[r_7 + r_8 cos\theta t + r_9 sin\theta t + r_{10}\cos 2\theta t + r_{11}\sin 2\theta t + r_{12}\cos 3\theta t + r_{13}\sin 3\theta t + r_{14}\cos 4\theta t + r_{15}\sin 4\theta t + r_{16}\cos 5\theta t + r_{17}\sin 5\theta t] \quad (86a)$$

$$U^{(5)}_{5m}(0,0) = U^{(5)}_{5m,t} = 0 \quad (86b)$$

The solution of (86a, b) is

$$U^{(5)}_{5m} = \alpha_7\cos\varphi t + \gamma_7\sin\varphi t + \frac{\beta}{16}\left[\frac{r_7}{\varphi^2} + \left(\frac{1}{\varphi^2-\theta^2}\right)(r_8 cos\theta t + r_9 sin\theta t) + \left(\frac{1}{\varphi^2-4\theta^2}\right)(r_{10}\cos 2\theta t + r_{11}\sin 2\theta t) + \left(\frac{1}{\varphi^2-9\theta^2}\right)(r_{12}\cos 3\theta t + r_{13}\sin 3\theta t) + \left(\frac{1}{\varphi^2-16\theta^2}\right)(r_{14}\cos 4\theta t + r_{15}\sin 4\theta t) + \left(\frac{1}{\varphi^2-25\theta^2}\right)(r_{14}\cos 5\theta t + r_{15}\sin 5\theta t)\right] \quad (87a)$$

$$\alpha_7(0) = B^5 \Omega_{30}^{(1)}, \quad \gamma_7(0) = 0 \tag{87b}$$

where,

$$\Omega_{30}^{(1)} = -\frac{\beta}{16}\left[\frac{63}{8\varphi^2} - \frac{105}{8(\varphi^2-\theta^2)} + \frac{25}{4(\varphi^2-4\theta^2)} - \frac{125}{16(\varphi^2-9\theta^2)} + \frac{5}{4(\varphi^2-16\theta^2)} - \frac{1}{8(\varphi^2-25\theta^2)}\right] \tag{87c}$$

The determination of $U_m^{(4)}$ and $U_{3m}^{(4)}$ in full will automatically depend on $U_m^{(2)}$ which vanishes as in (73c). Hence, we conclude that

$$U^{(4)} = U_m^{(4)} = U_{3m}^{(4)} \equiv 0$$

Thus, the buckling mode takes the form

$$U(x,t,\tau,\epsilon) = \epsilon U_m^{(1)} sinmx + \epsilon^3 \left(U_m^{(3)} sinmx + U_{3m}^{(3)} sin3mx\right) + \epsilon^5 \left(U_m^{(5)} sinmx + U_{3m}^{(5)} sin3mx + U_{5m}^{(5)} sin5mx\right) + \cdots \tag{88}$$

7. DYNAMIC BUCKLING LOAD

According to Amazigo [18], the dynamic buckling load λ_D is obtained through the maximization

$$\frac{d\lambda}{dU_a} = 0 \tag{89}$$

where U_a is the maximum displacement . The dynamic buckling load is the largest value of the load parameter for the solution to be bounded. The onus on us now is to first determine U_a subsequent upon which (89) shall be invoked to determine the value of λ_D. The conditions for the maximum U_a are

$$U_{,x} = 0, \quad U_{,t} + \epsilon^2 U_{,\tau} = 0 \tag{90a, b}$$

We see from (89) that the least nontrivial value of $x = x_a$, for $U_x = 0$ is

$$x_a = \frac{\pi}{2m}.$$

Thus, from (89), we have

$$U(x_a,t,\tau,\epsilon) = \epsilon U_m^{(1)} + \epsilon^3 \left(U_m^{(3)} - U_{3m}^{(3)}\right) + \epsilon^5 \left(U_m^{(5)} - U_{3m}^{(5)} + U_{5m}^{(5)}\right) + \cdots \tag{91}$$

Let the values of t and τ at maximum displacement be t_a and τ_a respectively and

$$t_a = t_0^{(1)} + \epsilon^2 t_2^{(1)} + \epsilon^4 t_4^{(1)} + \cdots$$

$$\tau_a = \epsilon^2 t_a = \epsilon^2 \left(t_0^{(1)} + \epsilon^2 t_2^{(1)} + \epsilon^4 t_4^{(1)} + \cdots \right)$$

The results will be given in two separate levels of approximations, first, by taking all the buckling modes in the shape of imperfection and secondly, by taking the buckling modes in the combined shapes of $\sin mx$ and $\sin 3mx$.

7.1 DYNAMIC BUCKLING LOAD FOR MODES IN THE SHAPE OF IMPERFECTION

In this case, (91) becomes

$$U = \epsilon U_m^{(1)} + \epsilon^3 U_m^{(1)} + \epsilon^5 U_m^{(1)} + \cdots \tag{92a}$$

By substituting (92a) into (90b) and equating coefficients of ϵ, we get

$$O(\epsilon): \; U_{m,t}^{(1)}\left(t_0^{(1)}, 0\right) = 0 \tag{92b}$$

$$O(\epsilon^2): \; t_2^{(1)} U_{m,tt}^{(1)} + U_{m,\tau}^{(1)} = 0 \tag{92c}$$

etc.

where, (92b, c) are evaluated at $\left(t_0^{(1)}, 0\right)$. From (92b), we get, $t_0^{(1)} = \frac{\pi}{\theta}$ and from (92c), we get

$$t_2^{(1)} = \frac{1}{\theta^2}$$

After evaluating (92a) at maximum value, we have the non – vanishing terms as

$$U_a = \Big[\epsilon U_m^{(1)} + \epsilon^3 \left(t_0^{(1)} U_{m,\tau}^{(1)} + U_m^{(3)} \right) + \epsilon^5 \left(t_2^{(1)} U_{m,\tau}^{(1)} + \frac{1}{2} {t_2^{(1)}}^2 U_{m,tt}^{(1)} + \frac{1}{2} {t_0^{(1)}}^2 U_{m,\tau\tau}^{(1)} + U_{m,\tau}^{(3)} + U_m^{(5)} \right) \Big] \Big|_{\left(t_0^{(1)}, 0\right)} + \cdots \tag{93a}$$

On simplifying (93a) we get

$$U_a = g_1 \epsilon + g_2 \epsilon^3 + g_3 \epsilon^5 + \cdots \tag{93b}$$

where,

$$g_1 = 2B, \qquad g_2 = \frac{-18B^3Q_{40}}{\theta^2}, \qquad Q_{40} = \left(1 + \frac{\pi\theta}{8\alpha B^2}\right) \tag{93c}$$

$$g_3 = \frac{2B^5\Omega_{38}Q_{41}}{\theta^2}, \tag{93d}$$

$$Q_{41} = \left[1 + \left(\frac{\theta^2}{25\Omega_{38}B^5}\right)\left\{\left(\frac{\pi}{\theta}\right)^2\left(\frac{B - 2025\alpha^2B^5}{128\theta^2}\right) + \left(\frac{\pi}{\theta}\right)\left(\frac{27\alpha B^3}{\theta^2}\right)\right\}\right] \tag{94a}$$

and

$$\Omega_{38} = \left[\Omega_8 - \frac{\Omega_9}{3} - \frac{\Omega_1}{5} - \frac{\theta^2\alpha^2}{2}\left\{\left\{\frac{3\Omega_1}{4\omega(w+2\theta)} + \frac{3\Omega_1}{4\omega(2\theta - w)} + 3\Omega_1\left\{\frac{1+\cos\left(\frac{\pi\omega}{\theta}\right)}{16(3\theta+w)(\theta+w)}\right\} - \frac{3\Omega_1\left\{1+\cos\left(\frac{\pi\omega}{\theta}\right)\right\}}{16(\omega-\theta)(3\theta-w)} - \frac{9\Omega_1\left\{1+\cos\left(\frac{\pi\omega}{\theta}\right)\right\}}{16(\theta+\omega)(\theta-w)}\right\}\right\}\right] \tag{94b}$$

Since the series (93b) does not converge when $U_a > U_{aD}$, where U_{aD} is the critical displacement at dynamic buckling, we, as in [18], have to reverse (93b) in the form

$$\epsilon = l_1U_a + l_2U_a^3 + l_3U_a^5 + \cdots \tag{95a}$$

By substituting in (95a) for U_a from (93b), and equating the coefficients of powers of ϵ, we have

$$l_1 = \frac{1}{g_1} = \frac{1}{2B}, \qquad l_2 = -\frac{g_2}{g_1^4} = \frac{9\alpha Q_{40}}{8B}$$

$$l_3 = \frac{3g_2^2 - g_1g_3}{g_1^7} = -\frac{g_1g_3}{g_1^7}\left(1 - \frac{3g_2^2}{g_1g_3}\right) = -\frac{g_3}{g_1^6}\left(1 - \frac{3g_2^2}{g_1g_3}\right) = -\frac{\Omega_{38}Q_{41}Q_{42}}{16B\theta^2}$$

where,

$$Q_{42} = \left(1 - \frac{243\alpha^2Q_{40}^2}{\Omega_{38}Q_{41}}\right)$$

In order to determine the dynamic buckling load λ_D, we now carry out the maximization (89) using (95a) to get

$$l_1 + 3l_2 U_{aD}^2 + 5l_3 U_{aD}^4 = 0 \tag{95b}$$

Where, $U_{aD} = U_a(\lambda_D)$.This yields

$$U_{aD}^2 = \frac{27}{5}\left(\frac{\left(-\frac{\theta^2}{\alpha}\right)}{\Omega_{38}Q_{41}Q_{42}}\right)\left[-1 - \sqrt{1 + \frac{2\Omega_{38}Q_{41}Q_{42}}{(9\alpha\theta Q_{40})^2}}\right]$$

$$\therefore \quad U_{aD} = \sqrt{\frac{27}{5}}\sqrt{\frac{\left(-\frac{\theta^2}{\alpha}\right)}{\Omega_{38}Q_{41}Q_{42}}}\left[-1 - \sqrt{1 + \frac{2\Omega_{38}Q_{41}Q_{42}}{(9\alpha\theta Q_{40})^2}}\right]^{\frac{1}{2}} \tag{95c}$$

To determine the dynamic buckling load λ_D, we evaluate (95a) at dynamic buckling condition where $U_a = U_{aD}$. Next, we multiply it by 5. By making $5l_3 U_{aD}^4$ the subject in (95b) and substituting same in (95a), (as multiplied by 5) and simplifying, we get

$$5\epsilon = \frac{2U_{aD}}{B}\left[1 + \frac{9\alpha Q_{40}U_{aD}^2}{8}\right] \tag{95d}$$

On simplifying (95d), we get

$$5m^2\bar{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_{aD}\left[1 + \frac{9\alpha Q_{40}U_{aD}^2}{8}\right] \tag{96}$$

It must be stressed that equation (96) is evaluated at $\lambda = \lambda_D$and it is implicit in λ_D.

7.2 DYNAMIC BUCKLING LOAD IN THE SHAPE OF BOTH $\boldsymbol{sin\ mx}$ and $\boldsymbol{sin\ 3mx}$

In this case, (91) becomes

$$U(x_a, t, \tau, \epsilon) = \epsilon U_m^{(1)} + \epsilon^3\left(U_m^{(3)} - U_{3m}^{(3)}\right) + \epsilon^5\left(U_m^{(5)} - U_{3m}^{(5)}\right) + \cdots \tag{97a}$$

Let the values of t and τ at maximum displacement of (97a) be t_c and τ_c respectively, where

$$t_c = t_0^{(2)} + \epsilon^2 t_2^{(2)} + \epsilon^4 t_4^{(2)} + \cdots \tag{97b}$$

$$\tau_c = \epsilon^2 t_c = \epsilon^2\left(t_0^{(2)} + \epsilon^2 t_2^{(2)} + \epsilon^4 t_4^{(2)} + \cdots\right) \tag{97c}$$

By substituting (97b, c) into (90b) and expanding as usual (similar to operations leading to (92b, c)), we get the non – vanishing terms as

$$O(\epsilon):\ U_{m,t}^{(1)} = 0 \tag{98a}$$

$$O(\epsilon^3):\ t_2^{(2)} U_{m,tt}^{(1)} - U_{3m,t}^{(3)} + U_{m,\tau}^{(1)} \ = 0 \tag{98b}$$

From (98a), we get

$$t_0^{(2)} = \frac{\pi}{\theta} \tag{98c}$$

and from (98b), we get

$$t_2^{(2)} = \frac{1}{\theta^2}(1 - B^2 \Omega_{34}) \tag{98d}$$

$$\Omega_{34} = \alpha\omega\left[\frac{4}{\omega^2} - \frac{15}{4(\omega^2-\theta^2)} + \frac{3}{(\omega^2-4\theta^2)} - \frac{1}{4(\omega^2-9\theta^2)}\right]\sin\left(\frac{\omega\pi}{\theta}\right)$$

Now, determining the maximum U_c of (97a), using (97b, c) and (98c, d), we get the non – vanishing terms as

$$\begin{aligned} U_c = {} & \epsilon U_m^{(1)} + \epsilon^3\left(t_0^{(2)} U_{m,\tau}^{(1)} + U_m^{(3)} - U_{3m}^{(3)}\right) \\ & + \epsilon^5\left[t_2^{(2)} U_{m,\tau}^{(1)} + \frac{1}{2} {t_2^{(2)}}^2 U_{m,tt}^{(1)} + {t_0^{(2)}}^2 U_{m,\tau\tau}^{(1)} + t_2^{(2)}\left(U_m^{(3)} - U_{3m}^{(3)}\right)_{,t}\right. \\ & \left. + t_0^{(2)}\left(U_m^{(3)} - U_{3m}^{(3)}\right)_{,\tau} + \left(U_m^{(5)} - U_{3m}^{(5)}\right)\right] + \cdots \end{aligned} \tag{99a}$$

where,

$$U_{3m}^{(3)}\left(t_0^{(2)}, 0\right) = \alpha B^3 \Omega_{35} \tag{99b}$$

$$\Omega_{35} = \left[4\left\{\frac{1-\cos\left(\frac{\omega\pi}{\theta}\right)}{\omega^2}\right\} - \frac{15\left\{1-\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{4(\omega^2-\theta^2\ )} + \frac{3\left\{1-\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{(\omega^2-4\theta^2\ )} - \frac{15\left\{1-\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{4(\omega^2-9\theta^2\ )}\right] \tag{99c}$$

$$U_{3m,t}^{(3)}\left(t_0^{(2)}, 0\right) = \alpha\omega B^5 \Omega_{36} \tag{100a}$$

$$\Omega_{36} = \left[\frac{4}{\omega^2} - \frac{15}{4(\omega^2-\theta^2\ )} + \frac{3}{(\omega^2-4\theta^2\ )} - \frac{15}{4(\omega^2-9\theta^2\ )}\right]\cos\left(\frac{\omega\pi}{\theta}\right)$$

$$U_{m,\tau}^{(3)}\left(t_0^{(2)}, 0\right) = \frac{27\alpha B^3}{\theta^2}, \qquad U_{3m,\tau}^{(3)}\left(t_0^{(2)}, 0\right) = \alpha B^3 \Omega_{37}$$

$$\Omega_{37} = \left[\Omega_1 \cos\left(\frac{\omega\pi}{\theta}\right) + 9\alpha\Omega_1 B^3 \sin\left(\frac{\omega\pi}{\theta}\right) - 3\left\{\frac{2}{\omega^2} + \frac{7}{4(\omega^2-\theta^2)} + \frac{2}{(\omega^2-4\theta^2)} + \frac{1}{4(\omega^2-9\theta^2)}\right\}\right] \quad (100b)$$

$$U_m^{(5)}\left(t_0^{(2)}, 0\right) = \frac{2B^5}{\theta^2}\Omega_{38} \quad (101a)$$

$$\Omega_{38} = \Omega_8 - \frac{\Omega_9}{3} - \frac{\Omega_1}{5} - \frac{\theta^2\alpha^2}{2}\left\{\left\{\frac{3\Omega_1}{4\omega(\omega+2\theta)} + \frac{3\Omega_1}{4\omega(2\theta-\omega)} + \frac{3\Omega_1\left\{1+\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{16(3\theta+\omega)(\theta+\omega)} - \frac{3\Omega_1\left\{1+\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{16(3\theta+\omega)(\theta+\omega)} - \frac{9\Omega_1\left\{1+\cos\left(\frac{\omega\pi}{\theta}\right)\right\}}{16(\theta+\omega)(\theta-\omega)}\right\}\right\} \quad (101b)$$

$$U_{3m}^{(5)}\left(t_0^{(2)}, 0\right) = B^5\Omega_{39} \quad (102a)$$

$$\Omega_{39} = \left[\Omega_{39}\cos\left(\frac{\omega\pi}{\theta}\right) + \frac{\Omega_{33}}{B^2}\sin\left(\frac{\omega\pi}{\theta}\right) - \frac{\Omega_{19}}{\omega^2-\theta^2} + \frac{\Omega_{21}}{(\omega^2-4\theta^2)} - \frac{\Omega_{23}}{4(\omega^2-9\theta^2)} + \frac{\Omega_{25}}{4(\omega^2-16\theta^2)} - \frac{\Omega_{26}}{4(\omega^2-25\theta^2)} + \frac{\Omega_{27}\cos\left(\frac{\omega\pi}{\theta}\right)}{\theta(2\omega+\theta)} + \frac{\Omega_{28}\cos\left(\frac{\omega\pi}{\theta}\right)}{\theta(2\omega-\theta)} - \frac{\Omega_{29}\cos\left(\frac{\omega\pi}{\theta}\right)}{4\theta(\theta+\omega)} + \frac{\Omega_{30}\cos\left(\frac{\omega\pi}{\theta}\right)}{4\theta(\omega-\theta)}\right] \quad (102b)$$

The maximum U_c of (99a) is now determined using (98c, d) and (97b, c), to get

$$U_c = 2B\epsilon + \epsilon^3\left[-\frac{B\pi}{\theta} - \alpha B^3\left(\frac{18}{\theta^2} + \Omega_{35}\right)\right] + \epsilon^5\left[-B\left(t_2^{(2)} + \frac{\theta^2 {t_2^{(2)}}^2}{2}\right) + \frac{{t_0^{(2)}}^2}{2}(B - 2025\alpha^2B^5) - t_2^{(2)}\omega\alpha B^5\Omega_{36} + B^3 t_0^{(2)}\alpha\left(\frac{27}{\theta^2} - \Omega_{37}\right) + \frac{2B^5\Omega_{38}}{\theta^2} + B^5\Omega_3\right] \quad (103a)$$

Further simplification of (103a) gives

$$U_c = \epsilon q_1^{(2)} + \epsilon^3 q_2^{(2)} + \epsilon^5 q_3^{(2)} \quad (103b)$$

$$q_1^{(2)} = 2B, \quad q_2^{(2)} = \frac{-18\alpha B^3}{\theta^2}Q_{56} \quad (103c)$$

$$Q_{56} = \left[1 + \frac{\theta^2}{18\alpha B^3}\left(\frac{B\pi}{\theta\alpha} + \alpha\Omega_{35}\right)\right]$$

$$q_3^{(2)} = \frac{2B^5\Omega_{38}Q_{57}}{\theta^2} \tag{103d}$$

$$Q_{57} = \left[1 + \frac{\theta^2}{2B^5\Omega_{38}}\left\{B\Omega_{39} - B\left(t_2^{(2)} + \frac{1}{2}{t_2^{(2)}}^2\theta\right) + \frac{{t_0^{(2)}}^2}{2}\left(B - 2025\alpha^2B^5\right) - t_2^{(2)}\alpha\omega B^5\Omega_{36} + t_0^{(2)}\left(\frac{27\alpha B^3}{\theta^2} - \alpha B^5\Omega_{37}\right)\right\}\right] \tag{103e}$$

We now reverse the series (103b) by letting

$$\epsilon = h_1U_c + h_2U_c^3 + h_3U_c^5 \tag{104}$$

where,

$$h_1 = \frac{1}{q_1^{(2)}} = \frac{1}{2B}, \qquad h_2 = \frac{q_2^{(2)}}{{q_1^{(2)}}^4} = \frac{9\alpha Q_{56}}{8B\theta^2}$$

$$h_3 = \frac{{q_2^{(2)}}^2 - q_1^{(2)}q_3^{(2)}}{{q_1^{(2)}}^7} = -\frac{q_2^{(2)}}{{q_1^{(2)}}^6}\left(1 - \frac{3q_2^{(2)}}{q_1^{(2)}q_3^{(2)}}\right) = \frac{-\Omega_{38}9\alpha Q_{57}Q_{58}}{32B\theta^2},$$

$$Q_{58} = \left[1 - \frac{243\alpha^2Q_{56}^2}{\theta^2\Omega_{38}Q_{57}}\right]$$

The maximization $\frac{d\lambda}{dU_c} = 0$ to obtain the dynamic buckling load λ_D yields, through (104),

$$5h_3U_{cD}^4 + 3h_2U_{cD}^2 + h_1 = 0 \tag{105a}$$

where, $U_{cD} = U_c(\lambda_D)$ and is the value of the displacement at dynamic buckling stage. This yields

$$U_{cD}^2 = \frac{-3h_2 \pm \sqrt{9h_2^2 - 20h_1h_3}}{10h_3}$$

$$= \left(\frac{3h_2}{10h_3}\right)\left[-1 \pm \sqrt{1 - \frac{20h_1h_3}{9h_2^2}}\right] \tag{105b}$$

$$= \frac{54}{5}\left(\frac{Q_{56}(-\alpha)}{\Omega_{38}Q_{57}Q_{58}}\right)\left[-1 \pm \sqrt{1 + \frac{20\theta^2\Omega_{38}Q_{57}Q_{58}}{729(\alpha Q_{56})^2}}\right]$$

Therefore, we have

$$U_{cD} = \left(\frac{54}{5}\right)^{\frac{1}{2}}\left(\frac{Q_{56}(-\alpha)}{\Omega_{38}Q_{57}Q_{58}}\right)^{\frac{1}{2}}\left[-1 \pm \left\{1 + \frac{20\theta^2\Omega_{38}Q_{57}Q_{58}}{729(\alpha Q_{56})^2}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad (105c)$$

The dynamic load λ_D in this case is obtained by first multiplying (104) by 5 to get

$$5\epsilon = U_c[5h_1 + 5h_2U_c^2 + 5h_3U_c^4] \qquad (106a)$$

From (105a) we make $5h_3U_c^4$ the subject and substitute same in (106a) and simplify to get

$$5m^2\bar{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_c\left[1 + \frac{9\alpha Q_{56}U_c^2}{8\theta^2}\right] \qquad (106b)$$

Equation (106b) gives an implicit equation for determining the dynamic buckling load λ_D in the case of buckling modes that are in the shapes of sinmx and sin3mx.

8. DISCUSSION OF RESULTS

The results (41), (48b), (96) and (106b) are all implicit in the corresponding load parameters and are valid provided the parameters of asymptotic expansions are really small compared to unity. Using equations (41) and (96) on one hand, and (48b) and (106b) on the other hand, we can easily determine the mathematical relationship between the dynamic buckling load λ_D and the corresponding static buckling load λ_S. These are respectively given as

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_{aD}}{w_a}\left[\frac{1 + \dfrac{9\alpha Q_{40}U_{aD}^2}{8}}{1 - \dfrac{3\alpha Q_{51}w_a^2}{8\theta^2}}\right] \qquad (107)$$

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_c}{w_{ac}}\left[\frac{1 + \dfrac{9\alpha Q_{56}U_c^2}{8}}{1 + \dfrac{3\alpha Q_{51}w_{ac}^2}{8\theta^2}}\right] \qquad (108)$$

9. MAIN RESULTS AND THEIR SIGNIFICANCE

In this analysis, we have been able to determine the static buckling load of the structure for the case in which the buckling mode is strictly in the case of imperfection. The result is

$$5\epsilon = \frac{4w_a(m^4 - 2m^2\lambda_S + 1)}{m^2\bar{a}_m\lambda_S}\left[1 - \frac{3\alpha w_a^2(\lambda_S)}{8\theta^2}\right] \tag{109}$$

We have also obtained the static buckling load for the case where the buckling mode is partly in the shape of imperfection and partly in the shape of $sin3mx$. The result is

$$5m^2\bar{a}_m\epsilon\lambda_S = 4w_{ac}(m^4 - 2m^2\lambda_S + 1)\left[1 + \frac{3\alpha w_{ac}^2 Q_{51}}{8\theta^2}\right] \tag{110}$$

In the same way, we have obtained the dynamic buckling load of the structure for the case in which the buckling mode is strictly in the case of imperfection. The result is

$$5m^2\bar{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_{aD}\left[1 + \frac{9\alpha Q_{40}U_{aD}^2}{8}\right] \tag{111}$$

Finally, the dynamic buckling load of the structure for the case in which the buckling mode is partly in the shape of imperfection and partly in the shape of $sin3mx$ is

$$5m^2\bar{a}_m\epsilon\lambda_D = (m^4 - 2m^2\lambda_D + 1)U_c\left[1 + \frac{9\alpha Q_{56}U_c^2}{8\theta^2}\right] \tag{112}$$

Using equations (109) and (111), we relate the dynamic buckling load to the static buckling load in the case where the buckling mode is strictly in the case of imperfection, as

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_{aD}}{w_a}\left[\frac{1 + \frac{9\alpha Q_{40}U_{aD}^2}{8}}{1 - \frac{3\alpha Q_{51}w_a^2}{8\theta^2}}\right] \tag{113}$$

In the same way, using equations (110) and (112), we relate the dynamic buckling load to the static buckling load in the case where the buckling mode is partly in the shape of imperfection and partly in the shape of $sin3mx$. The result is

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{1}{4}\left(\frac{m^4 - 2m^2\lambda_D + 1}{m^4 - 2m^2\lambda_S + 1}\right)\frac{U_c}{w_{ac}}\left[\frac{1 + \frac{9\alpha Q_{56}U_c^2}{8}}{1 + \frac{3\alpha Q_{51}w_{ac}^2}{8\theta^2}}\right] \tag{114}$$

The significance of $(113) and\ (114)$ is that, given any of λ_D or λ_S, we can find the other value without the labour of repeating the same asymptotic and perturbation procedures for the same imperfection parameter.

10. NUMERICAL RESULTS AND GRAPHICAL PLOTS

Numerical values and graphical plots of the results were obtained using Q-Basic codes and the results are hereby presented in Table1, Table2, Table3, Figure1, Figure2 and Figure3 below.

Table 1: Relationship between Imperfection Parameter $\bar{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1\ and\ \bar{a}_1 = 0.01$, using modes in the shape of imperfection, as in eqn (41).

Imperfection Parameter $\bar{a}_1\epsilon$	**Static Buckling Load λ_S**
0.01	0.593423
0.02	0.593421
0.03	0.59342
0.04	0.593419
0.05	0.593418
0.06	0.593417
0.07	0.593416
0.08	0.593415
0.09	0.593414
0.1	0.593413

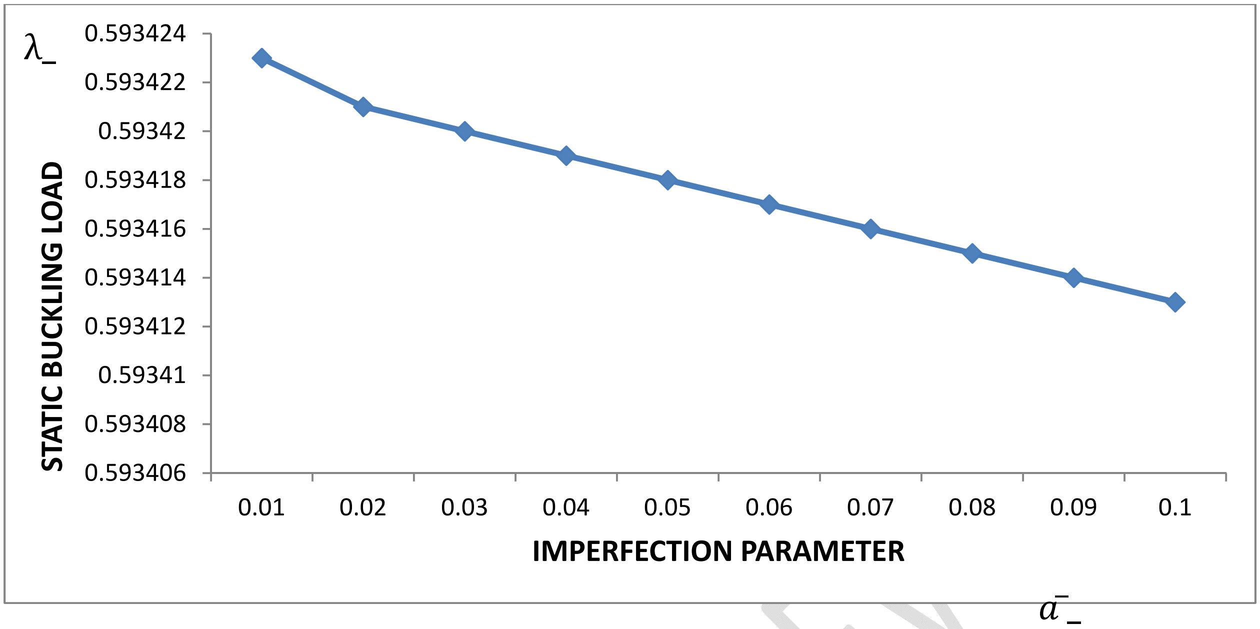

Figure 1: Graphical plot showing the relationship between Imperfection Parameter $\bar{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1$ and $\bar{a}_1 = 0.01$, using modes in the shape of imperfection.

Table 2: Relationship between Imperfection Parameter $\bar{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1$ and $\bar{a}_1 = 0.01$, in the case of modes in the shapes of combined $\sin mx$ and $\sin 3mx$, as in eqn (48b).

Imperfection Parameter $\bar{a}_1\epsilon$	Static Buckling Load λ_S
0.01	0.312078
0.02	0.281438
0.03	0.258314
0.04	0.239293
0.05	0.222829
0.06	0.208059
0.07	0.194414
0.08	0.181451
0.09	0.168739
0.1	0.155691

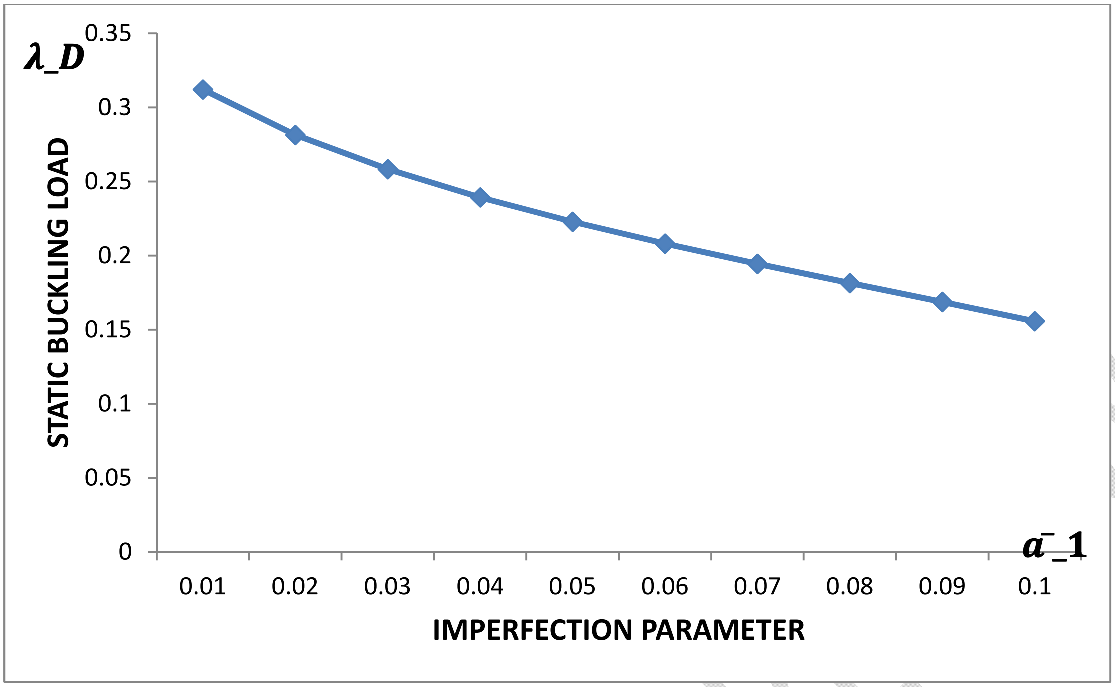

Figure 2: Relationship between Imperfection Parameter $\bar{a}_1\epsilon$ and Static Buckling Load λ_S for $m = 1, \alpha = 1, \beta = 1$ and $\bar{a}_1 = 0.01$, in the case of modes in the shapes of combined $\sin mx$ and $\sin 3mx$.

Table 3: Relationship between Imperfection Parameter $\bar{a}_1\epsilon$ and Dynamic Buckling Load λ_D for $m = 1, \alpha = 1, \beta = 1$ and $\bar{a}_1 = 0.01$, using modes in the shape of imperfection, as in eqn (96).

Imperfection Parameter $\bar{a}_1\epsilon$	Dynamic Buckling Load λ_D
0.01	0.250018
0.02	0.142869
0.03	0.100011
0.04	0.076929
0.05	0.062461
0.06	0.052657
0.07	0.045459
0.08	0.040004
0.09	0.035718
0.1	0.032261

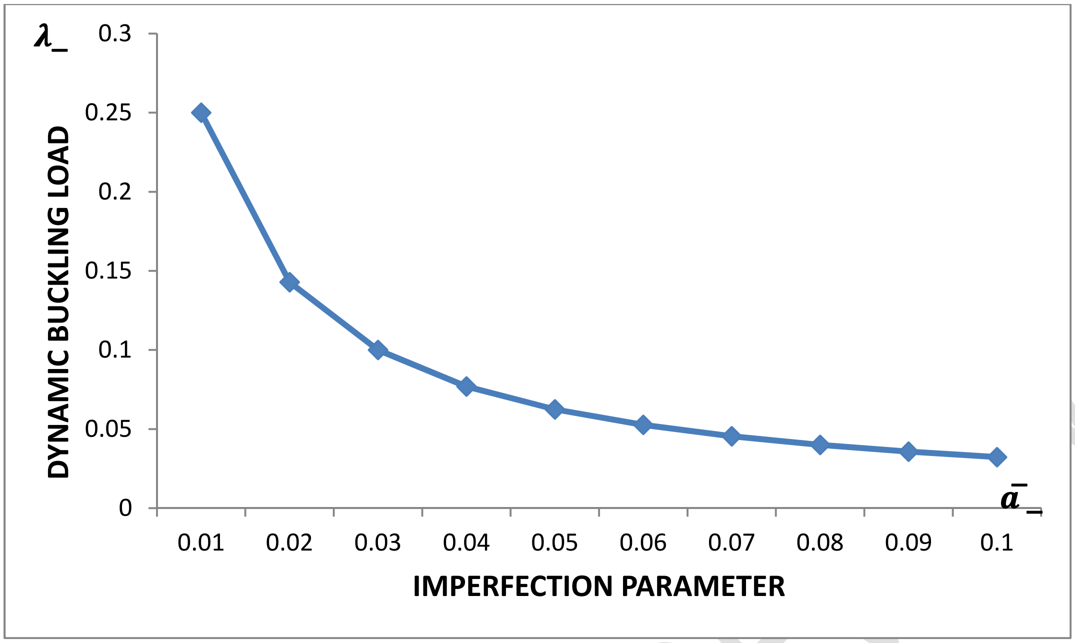

Figure 3: Graphical plot showing the relationship between Imperfection Parameter $\bar{a}_1\epsilon$ and Dynamic Buckling Load λ_D for $m = 1, \alpha = 1, \beta = 1$ and $\bar{a}_1 = 0.01$, using modes in the shape of imperfection.

11. CONCLUSION

In this paper, we have determined the static and dynamic buckling loads of a viscously damped column lying on a cubic – quintic nonlinear elastic foundation stressed by a step load (in the dynamic loading case). All results are asymptotic and implicit in the load parameters. The implicit nature of results notwithstanding, we are able to relate the dynamic buckling load λ_D to its corresponding static equivalent λ_S. This shows that if one of these buckling loads is known, then the other can be obtained easily. Specifically, the following are obtained from the graphical plots:

(a) This static and dynamic buckling loads decrease with increased imperfection,

(b) The static buckling load, for the case of buckling modes in the case of sinmx, appear to higher than the corresponding static buckling of the case of buckling modes that are in the combined shapes of sinmx and sin3mx,

(c) The dynamic load is significantly lower than the corresponding static buckling load for the same imperfection.

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