On the Insolubility of the 4-th Hilbert Problem

Abstract

The article provides two independent solutions to the 4-th Hilbert Problem. The first solution is the original (particular) solution, consisting of the fact that an infinite set of new geometries was found. These geometries have the following properties: *non-isometry* (non-preservation of lengths), *non-conformity* (non-preservation of angles), *non-aquirealirty* (non-preservation of areas) as among themselves as also in comparison with Lobachevsky's geometry, to which they arbitrarily close.

The new direction in geometry, the *Mathematics of Harmony* and its main component, the *Golden Section* are the main mathematical apparatus for this.

For the first approach, one and the same comparison algorithm was developed; the algorithm is based on the absolutely converging power Taylor series.

The second approach led to a global solution, which consists in the fact that, in the general case, the **4-th Hilbert Problem is insoluble**. This approach is that, in general, for all other possible comparative metrics, that implement non-Euclidean geometries, one and the same general algorithm is impossible, which allows to establish the coincidence of metric properties both between the comparative metrics themselves and the Lobachevsky metrics, to which they pleasingly close.

In this regard, this solution of the 4-th Hilbert Problem, proved in this article, similar to the solution method by Yuri Matiyasevich about the impossibility of the existence of one and the same general algorithm for solving the 10-th Hilbert Problem, which concerns to the formulated by Hilbert problem on solving Diophantine equations in integers [1], [2].

1. Hilbert's Problems [3] - [6]

David Hilbert is a German mathematician, who made a significant contribution to the development of many areas of mathematics. In 1900, from 6 to 12 August 1900, the II International Congress of Mathematicians was held in Paris. At this Congress, Hilbert presented his report "Mathematical Problems", in which he proposed his famous twenty-three problems of mathematics.

Currently, the 11 problems among the 23 problems have been solved. The 6 problems have been partially solved. For the two problems, mathematicians have no consensus, the 4-th and 23 problems are formulated too vaguely to judge whether they are solved or not (for more details see [3] - [6]).

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2. The 4-th Hilbert Problem

In the list of the 23 Hilbert Problems, the 4-th Problem is formulated as follows:

"*Enumerate the metrics, in which the lines are geodesic*."

The problem poses the task studying geometries, “close” (in a certain sense) to Euclidean geometry. Hilbert explains the meaning of the 4-th problem as follows:

“A more general question, arising in this case, is the following: is it possible from other fruitful points of view to construct geometries that with the same right could be considered closest to ordinary Euclidean geometry ...”

Under those closest to Euclidean geometry, Hilbert indicated *Lobachevsky geometry* (hyperbolic geometry), *Riemannian geometry* (elliptic geometry), *non-Archimedean geometry*, and *Murkowski geometry*.

3. The fifth postulate of Lobachevsky

If a straight line and a point lie on a plane, then at least two straight lines can be drawn through this point that do not intersect with the first straight line.

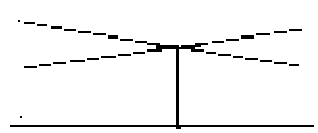

Figure 1. Illustration of the fifth postulate of Lobachevsky

Thus, the non-Euclidean geometry of Lobachevsky admits that on the same plane there can be several straight lines at once that do not intersect each other. But in the Euclidean geometry, through a point that does not belong to this straight line, we can draw one and only one straight line, that does not intersect with this straight line.

On February 11, 1826 at the Kazan University of on the meeting of the Physics and Mathematics Section Lobachevsky made a report about the discovery of new geometry. During

1829-30 he published five articles with the title “On the Principles of Geometry”, dedicated to this topic in the journal “Kazan Bulletin”, published at the Imperial Kazan University.

(see http://www.raruss.ru/russian-thought/597-lobachevsky.html).

The work "On the Principles of Geometry" was, at Lobachevsky’s request of, presented in 1832 by the Council of Kazan University to the Academy of Sciences. The Academy's conference meeting decided to give Lobachevsky's work to M.V. Ostrogradsky, the acknowledged leader of the Russian Empire mathematicians. В своем отзыве M.V. Ostrogradsky wrote the following:

“The author apparently set himself the goal of writing in such a way that he could not be understood. He achieved this goal. Everything that I understood in Lobachevsky's geometry is lower than mediocre. Lobachevsky’s book does not deserve the attention of the Academy. ”

(see: http://dfgm.math.msu.su/files/encyclopedia/Lobachevski220.pdf)

Among other colleagues, almost no one supported Lobachevsky either, misunderstanding and ignorant ridicule grew. Trying to find understanding abroad, in 1837 Lobachevsky published his article "Imaginary Geometry" in the German journal “Krell”.

Lobachevsky’s geometry was widely recognized and widely adopted only 12 years after his death, when it became clear that a scientific theory, built on the basis of a certain axiom system, is considered only fully completed when this axiom system satisfies to three conditions: *independence, consistency* and *completeness*. It is precisely these properties that Lobachevsky's geometry satisfies.

It is important to note that the Hungarian mathematician Janos Bolyai also came to similar conclusions about Lobachevsky's geometry, and the famous German mathematician Karl Friedrich Gauss came to such conclusions even earlier. Gauss generally refrained from publishing on this topic, and Bolyai’s works did not attract attention, and he soon abandoned this topic.

As a result, Nikolay Lobachevsky remained as the first and unique most consistent propagandist of new geometry.

4. Original (partial) solutions to the 4-th Hilbert problem. The dissertation of German mathematician Georg Hamel [7], defended in 1901 under Hilbert’s direction, was the first contribution to solving this problem.

As it is indicated later in the article of the American geometer Busemann [8], *“the work of Hamel, of course, did not exhaust everything that can be said about Hilbert’s fourth problem, other approaches to which were repeatedly proposed later”.*

Let's dwell in more detail on the important contribution to the solution of this problem, made by the outstanding Soviet mathematician A.V. Pogorelov [9]. The summary to Pogorelov's book [9] states the following:

"The book contains a solution to the well-known Hilbert's problem on the definition of all, up to isomorphism, realizations of the systems of axioms of classical geometries (Euclidean, Lobachevsky, elliptic), if we omit the congruence axioms, containing the concept of angle, and we supplement of these systems with the axiom of "triangle inequality": the length of any side of the triangle always does not exceed the sum of the lengths of its two other sides".

However, if Pogorelov replaces the axioms of congruence of angles with the axiom of "triangle inequality", for each of geometries: *Euclidean geometry* (Euclid), *hyperbolic geometry* (Lobachevsky), *elliptic geometry* (Riemann) , when we realize these geometries, the axiom of the congruence of angles becomes the theorem on the congruence of angles. Otherwise, Pogorelov's system of axioms cannot satisfy to three conditions: *independence, consistency* and *completeness*. Therefore, after the actual proof of this newly emerged theorem on the congruence of angles, when we realize Pogorelov's axioms, all previous systems of axioms for Euclidean, Lobachevsky and Riemann geometries are automatically restored.

This is Pogorelov's contribution to the 4-th Hilbert Problem, and, therefore, what he did is the original (particular), but not a complete solution to the 4-th Hilbert Problem.

5. Authors' original solution of the 4-th Hilbert Problem, based on the hyperbolic Fibonacci λ-functions [10-16]

Definitions and sourses. By considering possible directions for solving the 4-th Problem, as indicated above, Hilbert set the task of considering geometries, "close" (in a certain sense) to Euclidean geometry. In this regard, he recommended, above all, Lobachevsky's geometry. The purpose of this section is the original solution to the 4-th Hilbert Problem, which consists in constructing an infinite set of new geometries, "close" to Lobachevsky's geometry, but with other metric properties. The mathematical basis for such a solution is the creation by the authors of a general algorithm and authors used for this purpose the book "The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science" [11]. The Mathematics of Harmony and 4-th Hilbert Problem is the way to the Harmonic Hyperbolic and Spherical Words of Nature [12]. The «Golden» Non-Euclidean Geometry [13], the so-called "metallic" proportions of Vera Spinadel [14] and the hyperbolic Fibonacci λ-functions [15], [16] were used by authors in the study of the 4-th Hilbert Problem.

The “metallic” proportions [14], indicated by the symbol Φ_λ, are given by the formula $\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$. For all values of $\lambda \in (-\infty,+\infty)$, the function Φ_λ>0. For $\lambda \to -\infty$ $\Phi_\lambda \to 0$, for $\lambda = 0$ Φ_λ=1, for $\lambda \to +\infty$ $\Phi_\lambda \to +\infty$. For $\lambda = 1$, the formula $\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$ is reduced to the classical golden proportion $\Phi = \frac{1+\sqrt{5}}{2}$, that is, the metallic proportion Φ_λ is a generalization of the formula for the golden proportion $\Phi = \frac{1+\sqrt{5}}{2}$.

The hyperbolic Fibonacci λ-sine and λ- cosine have the following form, respectively [15]:

$$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{2}{\sqrt{4+\lambda^2}}\,\text{sh}\,[x\ln(\Phi_\lambda)],\ cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{2}{\sqrt{4+\lambda^2}}\,\text{ch}[x\ln(\Phi_\lambda)].$$

For the value of λ=1, the hyperbolic Fibonacci λ-sine and λ-cosine are reduced to the hyperbolic Fibonacci sine $sF(x)$ and cosine $cF(x)$, respectively (more in details see [11],[12]).

Lobachevsky metric and Lobachevsky classical metric. Denote by Π^+: $\{0 < u < +\infty, -\infty < v < +\infty\}$ the half-plane on the plane Π: $\{-\infty < u < +\infty, -\infty < v < +\infty\}$.

We equip the half-plane Π^+ with the metric, which, by following the terminology [17], is called the *Lobachevsky metric*. This metric has the form $(ds)^2 = R^2\left[(du)^2 + \text{sh}^2(u)(dv)^2\right]$, where ds is the length element. The coefficient $R > 0$ is called the radius of curvature of this metric, and the Gaussian curvature of this metric is $K = -\frac{1}{R^2} < 0$.

The concepts of Gaussian curvature and radius of curvature of a metric [12],[17]. Classical Lobachevsky’s metric is given on all the plane Π': $\{-\infty < u' < +\infty, -\infty < v' < +\infty\}$ and has the form $(ds')^2 = (du')^2 + \text{ch}^2\left(\frac{1}{R}u'\right)(dv')^2$, where R >0 is the radius of curvature of the classical Lobachevsky metric [18], [19] . There is shown in [18], that for the given radius R=1 the classical Lobachevsky metric is *isometric* to the Lobachevsky metric (the concept of *isometry* will be given below). In addition, according to the formulas, indicated below, it is easy to show that *Gaussian curvature* for the classical Lobachevsky metric with $R' = R$ is also equal to

$$K = -\frac{1}{R^2} < 0 .$$

Isometric displaying and isometry [20]. Let f be a *displaying* from the metric space A to the metric space A', that is, $f(A) \in A'$. If the *displaying* of f preserves the distance between the points, that is, from the conditions $\{x,y\} \in A$ and $\{x'=f(x),\ y'=f(y)\} \in A'$ it follows $\rho_\lambda(x,y) = \rho_{\lambda'}(x'=f(x), y'=f(y))$, then the *displaying* $f: A \to A'$ is called *isometric.*

The *isometric displaying* $f: A \to A'$ is called *isometry* of the metric space A to the metric space A', and the spaces A and A' are *isometric*. The isometric spaces A and A' are called *homeomorphic*, if the *displaying* $f: A \to A'$ is a one-to-one and mutually continuous displaying.

Isometric surfaces [20]. Isometric surfaces in Euclidean or Riemannian spaces are such surfaces, between which there is the *isometry* with respect to internal metrics, induced on them by the metric of the ambient space.

When we compare on the *isometry* (preservation of lengths) of two internal metrics on surfaces, the following property is important (*Gaussian theorem*) [21]:

"*For the displayings that preserve length (isometry), the Gaussian curvature at the corresponding points is the same, that is,* $K = K'$

There is explained in [21] that if the *displaying* is *isometric* (preserves the lengths of the curves), then it is also *conformal* (preserves angles) and *equiareal* (preserves areas). Conversely: if the *displaying* is *conformal* and *equiareal*, then it is *isometric*.

But then it follows from the *Gaussian theorem* that the *displaying* at the corresponding points their Gaussian curvatures K и K' are *inconsistent* ($K \neq K'$), then this *displaying* are *non-isometric* (don't preserves the lengths). Therefore, when $K \neq K'$, by virtue of the Gaussian theorem and the above remark about *isometry* [21], we get that if the *displaying* is *non-isometric* (does not save length), then, a priori, only the following situations are possible:

1) either the *displaying* is *non-conformal* (does not preserve angles) and *non-equiareal* (does not preserve areas);

2) either the *displaying* is *non-conformal* (does not preserve angles), but is *equiareal* (save areas);

3) either the *displaying* is *conformal* (preserve angles), is *non-equiareal* (does not preserve areas).

The first quadratic form. Let us give the necessary known facts of differential geometry of surfaces. Let the surface M^2 be given in parametric form:

$$M^2: \quad x = x(u, v), y = y(u, v), z = z(u, v),$$

where (u, v) belong to any area D of surface parameters.

The first quadratic form (that is, the differential of arc length) in this case looks as follows:

$$(ds)^2 = E(du)^2 + 2F du dv + G(dv)^2, \text{ where}$$

$$E = E(u, v) > 0,\ F = F(u, v),\ G = G(u, v) > 0,\ EG - F^2 > 0.$$

Let the surfaces of M^2: $\{x=x(u,v), y=y(u,v), z=z(u,v)\}$ and M'^2: $\{x'(u,v), y'=y'(u,v), z'=z'(u,v)\}$ are given in one and the same area Π: $\{0 < u < +\infty, -\infty < v < +\infty\}$ for the parameters u, v (possibly after changing the parameters).

The table below presents the necessary and sufficient condition on the metric form of a general form, induced from space, when the indicated metric properties under a one-to-one *displaying* $f: M^2 \Rightarrow M'^2$ of the surface M^2 on the surface M'^2 remain unchanged.

Table of the comparison of metric properties [21]

Displayings	Necessary and sufficient conditions imposed on the metric form
Preserving lengths (isometric)	$E = E'$, $F = F'$, $G = G'$
Preserving angles (conformal)	$E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$, $\lambda_0 > 0$
Preserving areas (equiarial)	$EG - (F)^2 = E'G' - (F')^2$

In the given Table E, F, G and E', F', G' are coefficients of the metrical forms, corresponding to the points $M(x, y, z) \in M^2$ and $M'(x', y', z') \in M'^2$. These metric forms have the following forms:

$(ds)^2 = E(du)^2 + 2F du dv + G(dv)^2$, where $E = E(u, v) > 0$, $F = F(u, v)$, $G = G(u, v) > 0$, $EG - F^2 > 0$,

$(ds')^2 = E'(du)^2 + 2F' du dv + G'(dv)^2$, where $E' = E'(u, v) > 0$, $F' = F'(u, v)$, $G' = G'(u, v) > 0$, $E'G' - F'^2 > 0$.

The construction of new metrics, "close" to the Lobachevsky metric, having other metric properties. As the basis metric we will consider the *Lobachevsky metric*:

$$(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2,$$

given in the half-plane Π^+: $\{0 < u < +\infty, -\infty < v < +\infty\}$. The coefficients of the *basic Lobachevsky metric* have the following form: E=1, F=0, G= $\mathrm{sh}^2(u)$ >0. This metric has the radius of curvature R=1 and, therefore, the Gaussian curvature $K = -\frac{1}{R^2} = -1$.

In this situation, the *Lobachevsky basic metric* is realised on the pseudo sphere M^2: $Z^2 - X^2 - Y^2 = 1$ in the three-dimensional Minkowski space (X, Y, Z), endowed with *Minkowski metric* $(dl)^2 = (dZ)^2 - (dX)^2 - (dY)^2$ for parameterization

$$X = \mathrm{sh}(u)\cos(v),\ Y = \mathrm{sh}(u)\sin(v),\ Z = \mathrm{ch}(u).$$

As metrics, which will be compared with the *basic Lobachevsky metric*, in order to study the discrepancy of metric properties with the *basic Lobachevsky metric*, we will consider the types of metrics, set for any values of the coefficients $\{\alpha \neq 0, \alpha \neq \pm 1\}$ and at the values of the parameters (u,v), on the half-plane Π:$\{0 < u < +\infty, -\infty < v < +\infty\}$, as the *basic Lobachevsky metric.* We will name them as *comparative metrics*. A view of these *comparative metrics* will be indicated below. More complex types of *comparative metrics* are given in the monograph of the authors [12], [13]).

In order to talk about the "proximity" of these *comparative metrics* to the basic Lobachevsky metric $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$, we first introduce the "distance" between them in the form $\rho = |\alpha^2 - 1| \geq 0$, where $\alpha \in \{-\infty, +\infty\}$ is the numeric coefficient for the *comparative metrics*. For $\alpha = \pm 1$ the distance between the *comparative metrics* and the basic metric is $\rho = 0$, that is, the *comparative metrics* coincide with the *basic Lobachevsky metric*. For the acceptable values $\{\alpha \neq 0, \alpha \neq \pm 1\}$ these metrics don't coincide, because $\rho > 0$. For the *comparative metrics,* given below, the above requirement of their "proximity" to the *basic Lobachevsky metric* is also fulfilled, because for $\alpha \to \pm 1$ the distance $\rho \to 0$. Important application of the metrics with the distance $\rho = |\alpha^2 - 1|$ is their interpretation in the terms of the

Mathematics of Harmony [11], when replacing $\alpha = \ln(\Phi_\lambda)$, where $\Phi_\lambda = \dfrac{\lambda + \sqrt{4+\lambda^2}}{2}$ ($\lambda \in \{-\infty, +\infty\}$) are the *metallic proportions* of Vera Spinadel [14] , from where it follows: $\lambda = 2\,\mathrm{sh}(\alpha)$. With such a replacement, the distance of the *comparative metrics* with the *basic Lobachevsky metric* is $\rho(\lambda) = \left|[\ln^2(\Phi_\lambda)] - 1\right| \geq 0$ ($\lambda \in \{-\infty, +\infty\}$). For the case $\lambda = \pm 2\,\mathrm{sh}(1) = \pm\, 2.3504$ the distance $\rho(\lambda) = 0$, that is, the *comparative metrics* coincide with the *basic Lobachevsky metric.* As it turned out, among the metallic proportions $\Phi_\lambda = \dfrac{\lambda + \sqrt{4+\lambda^2}}{2}$ for integer values $\lambda = 1,2,3,4,...$, closest in terms of distance $\rho(\lambda) = \left|[\ln(\Phi_\lambda)]^2 - 1\right|$ to the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ is not the *golden proportion* $\Phi_{\lambda=1} = \dfrac{1+\sqrt{5}}{2} \approx 1.61803$ ($\lambda = 1$), for which $\rho(\lambda) = 0.768435$, but the so-called *"silver proportion"* $\Phi_{\lambda=2} = 2.41421$ ($\lambda = 2$), for which $\rho(\lambda) = 0.223181$. This fact is of great interest for modern natural sciences.

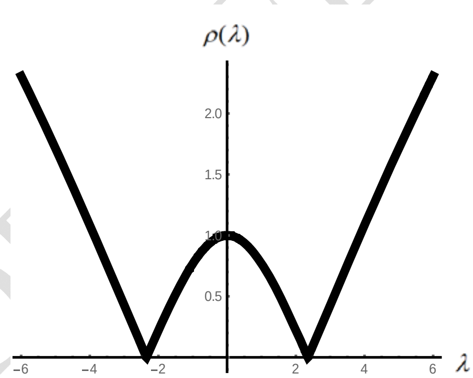

Figure 2. The distance $\rho(\lambda) = \left|[\ln(\Phi_\lambda)]^2 - 1\right|$ between the comparative metrics and the basic Lobachevsky metric $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$

The first type of comparison of metrics with $\{u > 0,\ v \in (-\infty, +\infty)\}$.

The *basic Lobachevsky metric* has the form: $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and its Gaussian curvature is equal $K = -1$). The *comparative metric* of the first type has the form: $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2, \{\alpha \neq 0, \alpha \neq \pm 1\}$ and its Gaussian curvature is equal $K' = -1$; in this case: $K' = K = -1$.

Representation of comparisons of metrics of the first type in terms of hyperbolic Fibonacci λ-functions.

Let's assume that the $\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\mathrm{sh}(\alpha), \{\alpha \neq 0, \alpha \neq \pm 1\} \Leftrightarrow \{\lambda \neq 0, \lambda \neq \pm 2.3504\}$. Then we get the metric $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ with the Gaussian curvature $K = -1$; and let's consider the next example of the metric

$$(ds')^2 = \ln^2(\Phi_\lambda)(du)^2 + \mathrm{sh}^2[u \bullet \ln(\Phi_\lambda)]\,(dv)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}\, sF_\lambda^2(u)\,(dv)^2$$

with the Gaussian curvature $K' = -1$, that is, $K' = K = -1$.

The second type of comparison of metrics with $\{u > 0,\ v \in (-\infty, +\infty)\}$

The *basic Lobachevsky metric* has the form: $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the Gaussian creature $K = -1$. The *comparative metric* of the second type has the following form:

$(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)(dv)^2$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ and the Gaussian curvature $K' = -\alpha^2 < 0$);

here we have: $K' \neq K$.

Representation of the second type of comparison of metrics in terms of hyperbolic Fibonacci λ-functions. Let's assume $\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\mathrm{sh}(\alpha), \{\alpha \neq 0, \alpha \neq \pm 1\} \Leftrightarrow \{\lambda \neq 0, \lambda \neq \pm 2.3504\}$. Then for this case we get the metric: $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ with the Gaussian curvature $K = -1$. By using the hyperbolic Fibonacci λ-functions, we can represent the example of the second type of the *comparative metric* as follows:

$$(ds')^2 = (du)^2 + \frac{1}{\ln^2(\Phi_\lambda)}\mathrm{sh}^2[u \bullet \ln(\Phi_\lambda)]\,(dv)^2 = (du)^2 + \frac{4+\lambda^2}{\ln^2(\Phi_\lambda^2)}\, sF_\lambda^2(u)\,(dv)^2$$

(the Gaussian curvature has the form: $K' = -\ln^2(\Phi_\lambda) < 0$ and consequently $K' \neq K$.

The *basic Lobachevsly metric* has the form: $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)$ and the geodesic curvature $K = -1$. Then for the condition $\Phi_{\lambda_0} = e$, were $e \approx 2.71828$, we get: $K'(\lambda_0) = K = -1$.

For this case we have: $\lambda_0 = \pm(e - \frac{1}{e}) = \pm 2.3504$.

The question of constructing other geometries with negative Gaussian curvatures, closest to the Lobachevsky geometry, but having different metric properties in comparison with it (*non-isometric, non-conformal, non-inequal*), is fundamental. Such geometries, closest to Lobachevsky's geometry, are also the closest geometries (in Hilbert sense) and to Euclidean geometry.

The Gaussian curvature of comparative metrics of the first kind.

Let the *comparative metric* of the first kind be given:

$(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$, where $\{\alpha \neq 0, \alpha \neq \pm 1, u > 0, -\infty < v < +\infty\}$.

The coefficients of this metric are as follows:

$E' = \alpha^2 > 0$, $F' = 0$, $G' = \mathrm{sh}^2(\alpha u) > 0$.

The Gaussian curvature for this case is equal: $K' = -1 < 0$. But then the radius of curvature R' of the first *comparative metric* $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$ is equal $R' = \frac{1}{\sqrt{-K'}} = 1$.

The first *comparative metric* is realized for parameterization:

$X' = \mathrm{sh}(\alpha u)\cos(v)$, $Y' = \mathrm{sh}(\alpha u)\sin(v')$, $Z' = \mathrm{ch}(\alpha u)$

on the pseudo-sphere $M'^2 : Z'^2 - X'^2 - Y'^2 = 1$, $Z' \geq 1$ in three-dimensional Minkowski space (X, Y, Z), endowed with the Minkowski metric $(dl)^2 = (dZ)^2 - (dX)^2 - (dY)^2$.

On the pseudo-sphere $M'^2 : Z'^2 - X'^2 - Y'^2 = 1$, $Z' \geq 1$ with the above parameterization of the *comparative metric* we get the relation:

$$-[(dZ')^2-(dX')^2-(dY')^2]=(ds')^2=\alpha^2(du)^2+\ \mathrm{sh}^2(\alpha u)(dv)^2 \quad .$$

Thus, with $\rho=\left|\alpha^2-1\right|>0$ the *comparative metric* of the first type

$$(ds')^2=\alpha^2(du)^2+\ \mathrm{sh}^2(\alpha u)(dv)^2,\ \{\alpha\neq 0, \alpha\neq\pm 1, u>0, -\infty<v<+\infty\}$$

has the same geodesic curvature $K'=-1$, as the geodesic curvature $K=-1$ of the *basic Lobachevsky metric* $(ds)^2=(du)^2+\mathrm{sh}^2(u)(dv)^2$. The carrier of these two metrics turned out to be the same pseudo-sphere: $Z^2-X^2-Y^2=1$, $Z\geq 1$

Gaussian curvature of comparative metrics of the second kind.

Let the *comparative metric* of the second type be given: $(ds')^2=(du)^2+\dfrac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)(dv)^2$,

where $\{\alpha\neq 0, \alpha\neq\pm 1, u>0, -\infty<v<+\infty\}$.

The coefficients of this metric are the following:

$$E'=1>0,\ F'=0,\ G'=\frac{1}{\alpha^2}\mathrm{sh}^2(\alpha\, u)>0.$$

The Gaussian curvature for this case is equal: $K'=-\alpha^2<0$. It follows from here that the radius of curvature R' of the second *comparative metric* of the second type is equal:

$$R'=\frac{1}{\sqrt{-K'}}=\frac{1}{\sqrt{\alpha^2}}=\frac{1}{|\alpha|}>0\Rightarrow R'^2=\frac{1}{\alpha^2}>0.$$

Because $\{\alpha\neq 0, \alpha\neq\pm 1\}$, then from the equalities $K'=-\alpha^2$, $R'=\dfrac{1}{|\alpha|}$, $R'^2=\dfrac{1}{\alpha^2}$ we get the following: $0>K'\neq -1, 0<R'\neq 1, 0<R'^2\neq 1$.

The second *comparative metric* is realized under parameterization

$$X'=\frac{1}{\alpha}\mathrm{sh}(\alpha\, u)\cos(v),\ Y'=\frac{1}{\alpha}\ \mathrm{sh}(\alpha\, u)\sin(v'),\ Z'=\frac{1}{\alpha}\mathrm{ch}(\alpha\, u)$$

on the pseudo-sphere $M'^2: Z'^2-X'^2-Y'^2=\dfrac{1}{\alpha^2}\neq 1$, $Z'\geq\dfrac{1}{|\alpha|}\neq 1$ in the three-dimensional Minkowski space (X, Y, Z), endowed with the Minkowski metric $(dl)^2=(dZ)^2-(dX)^2-(dY)^2$.

On the pseudo-sphere $M'^2 : Z'^2 - X'^2 - Y'^2 = \frac{1}{\alpha^2}$, $Z' \ge \frac{1}{|\alpha|}$ with the above parameterization of the *comparative metric*, we obtain the relationship:

$$-[(dZ')^2 - (dX')^2 - (dY')^2] = (ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\mathrm{sh}^2(\alpha u)(dv)^2 .$$

Thus, for $\rho = |\alpha^2 - 1| > 0$ the *comparative metric* of the second type has the following form: $(du)^2 + \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)(dv)^2$ $\{\alpha \ne 0, \alpha \ne \pm 1, u>0, -\infty < v < +\infty\}$ and has another geodesic curvature $K' = -\alpha^2$, than the geodesic curvature $K = -1$ of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$. Two different pseudo-spheres: $Z^2 - X^2 - Y^2 = 1, Z \ge 1$ (for the *basic Lobachevsky metric*) and $Z'^2 - X'^2 - Y'^2 = \frac{1}{\alpha^2}, Z' \ge \frac{1}{\alpha^2}$ (for the *comparative metric* of the second type) proved to be the carrier of these two metrics.

Comparison of metric properties for the metrics of the first type with $\{u>0, v \in (-\infty, +\infty)\}$.

Let us show that with $\rho = |\alpha^2 - 1| > 0$, $\{\alpha \ne 0, \alpha \ne \pm 1\}$ the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the *comparative metric* of the first type $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$ have opposite metric properties.

Non-isometry with $\rho = |\alpha^2 - 1| > 0$, $\{\alpha \ne 0, \alpha \ne \pm 1\}$ for the metric of the first type.

According to the metric table, in order that the displaying $f : M^2 \mapsto M'^2$ would be *isometric* (preserved lengths), it is necessary and sufficient that the coefficients of metric forms coincide for parameterization of one and the same area of the plane of the parameters of these surfaces [21]. In our situation when comparing the metric forms $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ (the *Lobachevsky metric*) and $(ds')^2 = [\alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2]$ (the *comparative metric*) at the parameterization of the surfaces M^2 and M'^2 in one and the same area $\{0 < u < +\infty, -\infty < v < +\infty\}$ of the plane of parameters, the following equalities $E = E'$, $F = F'$, $G = G'$ were performed. Here the coefficients of the metric forms have the following forms: $E = 1$, $F = 0$, $G = \mathrm{sh}^2(u) > 0$ and $E' = \alpha^2$, $F' = 0$, $G' = \mathrm{sh}^2(\alpha u) > 0$ with additional requirements $\{\alpha \ne 0, \alpha \ne \pm 1\}$.

According to the table of comparison of the metric properties [21], here and in the future, in order to establish *isometry*, *conformance* and *equiarity*, we can directly use the comparison of the coefficients of the corresponding metrics on surfaces. Let us apply a general algorithm, consisting in the use of expansion in absolutely convergent Taylor series.

Non-isometry. Suppose there is *isometry*. Then we get equalities:

$E=E'\Rightarrow 1=\alpha^2$, 0=0, $G=G'\Rightarrow$ $\text{sh}^2(u)=\text{sh}^2(\alpha u)$. Then we get equalities:

E=E' $\Rightarrow 1=\alpha^2\Rightarrow a=\pm 1$ what contradicts to the condition $\rho>0,\{\alpha\neq 0,\alpha\neq\pm 1\}$. Therefore, in this situation, we get the inequality: $E\neq E'$, that is, it is *no isometry*.

We also show that under the condition $\{\alpha\neq 0,\alpha\neq\pm 1,\ u>0\}$ we get also that $G=\text{sh}^2(u)\neq G'=\text{sh}^2(\alpha u)$. Let's suppose the contrary, that is, that the following equality exists::

$\text{sh}^2(u)=\text{sh}^2(\alpha u)$, where $\{\alpha\neq 0,\alpha\neq\pm 1,\ u>0\}$. Then we get: $\text{sh}^2(u)-\text{sh}^2(\alpha u)=0$. Let's decompose the function $P_1=\text{sh}^2(u)-\text{sh}^2(\alpha u)$ in a Taylor series on the variable u with the center of decomposition $u_0=0$. Then, we get:

$$P_1=(1-\alpha^2)u^2+(\frac{1}{3}(1-\alpha^4)u^4+\frac{2}{45}(1-\alpha^6)u^6+\frac{1}{315}(1-\alpha^8)u^8+...=0.$$

Because $\{\alpha\neq 0,\alpha\neq\pm 1, u>0\}$ and $P_1=0$, then we can divide this series by $(1-\alpha^2)u^2$. Then after all the cuts we get:

$$P_2=\frac{P_1}{(1-a^2)u^2}=1+\frac{1}{3}(1+\alpha^2)u^2+\frac{2}{45}(1+\alpha^2+\alpha^4)u^4+\frac{1}{315}(1+\alpha^2+\alpha^4+\alpha^6)u^6+...=0$$

All members of this series are positive and, moreover, $P_2>1$ what is contrary to $P_2=0$. But then under the conditions $\{\alpha\neq 0,\alpha\neq\pm 1, u>0\}$ not *isometry* (save lengths) between the *basic Lobachevsky metric* $(ds)^2=(du)^2+\text{sh}^2(u)(dv)^2$ and the *comparative metric of the first type* $(ds')^2=\alpha^2(du)^2+\text{sh}^2(\alpha u)(dv)$.

Non-conformity with $\rho=\left|\alpha^2-1\right|>0$, $\{\alpha\neq 0,\alpha\neq\pm 1\}$ for the metric of the first type.

The conformal displaying preserves the angles between curves at their intersection points (preservation of angles). Let's show that in the case of $\{\alpha\neq 0,\alpha\neq\pm 1\}$ for $\rho=\left|\alpha\ \ -1\right|>0$ there is no conformity. Suppose the contrary, that is, that there is a conformal displaying $f: M^2 \mapsto M'^2$ Then under the above conditions $\{\alpha \neq 0, \alpha \neq \pm 1\}, \{u>0\}$ there must be such a function

$\lambda_0 = \lambda_0\ (u,v) > 0$, so that $E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$. Because $E=1$, $F=0$, $G= \mathrm{sh}^2(u) > 0$ and $E' = \alpha^2$, $F'=0$, $G' = \mathrm{sh}^2(\alpha u) > 0$, that from the conditions $E = \lambda_0 E'$, $F = \lambda_0 F'$, $G = \lambda_0 G'$ we get the following equalities:

$$1 = \lambda_0 \alpha^2,\ \ \mathrm{sh}^2(u) = \lambda_0 \mathrm{sh}^2(\alpha u) \Rightarrow \lambda_0 = \frac{1}{\alpha^2},\ \mathrm{sh}^2(u) = \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u) \Rightarrow \alpha^2\ \mathrm{sh}^2(u) = \mathrm{sh}^2(\alpha u) \Rightarrow$$

$$P_1 = \alpha^2 \mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u) = 0.$$

Let's decompose the function $P_1 = \alpha^2 \mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u)$ in the Taylor series on the variable u with the center of the decomposition $u_0 = 0$. Then, we get:

$$P_1 = \frac{1}{3}(\alpha^2 - \alpha^4)u^4 + \frac{2}{45}(\alpha^2 - \alpha^6)u^6 + \frac{1}{315}(\alpha^2 - \alpha^8)u^8 + \frac{2}{14175}(\alpha^2 - \alpha^{10})u^{10} + \ldots = 0$$

Because $\{\alpha \neq 0, \alpha \neq \pm 1, u>0\}$, then $(\alpha^2 - \alpha^4)u^4 = \alpha^2(1-\alpha^2)u^4 \neq 0$. Then, the function

$P_2 = \dfrac{3}{(\alpha^2 - \alpha^4)u^4}\ P_1$ is decomposed into the Taylor series as follows:

$$P_2 = 1 + \frac{2}{15}(1 + a^2)u^2 + \frac{1}{105}(1 + \alpha^2 + \alpha^4)u^4 + \frac{2}{4725}(1 + \alpha^2 + \alpha^4 + \alpha^6)u^6 + \ldots = 0.$$

Since each member of this series is positive and, moreover, $P_2 > 1$, then we get a contradiction of the form: $0 = P_2 > 1$ what is impossible. Thus, *non-conformity* with $\rho > 0$ for the condition $\{\alpha \neq 0, \alpha \neq 1\}$ has been proved.

Aquirealirty for $\rho = |\alpha\ -1|$ >0, $\{\alpha \neq 0, \alpha \neq 1\}$ for the metrics of the first type. Aquireal displaying preserves the area of geometric figures. Let's show that in the case of $\{\alpha \neq 0, \alpha \neq 1\}$ for $\rho = |\alpha^2 - 1|$ there is *no aquirealirty* of metrics. Suppose the contrary, that is, that there is an aquireal displaying $f: M^2 \mapsto M'^2$. Then under the above conditions $\{\alpha \neq 0, \alpha \neq \pm 1\}, \{u>0\}$ the following equality will be performed: $E\,G-(F)^2 = E'G'-(F')^2$. The coefficients of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ have the following form:

E=1, F=0, G= $\mathrm{sh}^2(u) > 0$, but the coefficients of the *comparative metric*

$(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$ have the form: $E' = \alpha^2 > 0$, $F'=0$, $G' = \mathrm{sh}^2(\alpha u) > 0$.

Therefore, in this situation, the equality $E\,G-(F)^2 = E'G'-(F')^2$ has the form:

$\mathrm{sh}^2(u) - \alpha^2 \mathrm{sh}^2(\alpha u) = 0$. Let's decompose the function $P_1 = \mathrm{sh}^2(u) - \alpha^2 \mathrm{sh}^2(\alpha u)$ into Taylor series on the variable u with the center of decomposition $u_0 = 0$. Then we get:

$$P_1 = (1-\alpha^4)\,u^2 + \frac{1}{3}(1-\alpha^6)u^4 + \frac{2}{45}(1-\alpha^8)u^6 + \frac{1}{315}(1-\alpha^{10})u^8 + \ldots = 0.$$

Because $\{\alpha \neq 0, \alpha \neq \pm 1, u > 0\}$, then $(1-\alpha^4)u^2 \neq 0$. Then the function $P_2 = \frac{1}{(1-\alpha^4)u^2} P_1$ is decomposed into Taylor series as follows:

$$P_2 = 1 + \frac{1}{3}\left(\frac{1+\alpha^2+\alpha^4}{1+\alpha^2}\right)u^2 + \frac{2}{45}(1+\alpha^4)u^2 + \frac{1}{315}\left(\frac{1+\alpha^2+\alpha^4+\alpha^6+\alpha^8}{1+\alpha^2}\right)u^6 + \ldots = 0.$$

Because every member of this series is positive and, moreover, $P_2 > 1$, then we get the following contradiction: $0 = P_2 > 1$ what is impossible. Thus, the *non-aquirealirty* with $\rho > 0$ for the condition $\{\alpha \neq 0, \alpha \neq \pm 1\}$ has been proved.

So, when $\{\alpha \neq 0, \alpha \neq \pm 1\}$, $\{0 < u < +\infty, -\infty < v < +\infty\}$ for the case $\rho = \left|\alpha^2 - 1\right| > 0$, for the comparison of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ to the metric $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$, there is no *isometry* (the lengths are not preserved), there is no *conformity* (the angles are not preserved) and there is *non-aquirealirty* (the areas are not preserved).

The peculiarity of this result consists in the fact that in this case the Gaussian curvatures of the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the *comparative metrics* of the type $(ds')^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$ for the conditions $\{\alpha \neq 0, \alpha \neq \pm 1\}$,

$\{0 < u < +\infty, -\infty < v < +\infty\}$, are equal, that is, $K = K' = -1$.

Comparison of metric properties for metrics of the second type with $\{u>0,\ v \in (-\infty, +\infty)\}$

Let's show that with $\rho = |\alpha^2 - 1| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ the *basic Lobachevsky metric* $(ds^2) = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the *comparative metric* $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)(dv)^2$ have opposite metric properties

Non-isometry at $\rho = |\alpha^2 - 1| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ for the metrics of the second type.

According to the metric table, in order the *displaying* $f: M^2 \mapsto M'^2$ was *isometric,* it is necessary and sufficient, so that the coefficients of the metric forms coincide, when the parameterization in the same area of the plane of parameters of these surfaces was realized [21]. In our situation when comparing metrics $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ (the *basic Lobachevsky metric*) and $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)(dv)^2$ (the *comparative metric*) at parameterization of the surfaces M^2 and M'^2 in one and the same area $\{u > 0, -\infty < v < +\infty\}$ of parameters plane; this means that the following equalities are performed: $E = E'$, $F = F'$, $G = G'$. Here the coefficients of metric forms are the following:

$$E=1,\ F=0,\ G=\mathrm{sh}^2(u)>0 \text{ и } E'=1,\ F'=0,\ G'=\frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)>0$$

with additional requirements $\{\alpha \neq 0, \alpha \neq \pm 1\}$. Note that in this situation, the Gaussian curvature of the *basic Lobachevsky metric* is equal of $K = -1$, but the *comparative metric* is equal to $K' = -\alpha^2 < 0$. Because $\{\alpha \neq 0, \alpha \neq \pm 1\}$, then $K \neq K'$. Recall the Gaussian theorem [21] (a necessary condition for the constancy of the Gaussian curvature):

"*If isometry with the displaying (the lengths are preserved), then the Gaussian curvature at the corresponding points is the same".*

However, these conditions are necessary, but not sufficient, that is, if the Gaussian curvature at the corresponding points is the same, then the *displayings*, a priori, can be *non-isometric*. Namely, for the metrics of the first type, when $\{\alpha \neq 0, \alpha \neq \pm 1\}$, the Gaussian curvatures were the same ($K = K' = -1$), but there was *no isometry* (the preservation of lengths) and, moreover, there was also *no conformal* (the preservation of angles) and *aquireal* (the preservation of areas). If at the corresponding points the Gaussian curvatures do not coincide, then there is certainly *no isometry*, because, for example, in this case $K=-1,\ K'=-\alpha^2, K\neq K'$, where $\{\alpha\neq 0,\alpha\neq\pm1\}$. Therefore, in this situation, from the Gaussian theorem on *isometry* for the conditions E=1, F=0, G= $\mathrm{sh}^2(u)>0$ и E'=1, F'=0, $G'=\dfrac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)$ we get, that because

$E=E'$=1, $F=F'$=0, but *no isometry*, then we have: $G=\mathrm{sh}^2(u)\neq G'=\dfrac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)$.

Non-conformity with $\rho=\left|\alpha^2-1\right|>0$, $\{\alpha\neq 0,\alpha\neq\pm1\}$ for the metrics of the second type. Suppose the contrary, that is, that there is the *conformal displaying* $f: M^2\mapsto M'^2$.

Then under the above conditions $\{\alpha\neq 0,\ \alpha\neq\pm1, u>0, -\infty<v<+\infty\}$ when we compare the *basic Lobachevsky metric* $(ds)^2=(du)^2+\mathrm{sh}^2(u)(dv)^2$ with any fixed *comparative metric* $(ds')^2=(du)^2+\dfrac{1}{\alpha^2}\,\mathrm{sh}^2(\alpha u)(dv)^2$ there must be a such function $\lambda_0=\lambda_0(u,v)>0$, so that

$E=\lambda_0 E'$, $F=\lambda_0 F'$, $G=\lambda_0 G'$.

Because E=1, F=0, G= $\mathrm{sh}^2(u)>0$ and E'=1, F'=0, $G'=\dfrac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)>0$, then from the conditions $E=\lambda_0 E'$, $F=\lambda_0 F'$, $G=\lambda_0 G'$ we get the following equalities:

$$1=\lambda_0,\ \mathrm{sh}^2(u)=\lambda_0\frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)\Rightarrow \mathrm{sh}^2(u)-\frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)=0\Rightarrow \alpha^2\,\mathrm{sh}^2(u)-\mathrm{sh}^2(\alpha u)=0.$$

Next, apply the same algorithm for the Taylor expansion of the function $P_1=\alpha^2\,\mathrm{sh}^2(u)-\mathrm{sh}^2(\alpha u)$ and divide this series by the first coefficient and then we get that this series is greater than zero, but on the other hand, this series is zero what is impossible. Therefore, in this situation there is also *no conformity*.

Non-aquirealirty with $\rho=\left|\alpha^2-1\right|>0$, $\{\alpha\neq 0,\alpha\neq\pm1\}$ for the metrics of the second type.

The aquireal displaying preserves the areas of the corresponding geometric figures. Suppose the contrary, that is, that there is an *aquireal displaying* $f: M^2\mapsto M'$. Then under the above conditions $\{\alpha\neq 0,\ \alpha\neq\pm1,\ u>0,-\infty<v<+\infty\}$, according to [21] (a table of comparison of metric properties), for *aquirealirty* it is necessary and sufficient that when comparing the corresponding coefficients of metric forms satisfy to the equality: $EG - (F)^2 = E'G' - (F')^2$. In our situation $\{\alpha \neq 0, \alpha \neq \pm 1, u > 0, -\infty < v < +\infty\}$ by assuming *aquirealirty* between the *basic Lobachevsky metric* $(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)^2$ and the *comparative metric*

$(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\mathrm{sh}^2(\alpha u)(dv)^2$ it is necessary and sufficient so that the following equality is performed: $EG - (F)^2 = E'G' - (F')^2$. Here we have: E=1, F=0, G= $\mathrm{sh}^2(u)$ >0 , E'=1, F'=0, $G' = \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)$ >0. But then we get:

$$E\,G - (F)^2 = E'G' - (F')^2 \Rightarrow \mathrm{sh}^2(u) = \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u) \Rightarrow \alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u) = 0.$$

This situation $\alpha^2\,\mathrm{sh}^2(u) - \mathrm{sh}^2(\alpha u) = 0$ met when proving *non-conformity* with $\rho = \left|\alpha^2 - 1\right| > 0$, $\{\alpha \neq 0, \alpha \neq \pm 1\}$ for the metrics of the first type. It has been shown that this situation

is impossible. Therefore, we obtain that there follows from comparisons of the metrics

$(ds)^2 = (du)^2 + \mathrm{sh}^2(u)(dv)$ and $(ds')^2 = (du)^2 + \frac{1}{\alpha^2}\,\mathrm{sh}^2(\alpha u)(dv)^2$ that the metrics are *not aquirealirty* (the areas are not preserved) for the case $\{u > 0, v \in (-\infty, +\infty)\}$.

Therefore, when comparing the second type of the *comparative metrics* to the *basic Lobachevsky metrix*, we obtain *non-isometry*, *non-conformity* and *non-aquirealirty.*

The third type of metrics comparison for the case $\{u > 0,\ v \in (-\infty, +\infty)\}$

$(ds_1)^2 = \alpha^2 (du)^2 + \mathrm{sh}^2(\alpha u)(dv)^2$ (Gaussian curvature $K_1 = -1$),

$(ds_2)^2 = \beta^2 (du)^2 + \mathrm{sh}^2(\beta u)(dv)^2$ (Gaussian curvature $K_2 = -1$)

for the conditions $\{\alpha^2 \neq \beta^2, \alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u > 0,\ v \in (-\infty, +\infty)\}$.

Representation of the third type of metrics comparison in terms of hyperbolic Fibonacci λ-functions. Let's assume that

$\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\mathrm{sh}(\alpha)$, $\beta = \ln(\Phi_\mu) \Leftrightarrow \mu = 2\,\mathrm{sh}(\beta)$ for the conditions

$\alpha^2 \neq \beta^2, \alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u > 0,\ v \in (-\infty, +\infty)\} \Leftrightarrow$

$\{\ln^2(\Phi_\lambda) \neq \ln(\Phi_\mu),\ \lambda \neq 0, \lambda \neq \pm 2.3504,\ \mu \neq 0, \mu \neq \pm 2.3504,\ u>0,\ \ v \in (-\infty,+\infty)\}$.

Then we get the two metrics, the first *comparative metric*

$$(ds_1)^2 = \ln^2(\Phi_\lambda)(du)^2 + \mathrm{sh}^2[u \bullet \ln(\Phi_\lambda)]\,(dv)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}\ sF_\lambda^2(u)\ (dv)^2$$

(Gaussian curvature $K_1 = -1$), and the second *comparative metric*

$$(ds_2)^2 = \ln^2(\Phi_\mu)(du)^2 + \mathrm{sh}^2[u \bullet \ln(\Phi_\mu)]\,(dv)^2 = \ln^2(\Phi_\mu)(du)^2 + \frac{4+\mu^2}{4}\ sF_\mu^2(u)\ (dv)^2$$

(Gaussian curvature $K_2 = -1$).

The fours type of metrics comparison for the case $\{\ u>0,\ \ v \in (-\infty,+\infty)\}$

$$(ds_1)^2 = (du)^2 + \frac{1}{\alpha^2}\mathrm{sh}^2(\alpha u)(dv)^2 \ \ \text{(Gaussian curvature } K_1 = -\alpha^2 < 0\text{)}$$

$$(ds_2)^2 = (du)^2 + \frac{1}{\beta^2}\mathrm{sh}^2(\beta u)(dv)^2 \ \ \text{(Gaussian curvature } K_2 = -\beta^2 < 0\text{)}$$

for the conditions $\{\alpha^2 \neq \beta^2, \alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u>0,\ \ v \in (-\infty,+\infty)\}$.

Representation of the fourth kind of comparison of metrics in terms of hyperbolic Fibonacci λ-functions. Let's assume that

$\alpha = \ln(\Phi_\lambda) \Leftrightarrow \lambda = 2\,\mathrm{sh}(\alpha),\ \beta = \ln(\Phi_\mu) \Leftrightarrow \mu = 2\,\mathrm{sh}(\beta)$

for the conditions $\{\alpha^2 \neq \beta^2, \alpha \neq 0, \alpha \neq \pm 1, \beta \neq 0, \beta \neq \pm 1, u>0,\ \ v \in (-\infty,+\infty)\} \Leftrightarrow$

$\{\ln^2(\Phi_\lambda) \neq \ln(\Phi_\mu), \lambda \neq 0, \lambda \neq \pm 2.3504,\ \mu \neq 0, \mu \neq \pm 2.3504,\ u>0,\ \ v \in (-\infty,+\infty)\}$

Then we get the two metrics, which are expressed through the "metallic proportions" Φ_λ and Φ_μ:

$$(ds_1)^2 = (du)^2 + \frac{1}{\ln^2(\Phi_\lambda)}\mathrm{sh}^2[u \bullet \ln(\Phi_\lambda)]\,(dv)^2 = (du)^2 + \frac{4+\lambda^2}{\ln^2(\Phi_\lambda^2)}\ sF_\lambda^2(u)\ (dv)^2$$

(Gaussian curvature $K_1 = -\ln^2(\Phi_\lambda) < 0$), and $(ds_2)^2 = (du)^2 + \frac{1}{\ln^2(\Phi_\mu)} \text{sh}^2 [u \bullet \ln(\Phi_\mu)] \, (dv)^2 = (du)^2 + \frac{4+\mu^2}{\ln^2(\Phi_\mu^2)} \, sF_\mu^2(u) \, (dv)^2$

(Gaussian curvature $K_2 = -\ln^2(\Phi_\mu) < 0$), where $K_2 \neq K_1$.

Both types of these comparisons for the *comparative metrics* themselves on the subject of *non-isometry*, *non-conformity* and *non-aquirealirty* are conducted in a similar way (with a slight modification) by using the general algorithm for decomposition into Taylor series. In these last two cases, the function, taken as the distance between the comparative metrics is $\rho(\alpha, \beta) = \left|\alpha^2 - \beta^2\right|$.

In terms of the Harmony Mathematics after the replacements $\alpha = \ln(\Phi_\lambda), \beta = \ln(\Phi_\mu)$, the distance between the *comparative metrics* looks like $\rho(\lambda, \mu) = \left|\ln^2(\Phi_\lambda) - \ln^2(\Phi_\mu)\right|$. For the case $\lambda = \pm\mu$ we get: $\rho(\lambda, \mu) = \left|\ln^2(\Phi_\lambda) - \ln^2(\Phi_\mu)\right| = 0$. Note that with the above changes of the coefficients

$\alpha = \ln(\Phi_\lambda), \beta = \ln(\Phi_\mu)$, we get: $\lambda = 2\text{sh}(\alpha)$, $\mu = 2\text{sh}(\beta)$.

Conclusions

An original solution of the 4-th Hilbert Problem is obtained; it is based on the hyperbolic Fibonacci λ-functions. The originality of this solution consists in the following:

1). This original solution is based on the *Lobachevsky metric*, whose Gaussian curvature is equal to $K = -1$; this *Lobachevsky metric* is isometric to the *classical Lobachevsky metric* with the Gaussian curvature $K = -1$.

2). Two types of infinite set of the *comparative metrics*, based on hyperbolic Fibonacci λ-functions are considered. These metrics can be arbitrarily close to the *basic Lobachevsky metric* and in the limit they coincide with the *basic Lobachevsky metric*.

3) The first type of all these *comparative metrics* has Gaussian curvature $K = -1$, the same with the basic Lobachevsky metric. However, all these comparative metrics with respect to the *basic Lobachevsky metric* are *non-isometric* (do not preserve lengths), *non-conformal* (do not preserve angles), *non-aquireal* (do not preserve areas). Moreover, these comparative metrics themselves are also *non-isometric*, *non-conformal* and *non-aquireal*.

Thus, the important conclusion of this study is proving the existence of an infinite set of new geometries, arbitrarily close to Lobachevsky's geometry and having the same with Lobachevsky's geometry negative Gaussian curvature $K=-1$.

4) The second type of all these *comparative metrics* has negative Gaussian curvatures $K(\lambda) = -\ln^2(\Phi_\lambda) < 0)$, which differ from the Gaussian curvature $K = -1$ of the *basic Lobachevsky metric*. In relation to the *basic Lobachevsky metric*, all these *comparative metrics* are *non-isometric* (do not preserve lengths), *non-conformal* (do not preserve angles), *non-aquireal* (do not preserve areas).

Moreover, these *comparative metrics* themselves are also *non-isometric*, *non-conformal* and *non-aquireal*. In process of study, the authors of this article found metrics that clarify the Gaussian theorem about the intrinsic geometry of surfaces, that for *displaying*, that preserve length (*isometry*), the Gaussian curvature remains the same.

It follows from the Gaussian theorem that if two metrics, when compared, have different Gaussian curvatures, then they are *non-isometric*. It follows from our study (for some particular situations) a revision of the Gaussian theorem, that for any pairs of metrics, presented in the second type of specific metrics, not only between the *comparative metrics* and the *basic Lobachevsky's metric*, but also between specific metrics there exist not only *non-isometry* (according to the corollary to the Gaussian theorem), but also *non-conformity* and *non-aquirealirty*.

Thus, it is proved the existence of an infinite number of new geometries (*non-isometric*, *non-conformal*, *non-aquireal* each other) with different negative Gaussian curvature; these geometries are arbitrarily close to the Lobachevsky hyperbolic geometry, they have other negative Gaussian curvature and are in comparison to Lobchevsky geometry such properties as *non-isometry, non-conformy* and *non-aquirealirty* .

6. New Challenge for Theoretical Natural Sciences: Insolvability of the 4-th Hilbert Problem

Note that in this case on the significance of the authors' solution of the 4-th Hilbert Problem, obtained by the authors, can be spoken not only as on the original solution of this problem (the first approach), but also as on the **complete solution of the 4-th Hilbert Problem** (the second approach), unlike of the original solutions of Hamel, Pogorelov and others. Namely, the authors of this article proved the existence of an infinite number of new concrete non-Euclidean geometries, arbitrarily close to Lobachevsky's geometry, but having other metric properties in comparison with Lobachevsky's geometry (*non-isometric*, *non-conformal*, *non-aquirealirty* .

The authors indicated one and the same general algorithm for comparing these metrics to find their metric properties. This comparison algorithm allows to compare the concrete metrics to the Lobachevsky metric and the comparative metrics among themselves; this algorithm is based on the decomposition of the remainders between metrics into absolutely convergent Taylor series.

If such remainders after division on the first term have constant signs, then the corresponding metric properties do not match. If in such series, after division on the first term, we get alternating variables, then by using this general algorithm it is impossible to establish directly whether the corresponding metric properties of the compared metrics are coincident or differ. To do this, we always need to search for specific ways and algorithms.

But because the set of geometries, close to Lobachevsky's geometry, is infinite, we certainly come to the conclusion, that for an infinite set of geometries, close to Lobachevsky's geometry, it is impossible to find one and the same general algorithm, which makes possible for any metrics from this infinite set to define to have or don't have other metric properties than the metric properties of Lobachevsky's geometry or to draw a similar conclusion after comparison of the metrics to each other. This set of metrics can be used for comparison of the metric properties of both metrics with the same Gaussian curvature and metrics with different Gaussian curvatures.

In particular, this set contains all metrics with different Gaussian curvatures. When comparing any two such metrics of the form $(ds)^2 = E(du)^2 + 2Fdudv + G(dv)^2$ (there are an infinite set of such metrics), if we apply a general algorithm for comparing such metrics and we get the sign-alternating Taylor power series and therefore this general comparison method does not fit.

On the other hand, when comparing such metrics, the Gauss theorem is partially (but incomplete) applicable: if two such metrics have different Gaussian curvatures, then there is *no isometry*. But it does not at all follow from this that there is no possible *conformity* and *aquirealirty*. In order to establish the presence or absence of *conformity* and *aquirealirty*, it is necessary in this situation to search for a specific method every time.

This, in any sense, is analogous to the fact that in a binary graph the set of all vertices is countable, and the set of all paths is countable (the power of the continuum).

Such an approach is in some sense similar to the proof of the insolvability of the 10-th Hilbert problem (is there a universal algorithm for solving arbitrary Diophantine equations), made by the Russian mathematician Yuri Matiyasevich in 1970. Recall that the basic idea of the proof of the insolvability of the 10-th Hilbert Problem consisted in the fact that since the set of all Diophantine equations is uncountable and then, according to the main Matiyasevich theorem, *"the same general method (algorithm) is impossible, which allows for any Diophantine equations determining, whether they have a solution in integers or not. "*

That is why, the authors recommend to the readers to pay attention to the importance of the article [22], in which the 4-th Hilbert Problem was called **the MILLENIUM PROBLEM in Geometry.**

References

[1] Matiyasevich, Y.V. *Diopentolate enumerable sets*. DAN SSSR (1970),Vol. 191, No. 2. (Russian)

[2] Matiyasevich, Y.V. *Hilbert's tenth problem*. Moscow: Science (Mathematical logic and foundations of geometry) (1993).(Russian)

[3] Aleksandrov, P.S. (General Editor). *Hilbert's Problems*. Moscow:(1969) (Russian) .

[4]. Demidov S.S. *To the history of Hilbert problems*. *Historical and mathematical research.* Moscow: Science (1966), No. 17. P. 91-122. (Russian)

[5] Demidov S.S. "*Mathematical problems*" *of Hilbert and mathematics of the* XX *century. Historical and mathematical research.* Moskow: Janus-K (2001), No. 41 (6). P. 84-99. (Russian)

[6] Bolibruh, A.A. *Hilbert Problems* (100 *years later*). Moscow: Science (Publishing house of the Moscow center for continuous mathematical education) (1999),V.2, 24 p. (Library "Mathematical education" (Russian))

[7] Hamel, G. *On the geometries in which the straight lines are the shortest*. Math. Ann. (1903), No.67, P.231-264 (German)

[8] Buseman, G. *On the Fourth Hilbert Problem.* Moscow: Advances in mathematical Sciences (1966), Vol. 21, No.1 (127), P.155-164 (Russian)

[9] Pogorelov, A.V. *The Fourth Hilbert Problem*, Moscow: Science (1974), 80 p. (Russian) (http://padaread.com/?book=37757)

[10] Aranson, S. Kh. *Again on the* 4-*th Hilbert Problem*. Moscow: Academy of Trinitarism (2009), Electronic No. 77-6567, publ.15677 (Russian) (http://www.trinitas.ru/rus/doc/0232/009a/1180-ar.pdf)

[11] Stakhov, A.P. Assisted by Scott Olsen. *The Mathematic of Harmony: from Euclid to Contemporary Mathematics and Computer Science*. Singapore: World Scientific (2009), 694 p.

[12] Stakhov,A.P, Aranson, S.Kh. *The Mathematics of Harmony and Hilbert's Fourth Problem. The Way to the Harmonic Hyperbolic and Spherical Words of Nature.* Germany: Lambert Academic Publishing (2014), 237 p.

[13] Stakhov, A.P., Aranson S.Kh. Assisted by Scott Olsen. *The «Golden» Non-Euclidean Geometry. Hilbert's Fourth Problem, «Golden» Dynamical Systems, and the Fine-Structure Constant*. Singapore: World Scientific (2016),Vol.7, 284 p. (http://www.worldscientific.com/worldscibooks/10.1142/9603)

[14] de Spinadel ,V.W. *From the Golden Mean to Chaus.* Editorial Nueva Librería, Buenos Aires, Argentina (1998).

[15] Stakhov, A.P. *Gasale formulas, a new class of the hyperbolic Fibonacci and Lucas functions, and the improved method of the "golden" cryptography.* Academy of Trinitarizm. Moscow: Electronic number 77-6567, publication 14098, 21.12.2006 (Russian) (http://www.trinitas.ru/rus/doc/0232/004a/02321063.htm)

[16] Stakhov, A.P. *On the general theory of hyperbolic functions based on the hyperbolic Fibonacci and Lucas functions and on Hilbert's Fourth Problem.* Visual Mathematics (2013), Vol.15, No.1.(http://www.mi.sanu.ac.rs/vismath/pap.htm)

[17] Dubrovin, B.A., Novikov, S.P., and Fomenko A.T. *Modern Geometry* : *Methods and Applications.* Moscow: Science (1965),760 p. (Russian)

[18] Novikov, S.P., Mishchenko, A.S., Soloviev, Yu.P., Fomenko, A.T. *Tasks for Geometry* . Moscow: Moscow University Press (1978), 164 p. (Russian)

[19] *Non-euclidean geometry* (Russian) (http://www.newreferat.com/ref-16006-2.html)

[20] Wojciechowski, M.I. *Isometric display* (p.506). *Isometric surfaces* (p.511). Moscow: Mathematical Encyclopedia (1979), Vol. 2, 1103 p. (Russian)

[21] Bronstein, I. N. and Semendyaev, K.A. *Handbook of Mathematics*. Moscow: Science (1986), 344 p. (Russian)

[22] Stakhov, A.P., Aranson S.Kh. *Hilbert's Fourth Problem as a Possible Candidate on the MILLENIUM PROBLEM in Geometry.* British Journal of Mathematics & Computer Shience (2016), 12(4), p.1-25 (http://www.journalrepository.org/media/journals/BJMCS_6/2015/Nov/Stakhov1242015BJMCS21849.pdf)