Reliability Analysis of a Commodity-Supply Multi-state System Using the Map Method

Abstract

A multi-state k-out-of-n: G system is a multi-state system whose multi-valued success is greater than or equal to a certain value j (lying between 1 (the lowest non-zero output level) and M (the highest output level)) whenever at least k_m components are in state m or above for all m such that $1 \leq m \leq j$. This paper is devoted to the analysis of a commodity-supply system that serves as a standard gold example of a non-repairable multi-state k-out-of-n: G system with independent non-identical components. We express each instance of the multi-state system output as an explicit function of the multi-valued inputs of the system. The ultimate outcome of our analysis is a Multi-Valued Karnaugh Map (MVKM), which serves as a natural, unique, and complete representation of the multi-state system. To construct this MVKM, we use "binary" entities to relate each of the instances of the output to the multi-valued inputs. These binary entities are represented *via* an eight-variable Conventional Karnaugh Map (CKM) that is adapted to a map representing four variables that are four-valued each. Despite the relatively large size of the maps used, they are still very convenient, thanks to their regular structure. No attempt was made to draw loops on the maps or to seek minimal formulas. The maps just served as handy tools for combinatorial representation and for collectively implementing the operations of ANDing, ORing, and complementation. Our symbolic analysis yields results that agree numerically with those obtained earlier.

Keywords: System reliability, k-out-of-n system, Multi-state system, Multiple-valued logic, Eight-variable Karnaugh Map, Multi-Valued Karnaugh Map.

1. Introduction

A binary *k*-out-of-*n*: G system is *uniquely* defined as a dichotomous system that is successful if and only if at least *k* out of its *n* components are successful [1-23], By contrast, a multi-state *k*-out-of-*n*: *G* system does not possess a unique definition [24-43]. The definition adopted herein is that this system is a multi-state system (MSS) whose multi-valued success is greater than or equal to a certain value *j* (lying between *1* (the lowest non-zero output level) and *M* (the highest output level)) whenever at least k_m components are in state *m* or above for all *m* such that $1 \leq m \leq j$ [34, 40-43].

In this paper, we a study a standard multi-state system, which was proposed and studied by Tian *et al*. [34], and further studied by Fadhel *et al*. [44], Mo *et al*. [40], Rushdi [41], Rushdi & Al-Amoudi [42, 43]. The system (shown in Fig. 1) is a supply system of a certain commodity (e.g., oil, water, energy, transportation traffic, or communication traffic, *etc*.) that employs four pipelines to transport the given commodity from the given *source* to three *sink* nodes called *stations*. Both the system and each pipeline have four states, which are defined as shown in Table 1. The states of the system are defined according to whether the demands of *up to* a certain station can be met. We use $S\{k\}$ $\{0 \leq k \leq 3\}$ to denote a binary indicator that the system can meet the commodity demand *up to* the station number k, *i.e.*, for all stations m $(1 \leq m \leq k)$. The states of each pipeline are defined according to which station/stations can be reached by the commodity supply *via* this pipeline. Therefore, pipeline number i is represented by a multi-valued variable X_i, which has four values or instances $X_i\{j\}, (1 \leq i \leq 4, 0 \leq j \leq 3)$. The instance $X_i\{j\}$ is a binary indicator that the commodity can reach *up to* station j through pipeline i.

We have recently reported several solutions of the aforementioned problem, and our present paper offers yet another solution of this problem. In our earlier solutions, we employed *purely-algebraic methods* of multi-valued logic, in which we handled multi-valued variables either directly [41] or through some binary encoding [42, 43], with various map versions used occasionally for verification. In this paper, however, we deliberately avoid the mathematically-demanding algebraic manipulations in [41-43] by employing the Karnaugh map [45-50] as the sole vehicle for our manipulations. There is a long history of utilization of the Karnaugh map as a probability map (or reliability map) in the binary case [51-59]. There are also some notable applications of the Karnaugh map as a multi-value map [60-61]. Our work herein combines the probability and multi-value notions by adapting the map to multi-valued reliability calculations. We modify a regular form of the binary eight-variable Karnaugh map (of $2^8 = 256$ cells) [62-64] for use as a map of $256 = 4^4$ cells representing four variables that are four-valued each.

The organization of the remainder of this paper is as follows. Section 2 retrieves from Rushdi [41] a mathematical description of the example multi-state k-out-of-n system. Section 3 implements a purely-map analysis of the system. Section 4 shows that our numerical results exactly agree with those obtained by earlier authors. Section 5 discusses certain advantages of using the map, while Section 6 concludes the paper.

2. Mathematical Description of the Example Multi-State *k*-out-of-*n* System

In this Section, we summarize from Rushdi [41] a mathematical description of the example multi-state k-out-of-n system . We use $S_m \{1 \leq m \leq 3\}$ to depict the success of station m (the indicator that the commodity demand of station m is met). The successes of the three stations are given by

$$\begin{aligned} S_1 &= Sy(4;\{4\};\ \bar{X}_1\{0\}, \bar{X}_2\{0\}, \bar{X}_3\{0\}, \bar{X}_4\{0\}) \\ &= \bar{X}_1\{0\}\ \bar{X}_2\{0\}\ \bar{X}_3\{0\}\ \bar{X}_4\{0\}, \end{aligned} \tag{1a}$$

$$\begin{aligned} S_2 &= Sy(4;\{2,3,4\};\ X_1\{2\} \vee X_1\{3\}, X_2\{2\} \vee X_2\{3\}, X_3\{2\} \vee X_3\{3\}, X_4\{2\} \vee X_4\{3\}) \\ &= (X_1\{2\} \vee X_1\{3\})(X_2\{2\} \vee X_2\{3\}) \vee (X_1\{2\} \vee X_1\{3\})(X_3\{2\} \vee X_3\{3\}) \vee (X_1\{2\} \vee X_1\{3\})(X_4\{2\} \vee X_4\{3\}) \\ &\quad \vee (X_2\{2\} \vee X_2\{3\})(X_3\{2\} \vee X_3\{3\}) \\ &\quad \vee (X_2\{2\} \vee X_2\{3\})(X_4\{2\} \vee X_4\{3\}) \vee (X_3\{2\} \vee X_3\{3\})(X_4\{2\} \vee X_4\{3\}), \end{aligned} \tag{1b}$$

$$\begin{aligned} S_3 &= Sy(4;\{3,4\};\ X_1\{3\}, X_2\{3\}, X_3\{3\}, X_4\{3\}) \\ &= X_1\{3\}\,X_2\{3\}\,X_3\{3\} \vee X_1\{3\}\,X_2\{3\}\,X_4\{3\} \vee X_1\{3\}\,X_3\{3\}\,X_4\{3\} \vee X_2\{3\}\,X_3\{3\}\,X_4\{3\}. \end{aligned} \tag{1c}$$

The notation $Sy(n;\boldsymbol{A};\boldsymbol{X})$ denotes a symmetric switching function (*SSF*), which is defined as [1, 4, 20, 41-43, 46-48, 65-68]:

$$f = Sy(n; \boldsymbol{A}; \boldsymbol{X}) = Sy(\, n; \{a_1, a_2, \dots, a_m\};\ X_1, X_2, \dots, X_n), \tag{2}$$

and is specified *via* its number of inputs n, its characteristic set

$$\boldsymbol{A} = \{a_0, a_1, \dots, a_m\} \subseteq \boldsymbol{I}_{n+1} = \{0, 1, 2, \ \dots, n\},\ \{m \le n\}, \tag{3}$$

and its inputs $\boldsymbol{X} = [X_1, X_2, \dots, X_n]^{\mathrm{T}}$. This function has the value 1 if and only if

$$\sum_{i=1}^{n} X_i = a_i, \tag{4}$$

for all integers i such that $0 \le i \le m$, and has the value 0, otherwise.

The four instances of the system output variable S are related to station successes by [41]

$$S\{0\} = \bar{S}_1, \tag{5a}$$

$$S\{1\} = S_1\, \bar{S}_2, \tag{5b}$$

$$S\{2\} = S_1\, S_2\, \bar{S}_3, \tag{5c}$$

$$S\{3\} = S_1\, S_2\, S_3\,. \tag{5d}$$

3. Karnaugh-map Construction and Analysis

This Section describes how the current problem is solved through the construction of a series of Karnaugh maps. Each of Figs. 2-11 is a Karnaugh map of four four-valued inputs X_1, X_2, X_3 and X_4. This map is considerably large as it has $4^4 = 256$ cells, and is simply an adaptation of a map of eight binary variables that has the same number of cells ($2^8 = 256$), introduced earlier in [62-64]. Each of the maps in Figs. 2-10 has binary outputs belonging to $\{0, 1\}$, while the map in Fig. 11 alone has four-valued entries belonging to $\{0, 1, 2, 3\}$. In Figs. 2-10, every 1-entry is written explicitly, while all 0-entered cells are left blank (as usual).

The maps in Figs. 2-4 represent the two-valued station successes S_1, S_2 and S_3, as given by equations (1). These maps are filled-in *collectively* (and not in a cell-by-cell fashion), as we explain now. Equation (1a) sets to 1 (positively asserts) S_1 unless any of the four inputs X_1, X_2, X_3 or X_4 is negatively asserted (equated to 0). Excluding $\{X_1 = 0\}$ in Fig. 2 amounts to setting to 0 all cells in the first four columns of the map in Fig. 2, while avoiding $\{X_2 = 0\}$ assigns 0 to every cell in the first four rows of this map. Avoiding $\{X_3 = 0\}$ requires that 0 be entered in every cell in the first column of every group of four consecutive columns in Fig. 2, while rejecting $\{X_4 = 0\}$ does the same for every cell in the first row of every group of four consecutive rows in Fig. 2. For illustrative purposes, we highlight in yellow the blank (implicitly 0-entered) cells comprising $\{X_3 = 0\}$ in Fig. 2.

Equation (1b) sets to 1 (positively asserts) S_2 for six terms, the first of which is $(X_1\{2\} \vee X_1\{3\})(X_2\{2\} \vee X_2\{3\})$. The four columns covered by this term are highlighted in yellow in Fig. 3. Equation (1c) sets to 1 (positively asserts) S_3 for four terms, the first of which is $X_1\{3\}\, X_2\{3\}\, X_3\{3\}$. The four cells covered by this term are highlighted in yellow in Fig. 4. Figures 5-7 are obtained by *collective* cell-wise complementation of the maps in Figs. 3, 4, and 2, respectively. Figures 7-10 express the four instances of the system output S *via* equations (5). Figures 8-10 use *collective* cell-wise ANDing of maps in the appropriate earlier figures. Figure 11 is a map of multi-valued entries, which represents the multi-valued output S. This map combines the results of the binary-entered maps in Figs. 7-10, which represent the four binary instances $S\{0\}, S\{1\}, S\{2\}$, and $S\{3\}$ of S. Either the four maps in Figs. 7-10, or (equivalently) the individual map in Fig. 11 can be read immediately to express the expectation of each instance (its probability of being equal to 1) as follows.

$$E\{S\{0\}\} = 1 - E\{\bar{X}_1\{0\}\}\, E\{\bar{X}_2\{0\}\}\, E\{\bar{X}_3\{0\}\}\, E\{\bar{X}_4\{0\}\}. \quad (6a)$$

$$E\{S\{1\}\} = E\{X_1\{1\}\}\, E\{X_2\{1\}\}\, E\{X_3\{1\}\}\, (E\{X_4\{2\}\} + E\{X_4\{3\}\}) +$$
$$E\{X_1\{1\}\}\, E\{X_2\{1\}\}\, (E\{X_3\{2\}\} + E\{X_3\{3\}\})\, E\{X_4\{1\}\} +$$

$$E\{X_1\{1\}\}\ (E\{X_2\{2\}\} + E\{X_2\{3\}\})\ E\{X_3\{1\}\}\ E\{X_4\{1\}\} +$$

$$(E\{X_1\{2\}\} + E\{X_1\{3\}\})\ E\{X_2\{1\}\}\ E\{X_3\{1\}\}\ E\{X_4\{1\}\} +$$

$$E\{X_1\{1\}\}\ E\{X_2\{1\}\}\ E\{X_3\{1\}\}\ E\{X_4\{1\}\} \qquad (6b)$$

$$E\{S\{3\}\} = E\{X_1\{3\}\}\ E\{X_2\{3\}\}\ E\{X_3\{3\}\}\ (E\{X_4\{2\}\} + E\{X_4\{1\}\}) +$$

$$E\{X_1\{3\}\}\ E\{X_2\{3\}\}\ (E\{X_3\{2\}\} + E\{X_3\{1\}\})\ E\{X_4\{3\}\} +$$

$$E\{X_1\{3\}\}\ (E\{X_2\{2\}\} + E\{X_2\{1\}\})\ E\{X_3\{3\}\}\ E\{X_4\{3\}\} +$$

$$(E\{X_1\{2\}\} + E\{X_1\{1\}\})\ E\{X_2\{3\}\}\ E\{X_3\{3\}\}\ E\{X_4\{3\}\} +$$

$$E\{X_1\{3\}\}\ E\{X_2\{3\}\}\ E\{X_3\{3\}\}\ E\{X_4\{3\}\}. \qquad (6c)$$

$$E\{S\{2\}\} = 1 - \left(E\{S\{0\}\} + E\{S\{1\}\} + E\{S\{3\}\}\right). \qquad (6e)$$

4. Comparison with Previous Work

The problem handled herein was solved *via* various techniques by Tian *et al.* [34], Mo. *et al.* [40], Rushdi [41], and Rushdi & Al-Amoudi [42, 43]. In all cases, the results were tested by the following input matrix, in which the sum of entries in each row is 1, since such entries are the probabilities of mutually exclusive and exhaustive events.

$$\{E\{X_i\{j\}\}\} = \begin{bmatrix} .0500 & .0950 & .0684 & .7866 \\ .0500 & .0950 & .0684 & .7866 \\ .0300 & .0776 & .0446 & .8478 \\ .0300 & .0776 & .0446 & .8478 \end{bmatrix} \qquad (1 \le i \le 4,\ 0 \le j \le 3) \qquad (7)$$

Table 2 compares our results for this specific input with the results of the earlier teams of authors. The six sets of results are essentially the same, despite the existence of minor differences in precision.

5. Discussion

The ultimate outcome of our analysis is the Multi-Valued Karnaugh Map (MVKM) of Fig. 11, which serves as a natural, unique, and complete representation of the multi-state system. One can obtain many useful insights and deduce certain (not-so-obvious) facts from this map.

- The map reveals the nature of the four binary instances $S\{0\}, S\{1\}, S\{2\}$, and $S\{3\}$ of S, when these instances are viewed as individual binary reliability systems. The instance $S\{0\}$ acts like a coherent binary *failure* while the instance $S\{3\}$ behaves like a coherent binary *success*. Both $S\{1\}$ and $S\{2\}$ have a general *non-coherent* behavior, which somewhat mimics that of a k-to-l-out-of-n: G system [65, 66], or a double-threshold system [67, 68]. It is interesting to note that the instances $S\{0\}$, $S\{1\}$, and $S\{2\}$ are non-coherent in a binary sense, though each of the station successes S_1, S_2 and S_3 is coherent in the same sense. By contrast, the overall system output S is coherent in a multi-state sense.

- The map offers a convenient pictorial mechanism for *decomposing* its output function into various sub-functions, thereby constructing a multi-valued expansion tree or decision diagram for this function [1-4, 19-23, 41, 65-71].

- The map is a tool to visualize each of the properties of *causality*, *monotonicity*, and *relevancy*, which when combined together amount to labelling the present multi-state system as a *coherent* one [43].

- The map demonstrates *total symmetry* of the system function S with respect to its four arguments X_1, X_2, X_3 and X_4. Total symmetry means that the map entries are invariant to interchanging any two of the four arguments [46].

- The map in Fig. 11 is a valuable resource for computing a plethora of Importance Measures [72-96] for the current multi-state system. Importance Measures are used to assess the criticality of individual components within the system, identify system weaknesses, and rank components so as to prioritize potential reliability improvements A crucial map feature in this respect is the capability of the map to perform "Boolean differentiation" or "Boolean differencing" through appropriate map folding [87-100].

Tedious algebraic manipulations were needed in [41-43] to prove that

$$S_1 S_3 \leq S_1 S_2 , \tag{8}$$

Equation (8) is a useful result, since it facilitates the derivation of an algebraic expression for $S\{3\}$. However, inspection of Figs. 2-4 reveals not only (8) but also the more powerful result

$$S_3 \leq S_2 , \tag{9}$$

Direct inspection of Figs. 2-4 also attests that S_1 is neither comparable to S_2 nor comparable to S_3. Figures 7-10 confirm that the four instances $S\{0\}, S\{1\}, S\{2\}$, and $S\{3\}$ of S form an orthonormal set, thereby allowing a consistent construction of the MVKM in Fig. 11.

5. Conclusions

This paper demonstrated how *MSS* reliability can be handled solely *via* Karnaugh maps of multi-valued inputs, and of binary or multi-valued entries. A classical *MSS* problem was manually analyzed by maps that resemble eight-variable Karnaugh maps. Despite the relatively large size of the maps used, they were very convenient, indeed. No attempt was made to draw loops on the maps or to seek minimal formulas. The maps just served as handy tools for combinatorial representation and for collective implementation of the operations of ANDing, ORing, and complementation. Results obtained are satisfactory as they exactly replicate earlier results obtained by various automated and manual means.

References

[1] Rushdi, A. M. (1986). Utilization of symmetric switching functions in the computation of k-out-of-n system reliability. *Microelectronics and Reliability*, *26*(5), 973-987.

[2] Rushdi, A. M. (1990). Threshold systems and their reliability. *Microelectronics and Reliability*, *30*(2), 299-312.

[3] Rushdi, A. M. (1991). Comments on "An efficient non-recursive algorithm for computing the reliability of k-out-of-n systems." *IEEE Transactions on Reliability*, *40*(1), 60-61.

[4] Rushdi, A. M. (1993). *Reliability of k-out-of-n systems*, Chapter 5 in K. B. Misra (Editor), *New Trends in System Reliability Evaluation*, Vol. 16, *Fundamental Studies in Engineering*, Amsterdam, The Netherlands: Elsevier Science Publisher, pp. 185-227.

[5] Sarje, A. K. (1993). On the reliability computation of a k-out-of-n system. *Microelectronics and Reliability*, *33*(2), 267-269.

[6] Rushdi, A. M., & Al-Hindi, K. A. (1993). A table for the lower boundary of the region of useful redundancy for k-out-of-n systems. *Microelectronics and Reliability*, *33*(7), 979-992.

[7] Rushdi, A. M., & Al-Thubaity, A. O. (1993). Efficient computation of the sensitivity of k-out-of-n system reliability, *Microelectronics and Reliability*, 33(13), 1963–1979.

[8] Belfore, II, L. A. (1995). An $O(n (\log_2 n)^2)$ algorithm for computing the reliability of k-out-of-n: G & k-to-*l*-out-of-n: G systems, *IEEE Transactions on Reliability*, 44(1), 132–136.

[9] Zuo, M., Chiovelli, S., & Huang, J. (1999). Reliability evaluation of furnace systems. *Reliability Engineering & System Safety*, *65*(3), 283-287.

[10] Dutuit, Y., & Rauzy, A. (2001). New insights into the assessment of k-out-of-n and related systems. *Reliability Engineering & System Safety*, *72*(3), 303-314.

[11] Kuo, W., & Zuo, M. J. (2003). The k-out-of-n System Model, Chapter 7 in *Optimal Reliability Modelling: Principles and Applications*, Wiley, New York, NY, USA, pp. 231-280.

[12] Papadimitratos, P., & Haas, Z. J. (2006). Secure data communication in mobile ad hoc networks. *IEEE Journal on Selected Areas in Communications*, *24*(2), 343-356.

[13] Rushdi, A. M., & Alsulami, A. E. (2007). Cost elasticities of reliability and MTTF for k-out-of-n systems. *Journal of Mathematics and Statistics*, *3*(3), 122-128.

[14] Misra, K. B. (2008). Reliability engineering: A perspective. In *Handbook of Performability Engineering* (pp. 253-289). Springer, London.

[15] Amari, S. V., Zuo, M. J., & Dill, G. (2008). O(kn) algorithms for analyzing repairable and non-repairable k-out-of-n: G systems. In *Handbook of Performability Engineering* (pp. 309-320). Springer, London.

[16] Rushdi, M. A. & Rushdi, A. M. (2010), A tutorial overview of embedded systems, *Electronic Proceedings of the 2nd Conference of the Egyptian Engineering Association (EEA)*, Riyadh, Saudi Arabia.

[17] Rushdi, A. M. A., Partially-redundant systems: Examples, reliability, and life expectancy, *International Magazine on Advances in Computer Science and Telecommunications,* 1(1), 1-13 (2010).

[18] Zuo, M. J., & Tian, Z. (2010). *k- out- of- n Systems*. In Cochran, J. J., Cox Jr., L. A., Jeffrey, P. K., Kharoufeh, P., & Smith, J. C. (Editors), *Wiley Encyclopedia of Operations Research and Management Science*.

[19] Rushdi, M. A. M., Ba-Rukab, O. M., & Rushdi, A. M. (2016). Multi-dimensional recursion relations and mathematical induction techniques: The case of failure frequency of k-out-of-n systems, *Journal of King Abdulaziz University: Engineering Sciences*, *27*(2): 15-31.

[20] Rushdi, A. M., & Rushdi, M. A. (2017). *Switching-Algebraic Analysis of System Reliability*, Chapter 6 in M. Ram and P. Davim (Editors), *Advances in Reliability and System Engineering*, Management and Industrial Engineering Series, Springer International Publishing, Cham, Switzerland, 139-161.

[21] Rushdi, A. M. A., & Alturki, A. M. (2017). Computation of k-out-of-n system reliability via reduced ordered binary decision diagrams. *British Journal of Mathematics & Computer Science*. *22*(3), 1-9.

[22] Rushdi, A. M., & Alturki, A. M. (2018). Novel representations for a coherent threshold reliability system: a tale of eight signal flow graphs. *Turkish Journal of Electrical Engineering & Computer Sciences*, *26*(1), 257-269.

[23] Rushdi, A. M. A., & Alturki, A. M. (2018). Unification of mathematical concepts and algorithms of k-out-of-n system reliability: A perspective of improved disjoint products, *Journal of Engineering Research*, *6*(4), 1-31.

[24] El-Neweihi, E., Proschan, F. & Sethuraman, J. (1978). Multi-state coherent system. *Journal of Applied Probability*, 15, 675–688.

[25] Barlow, R.E. & Wu, A.S. (1978) Coherent systems with multi-state components. *Mathematics of Operations Research*, 3(4), 275–281.

[26] Natvig, B. (1982). Two suggestions of how to define a multi-state coherent system. *Applied Probability*, *14*, 391–402.

[27] Pham, H., Suprasad, A., & Misra, R. B. (1996). Reliability and MTTF prediction of k-out-of-n complex systems with components subjected to multiple stages of degradation. *International Journal of Systems Science*, *27*(10), 995-1000.

[28] Huang, J., & Zuo, M. J. (2000). Multi-state k-out-of-n system model and its applications. In *Reliability and Maintainability Symposium, 2000. Proceedings. Annual* (pp. 264-268). IEEE.

[29] Huang, J., Zuo, M. J., & Wu, Y. (2000). Generalized multi-state k-out-of-n: G systems. *IEEE Transactions on Reliability*, *49*(1), 105-111.

[30] Cochran, J. K., & Lewis, T. P. (2002). Computing small-fleet aircraft availabilities including redundancy and spares. *Computers & Operations Research*, *29*(5), 529-540.

[31] Zuo, M. J., Huang, J., & Kuo, W. (2003). Multi-state k-out-of-n systems. In *Handbook of Reliability Engineering* (pp. 3-17). Springer, London.

[32] Zuo, M. J., & Tian, Z. (2006). Performance evaluation of generalized multi-state k-out-of-n systems. *IEEE Transactions on Reliability*, *55*(2), 319-327.

[33] Yamamoto, H., Zuo, M. J., Akiba, T., & Tian, Z. (2006). Recursive formulas for the reliability of multi-state consecutive-k-out-of-n: G systems. *IEEE Transactions on Reliability*, *55*(1), 98-104.

[34] Tian, Z., Zuo, M. J., & Yam, R. C. (2008). Multi-state k-out-of-n systems and their performance evaluation. *IIE Transactions*, *41*(1), 32-44.

[35] Tian, Z., Li, W., & Zuo, M. J. (2008). Modeling and reliability evaluation of multi-state k-out-of-n systems. In *Recent Advances in Reliability and Quality in Design* (pp. 31-56). Springer, London.

[36] Tian, Z., Yam, R. C., Zuo, M. J., & Huang, H. Z. (2008). Reliability bounds for multi-state k-out-of-n systems. *IEEE Transactions on Reliability*, *57*(1), 53-58.

[37] Ding, Y., Zuo, M. J., Tian, Z., & Li, W. (2010). The hierarchical weighted multi-state k-out-of-n system model and its application for infrastructure management. *IEEE Transactions on Reliability, 59*(3), 593-603

[38] Zhao, X., & Cui, L. (2010). Reliability evaluation of generalised multi-state k-out-of-n systems based on FMCI approach. *International Journal of Systems Science*, *41*(12), 1437-1443.

[39] Yamamoto, H., Akiba, T., Yamaguchi, T., & Nagatsuka, H. (2011). An evaluating algorithm for system state distributions of generalized multi-state k-out-of-n: F systems. *Journal of Japan Industrial Management Association*, *61*(6), 347-354.

[40] Mo, Y., Xing, L., Amari, S. V., & Dugan, J. B. (2015). Efficient analysis of multi-state k-out-of-n systems. *Reliability Engineering & System Safety*, *133*, 95-105.

[41] Rushdi, A. M. A. (2019). Utilization of symmetric switching functions in the symbolic reliability analysis of multi-state k-out-of-n systems, *International Journal of Mathematical, Engineering and Management Sciences* (*IJMEMS*), *4*(2), 306-326.

[42] Rushdi, A. M. A. & Al-Amoudi M. A. (2018). Switching-algebraic analysis of multi-state system reliability, *Journal of Engineering Research and Reports, 3*(3),1-22.

[43] Rushdi, A. M. A. & Al-Amoudi M. A. (2019), Reliability analysis of a multi-state system using multi-valued logic, *IOSR Journal of Electronics and Communication Engineering (IOSR-JECE)*, *14*(1), 1-10.

[44] Fadhel S. F., Alauldin N. A., & Ahmed Y. Y. (2014). Reliability of dynamic multi-state oil supply system by structure function. *International Journal of Innovative Research in Science, Engineering and Technology*, *3(6)*, 13548-13555.

[45] Karnaugh, M. (1953). The map method for synthesis of combinational logic circuits. *Transactions of the American Institute of Electrical Engineers, Part I: Communication and Electronics*, *72*(5), 593-599.

[46] Lee, S. C. (1978). *Modern Switching Theory and Digital Design*, Prentice-Hall, Englewood Cliffs, New Jersey, NJ, USA.

[47] Muroga, S., (1979). *Logic Design and Switching Theory*, John Wiley, New York, NY, USA.

[48] Hill, F. J., & Peterson, G. R. (1993). *Computer Aided Logical Design with Emphasis on VLSI*, 4th Edition, Wiley, New York, NY, USA.

[49] Rushdi, A. M. A., (1997). Karnaugh map, Encyclopedia of Mathematics, Supplement Volume I, M. Hazewinkel (Editor), Boston, Kluwer Academic Publishers, pp. 327-328. Available at http://eom.springer.de/K/k110040.html.

[50] Roth, C. & Kinney, L., (2014). *Fundamentals of Logic Design*,7th Edition, Cengage Learning, Stamford, CT, USA.

[51] Hurley, R. B., (1963). Probability maps, *IEEE Transactions on Reliability*, *R-12*(3): 39-44.

[52] Rushdi, A. M. (1983). Symbolic reliability analysis with the aid of variable-entered Karnaugh maps, *IEEE Transactions on Reliability*, *R-32*(2): 134-139.

[53] Rushdi, A. M., & Al-Khateeb, D. L. (1983). A review of methods for system reliability analysis: A Karnaugh-map perspective. In *Proceedings of the First Saudi Engineering Conference*, Jeddah, Saudi Arabia, Vol. 1, pp. 57-95.

[54] Rushdi, A. M. (1984). Overall reliability analysis for computer-communication networks. In *Proceedings of the Seventh National Computer Conference*, Riyadh, Saudi Arabia, pp. 23-38.

[55] Rushdi A. M. (1984). On reliability evaluation by network decomposition. *IEEE Transactions on Reliability*, 33(5), 379-384. Corrections: *ibid*., 1985, 34(4), 319.

[56] Rushdi, A. M. A., & Ghaleb, F. A. M. (2014). The Walsh spectrum and the real transform of a switching function: A review with a Karnaugh-map perspective. *Journal of Engineering and Computer Sciences, Qassim University*, *7*(2), 73-112.

[57] Rushdi, A. M., & Talmees, F. A. (2018). An exposition of the eight basic measures in diagnostic testing using several pedagogical tools. *Journal of Advances in Mathematics and Computer Science*, *26*(3), 1-17.

[58] Rushdi, R. A., Rushdi, A. M., & Talmees, F. A. (2018). Novel pedagogical methods for conditional-probability computations in medical disciplines. *Journal of Advances in Medicine and Medical Research*, *25*(10), 1-15.

[59] Rushdi, R. A., & Rushdi, A. M. (2018). Karnaugh-map utility in medical studies: The case of Fetal Malnutrition. *International Journal of Mathematical, Engineering and Management Sciences (IJMEMS)*, *3*(3), 220-244.

[60] Bahraini, M. & Epstein, G. (1988). Three-valued Karnaugh maps. In *International Symposium on Multiple-Valued Logic (ISMVL),18,* 178-185.

[61] Rushdi, A. M. A. (2018). Utilization of Karnaugh maps in multi-value qualitative comparative analysis, *International Journal of Mathematical, Engineering and Management Sciences* (*IJMEMS*), *3*(1), 28-46.

[62] Halder, A. K. (1982). Karnaugh map extended to six or more variables. *Electronics Letters*, *18*(20), 868-870.

[63] Motil, J. M. (2017). Views of digital logic & probability via sets, numberings. Available at: http://www.csun.edu/~jmotil/ccSetNums2.pdf.

[64] Rushdi, A. M., Zagzoog, S. & Balamesh, A. S. (2019). Derivation of a scalable solution for the problem of factoring an n-bit integer, *Journal of Advances in Mathematics and Computer Science*, 30(1), 1-22.

[65] Rushdi, A. M. (1987). Efficient computation of k-to-*l*-out-of-n system reliability, *Reliability Engineering*, 17(3), 157-163, (1987), Erratum: *ibid.*, 1987, 19(4): 321.

[66] Rushdi, A. M. & Dehlawi F. (1987). Optimal computation of k-to-*l*-out-of-n system reliability, *Microelectronics and Reliability*, 27(5): 875-896, Erratum: *ibid*., 1988, 28(4), 671.

[67] Rushdi, A. M. A., & Alturki, A. M. (2015). Reliability of coherent threshold systems. *Journal of Applied Sciences*, *15*(3), 431-443.

[68] Rushdi, A. M. A., & Bjaili, H. A. (2016). An ROBDD algorithm for the reliability of double-threshold systems. *British Journal of Mathematics and Computer Science*, *19*(6), 1-17.

[69] Rushdi, A. M. A., & Hassan A. K. (2015). Reliability of migration between habitat patches with heterogeneous ecological corridors, *Ecological modelling*, 304, 1-10.

[70] Rushdi, A. M. A., & Hassan A. K. (2016). An exposition of system reliability analysis with an ecological perspective, *Ecological Indicators*, 63, 282-295.

[71] Rushdi, A. M. A., & Al-Amoudi, M. A. (2018). Recursively-defined combinatorial functions: the case of binomial and multinomial coefficients and probabilities, *Journal of Advances in Mathematics and Computer Science*, 27(4), 1-16.

[72] Bossche, A. (1987). Calculation of critical importance for multi-state components. *IEEE Transactions on Reliability*, *36*(2), 247-249.

[73] Tapia, M. A., Guima, T. A., & Katbab, A. (1991). Calculus for a multivalued-logic algebraic system. *Applied Mathematics and Computation*, *42*(3), 255-285.

[74] Boland, P. J., & El-Neweihi, E. (1995). Measures of component importance in reliability theory. *Computers & operations research*, *22*(4), 455-463.

[75] Levitin, G., & Lisnianski, A. (1999). Importance and sensitivity analysis of multi-state systems using the universal generating function method. *Reliability Engineering & System Safety*, *65*(3), 271-282.

[76] Levitin, G., Podofillini, L., & Zio, E. (2003). Generalised importance measures for multi-state elements based on performance level restrictions. *Reliability Engineering & System Safety*, *82*(3), 287-298.

[77] Zio, E., & Podofillini, L. (2003). Monte Carlo simulation analysis of the effects of different system performance levels on the importance of multi-state components. *Reliability Engineering & System Safety*, *82*(1), 63-73.

[78] Zio, E., & Podofillini, L. (2003). Importance measures of multi-state components in multi-state systems. *International Journal of Reliability, Quality and Safety Engineering*, *10*(03), 289-310.

[79] Zio, E., Podofillini, L., & Levitin, G. (2004). Estimation of the importance measures of multi-state elements by Monte Carlo simulation. *Reliability Engineering & System Safety*, *86*(3), 191-204.

[80] Ramirez-Marquez, J. E., & Coit, D. W. (2005). Composite importance measures for multi-state systems with multi-state components. *IEEE Transactions on Reliability*, *54*(3), 517-529.

[81] Hwang, F. K. (2005). A hierarchy of importance indices. *IEEE Transactions on Reliability*, *54*(1), 169-172.

[82] Ramirez-Marquez, J. E., Rocco, C. M., Gebre, B. A., Coit, D. W., & Tortorella, M. (2006). New insights on multi-state component criticality and importance. *Reliability Engineering & System Safety*, *91*(8), 894-904.

[83] Barabady, J., & Kumar, U. (2007). Availability allocation through importance measures. *International journal of quality & reliability management*, *24*(6), 643-657.

[84] Ramirez-Marquez, J. E., & Coit, D. W. (2007). Multi-state component criticality analysis for reliability improvement in multi-state systems. *Reliability Engineering & System Safety*, *92*(12), 1608-1619.

[85] Zhu, X., & Kuo, W. (2008). Comments on "A hierarchy of importance indices." *IEEE Transactions on Reliability*, *57*(3), 529-531.

[86] Peng, H., Coit, D. W., & Feng, Q. (2012). Component reliability criticality or importance measures for systems with degrading components. *IEEE Transactions on Reliability*, *61*(1), 4-12.

[87] Zaitseva, E. (2012). Importance analysis of a multi-state system based on multiple-valued logic methods. In *Recent Advances in System Reliability* (pp. 113-134). Springer, London.

[88] Ramirez-Marquez, J. E. (2012). Innovative approaches for addressing old challenges in component importance measures. *Reliability Engineering & System Safety*, *108*, 123-130.

[89] Si, S., Dui, H., Cai, Z., & Sun, S. (2012). The integrated importance measure of multi-state coherent systems for maintenance processes. *IEEE Transactions on Reliability*, *61*(2), 266-273.

[90] Si, S., Levitin, G., Dui, H., & Sun, S. (2013). Component state-based integrated importance measure for multi-state systems. *Reliability Engineering & System Safety*, *116*, 75-83.

[91] Zaitseva, E., Kvassay, M., Levashenko, V., Kostolny, J., & Pancerz, K. (2015). Estimation of a healthcare system based on the importance analysis. In *Computational Intelligence, Medicine and Biology* (pp. 3-22). Springer, Cham.

[92] Kvassay, M., Zaitseva, E., & Levashenko, V. (2017). Importance analysis of multi-state systems based on tools of logical differential calculus. *Reliability Engineering & System Safety*, *165*, 302-316.

[93] Vujošević, M., Makajić-Nikolić, D., & Pavlović, P. (2017). A new approach to determination of the most critical multi-state components in multi-state systems. *Journal of Applied Engineering Science*, *15*(4), 401-405.

[94] Kvassay, M., Levashenko, V., Rabcan, J., Rusnak, P., & Zaitseva, E. (2018). Structure function in analysis of Multi-State System availability. In *Safety and Reliability–Safe Societies in a Changing World* (pp. 897-905). CRC Press.

[95] Kvassay, M., & Zaitseva, E. (2018). Topological Analysis of Multi-state Systems Based on Direct Partial Logic Derivatives. In *Recent Advances in Multi-state Systems Reliability* (pp. 265-281). Springer, Cham.

[96] Markopoulos, T., & Platis, A. N. (2018). Reliability Analysis of a Modified IEEE 6BUS RBTS Multi-state System. In *Recent Advances in Multi-state Systems Reliability* (pp. 301-319). Springer, Cham.

[97] Rushdi, A. M. (1986). Map differentiation of switching functions. *Microelectronics and Reliability*, *26*(5), 891-907.

[98] Yamamoto, Y. (2012). Banzhaf index and Boolean difference. In *2012 IEEE 42nd International Symposium on Multiple-Valued Logic* (pp. 191-196). IEEE.

[99] Alturki, A. M., & Rushdi, A. M. A. (2016). Weighted voting systems: A threshold-Boolean perspective. *Journal of Engineering Research*, *1*(4), 1-19.

[100] Rushdi, A. M. A., & Ba-Rukab, O. M. (2017). Calculation of Banzhaf voting indices utilizing variable-entered Karnaugh maps. *British Journal of Mathematics & Computer Science*, 20(4), 1-17.

Table 1. Defintion of the four-valued input variable X_i, which detemines the status of pipeline i ($1 \leq i \leq 4$), and the four-valued output variable S, which detemines the overall system status.

Value of X_i	Meaning
0	Pipeline i cannot transmit the commodity to any station.
1	Pipeline i can transmit the commodity *up to* station 1.
2	Pipeline i can transmit the commodity *up to* station 2.
3	Pipeline i can transmit the commodity *up to* station 3.
Value of S	**Meaning**
0	The system cannot meet the commodity demand of any station.
1	The system can meet the commodity demand of *up to* station 1.
2	The system can meet the commodity demand of *up to* station 2.
3	The system can meet the commodity demand of *up to* station 3.

Table 2. Comparison of the present results with those in earlier work.

	Tian *et al.* [34]	**Mo *et al.* [40]**	**Rushdi [41]**	**Rushdi & Al-Amoudi [42, 43]**	**Present Results**
E{S(0)}	0.**1508**	0.**150838**	0.**15083775**0000	0.**15083775**0000000	0.**15083775**0000
E{S(1)}	0.00**23**	0.00**2282**	0.00**2282548128**	0.00**2282548128**000	0.00**2282548128**
E{S(2)}	0.0**892**	0.0**89181**	0.0**89180866436**	0.0**89180866435**691	0.0**89180866436**
E{S(3)}	0.**7577**	0.**757699**	0.**757698835436**	0.**757698835436**309	0.**757698835436**
Total	1.0000	1.000000	1.000000000000	1.000000000000000	1.000000000000

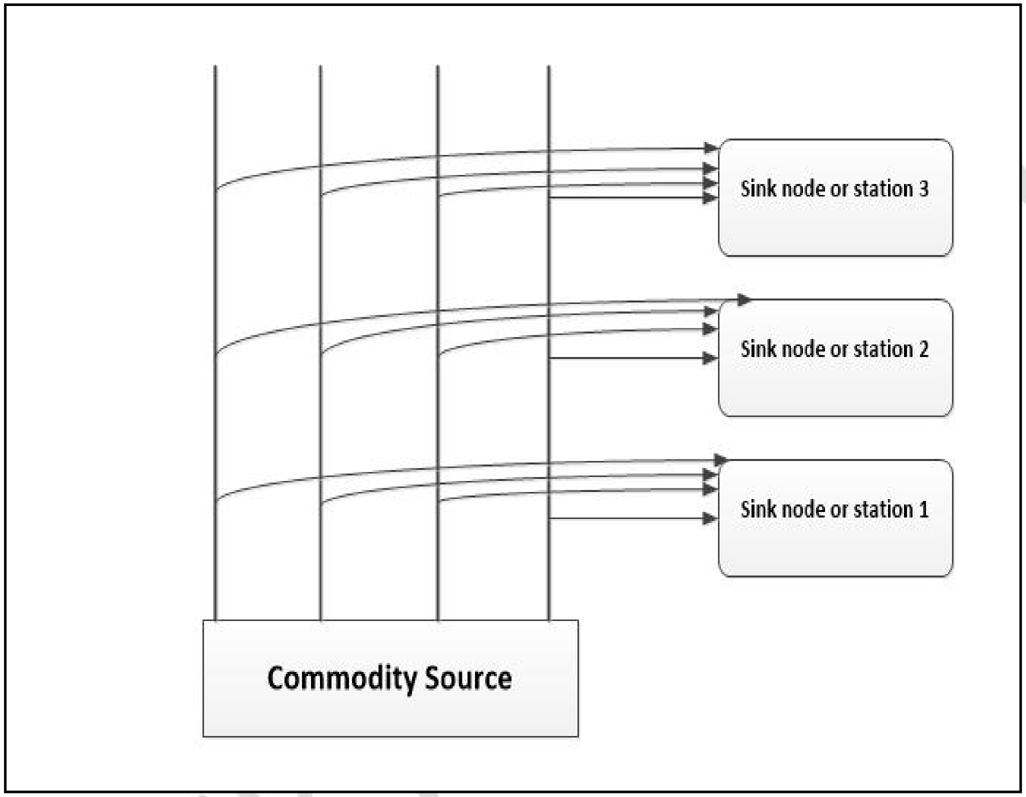

Fig. 1. A commodity-supply system that is modeled as a multi-state k-out-of-n: G system (Adapted from *Tian et al.* (2008)).

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																	3	
																	0	1
						1	1	1		1	1	1		1	1	1	1	
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1	1	3	
																	0	2
						1	1	1		1	1	1		1	1	1	1	
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1	1	3	
																	0	3
						1	1	1		1	1	1		1	1	1	1	
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1	1	3	
																	X_4	X_2

Fig .2. A Karnaugh map (of four four-valued inputs) representing the success of station 1.

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
											1	1			1	1	0	0
											1	1			1	1	1	
			1	1			1	1	1	1	1	1	1	1	1	1	2	
			1	1			1	1	1	1	1	1	1	1	1	1	3	
											1	1			1	1	0	1
											1	1			1	1	1	
			1	1			1	1	1	1	1	1	1	1	1	1	2	
			1	1			1	1	1	1	1	1	1	1	1	1	3	
			1	1			1	1	1	1	1	1	1	1	1	1	0	2
			1	1			1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	
			1	1			1	1	1	1	1	1	1	1	1	1	0	3
			1	1			1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	
																	X_4	X_2

Fig .3. A Karnaugh map (of four four-valued inputs) representing the success of station 2.

396

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																1	3	
																	0	1
																	1	
																	2	
																1	3	
																	0	2
																	1	
																	2	
																1	3	
																1	0	3
																1	1	
																1	2	
				1				1					1	1	1	1	3	
																	X_4	X_2

Fig .4. A Karnaugh map (of four four-valued inputs) representing the success of station 3.

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
	1	1	1	1	1	1	1	1	1	1			1	1			0	0
	1	1	1	1	1	1	1	1	1	1			1	1			1	
	1	1			1	1											2	
	1	1			1	1											3	
	1	1	1	1	1	1	1	1	1	1			1	1			0	1
	1	1	1	1	1	1	1	1	1	1			1	1			1	
	1	1			1	1											2	
	1	1			1	1											3	
	1	1			1	1											0	2
	1	1			1	1											1	
																	2	
																	3	
	1	1			1	1											0	3
	1	1			1	1											1	
																	2	
																	3	
																	X_4	X_2

Fig .5. A Karnaugh map (of four four-valued inputs) representing the failure of station 2, obtained by cell-wise complementation of the map in Fig. 3.

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		3	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		3	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	2
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		3	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		0	3
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		2	
	1	1	1		1	1	1		1	1	1						3	
																	X_4	X_2

Fig .6. A Karnaugh map (of four four-valued inputs) representing the failure of station 3, obtained by cell-wise complementation of the map in Fig. 4.

399

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1
	1	1	1	1	1				1				1				1	
	1	1	1	1	1				1				1				2	
	1	1	1	1	1				1				1				3	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	2
	1	1	1	1	1				1				1				1	
	1	1	1	1	1				1				1				2	
	1	1	1	1	1				1				1				3	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	3
	1	1	1	1	1				1				1				1	
	1	1	1	1	1				1				1				2	
	1	1	1	1	1				1				1				3	
																	X_4	X_2

Fig. 7. A Karnaugh map for the binary indicator of instant $S\{0\} = \overline{S}_1$ of system output, obtained by cell-wise complementation of the map in Fig. 2.

X_2	X_1	0				1				2				3			
	X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	0																
	1																
	2																
	3																
1	0																
	1						1	1	1		1				1		
	2						1										
	3						1										
2	0																
	1						1										
	2																
	3																
3	0																
	1						1										
	2																
	3																
X_2	X_4																

Fig. 8. A Karnaugh map for the binary indicator of instant $\overline{S\{1\}} = \overline{S_1}\,\overline{S_2}$ of system output, obtained by cell-wise ANDing of the maps in Figs. 2 and 5.

402

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	
																	1	0
																	2	
																	3	
																	0	
											1	1			1	1	1	1
							1	1		1	1	1		1	1	1	2	
							1	1		1	1	1		1	1		3	
																	0	
							1	1		1	1	1		1	1	1	1	2
						1	1	1		1	1	1		1	1	1	2	
						1	1	1		1	1	1		1	1		3	
																	0	
							1	1		1	1	1		1	1		1	3
						1	1	1		1	1	1		1	1		2	
						1	1			1	1						3	
																	X_4	X_2

Fig. 9. A Karnaugh map for the binary indicator of instant $S\{2\} = S_1 S_2 \bar{S}_3$ of system output obtained by cell-wise ANDing of the maps in Figs. 2, 3 and 6.

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
																	0	0
																	1	
																	2	
																	3	
																	0	1
																	1	
																	2	
																1	3	
																	0	2
																	1	
																	2	
																1	3	
																	0	3
																1	1	
																1	2	
								1				1		1	1	1	3	
																	X_4	X_2

Fig. 10. A Karnaugh map for the binary indicator of instant $s\{3\} = s_1 s_2 s_3$ of system output, obtained by cell-wise ANDing of the maps in Figs. 2, 3 and 4.

X_1	0				1				2				3					
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3		
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	1	1	1	0	1	2	2	0	1	2	2	1	
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	2	2	
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	3	3	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	2	1	
	0	0	0	0	0	2	2	2	0	2	2	2	0	2	2	2	2	
	0	0	0	0	0	2	2	2	0	2	2	2	0	2	2	3	3	
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
	0	0	0	0	0	1	2	2	0	2	2	2	0	2	2	3	1	
	0	0	0	0	0	2	2	2	0	2	2	2	0	2	2	3	2	
	0	0	0	0	0	2	2	3	0	2	2	3	0	3	3	3	3	
																	X_4	X_2

Fig. 11. A MVKM representing the multi-valued output S, obtained by combining information from the four maps in Figs. 7-10.

403