ANALYSIS AN MODELING OF TUBERCULOSIS TRANSMISSION IN KENYA

1.0 Abstract

We develop a mathematical model that explains the transmission of Tuberculosis Consisting of four compartments; the susceptible, the infectious, the latently infected, and the recovered humans. We then analyzed the disease free and endemic equilibrium points. We then compute the basic reproduction number using the next generation matrix approach. The Tuberculosis model is analyzed in order to give a proper account of the impact of its transmission dynamics and the effect of latent stage in TB transmission. The basic reproduction number is greater than one, TB will continue to persist in the environment. This is due to the fact that the rate of contact with the infectious is greater than the recovery rate. The findings show that as more people come into contact with infectious individuals, the spread of TB would increase.

Key words: latent TB, Transmission dynamics, Basic reproduction number, Stability analysis and equilibrium points.

2.0 Introduction

Tuberculosis (TB) is an airborne disease caused by the bacterium *Mycobacterium tuberculosis*. *Mycobacterium* is carried in air particles called droplets nuclei. Depending on the environment, these tiny particles can remain suspended in the air for several hours, potentially infecting anyone who breathes them in. However, not everyone who inhales the bacteria gets sick because some people's immune system immediately kills the bacteria. In others the bacteria remains in a latent or dormant The bacteria become inactive, but they remain alive in body. People with latent tuberculosis have no symptoms of TB; they don't feel sick and can't spread the disease to others.Once infected, an individual stays infected for many years possibly latently-infected for life.

The population at a given time t is denoted by $N(t)$. The model divides the population into four epidemiological classes with respect to their disease status in the environment. The total population, represented by N(t), is divided into the sub- population of susceptible humans (S), infectious humans (I), latent (L), and recovered (R).the total population becomes;

$N(t)=S(t)+ L(t)+ I(t)+ R(t)$, where;

Contact with infectious humans is at a rate β. Individuals recover from the disease at a rate α. Humans who are infected with the disease die at a rate δ and the recovered humans may lose immunity and return to the susceptible compartments at a rateγ. The natural death rate of the entire human compartments is μ. susceptible humans become latently infected at the rate θ and the latently infected become infectious at the rate γ.

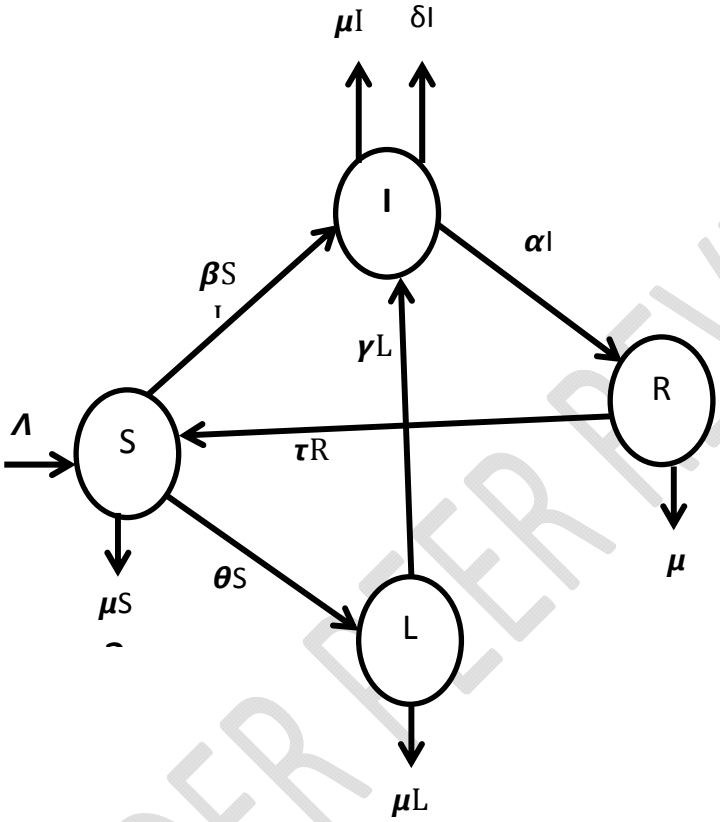

Figure 3.1: Model flow chart showing the compartments

From the figure above, the model equation become;

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta SI + \tau R - (\mu + \theta) S \\
\frac{dI}{dt} &= \beta SI + \gamma L - (\mu + \delta + \alpha) I \\
\frac{dL}{dt} &= \theta S - (\mu + \gamma) L \\
\frac{dR}{dt} &= \alpha I - (\mu + \tau) R
\end{aligned} \qquad (1)$$

The Disease Free Equilibrium

At the disease free equilibrium, there is no infection hence no recovery that is; I=L=R=0.

Therefore at the equilibrium, we have, $\frac{dS}{dt}=\frac{dI}{dt}=\frac{dL}{dt}=\frac{dR}{dt}=0$ (2)

From equation (1) we have:

$$\Lambda-(\mu+\theta)S=0$$

$$S=\frac{\Lambda}{}\ S=\frac{\Lambda}{\mu+\theta} \qquad (3)$$

The disease free equilibrium points from the model is expressed as follows;

$$E(S,I,L,R)=\left(\frac{\Lambda}{\mu+\theta},0,0,0\right) \quad (4)$$

Stability of the DiseaseFree Equilibrium

The stability of the disease free equilibrium solved using the Jacobian matrix is given by the following eigen values;

$$\lambda_1=-(\mu+\theta),\lambda_2=-\frac{\beta\Lambda}{\mu+\theta}+(\mu+\delta+\alpha),\lambda_3=-(\mu+\gamma),\lambda_4=-(\mu+\tau) \qquad (5)$$

Since all Eigen values are negative, then the disease free equilibrium is locally asymptotically stable.

The basic reproduction number

In order to get the reproductive number, we calculate it using the next generation matrix from the model equations. Therefore the basic reproduction number is given by;

$$R_0=\frac{\beta\gamma}{(\mu+\delta+\alpha)(\mu+\gamma)} \qquad (6)$$

Theorem

If $\gamma<\frac{\mu(\mu+\delta+\alpha)}{\beta-(\mu+\delta+\alpha)}$, the disease will not take hold in the population.

Proof

At the disease free equilibrium, $R_0 = \frac{\beta\gamma}{(\mu+\delta+\alpha)(\mu+\gamma)}$ then $R_0 < 1$

$$\frac{\beta\gamma}{(\mu+\delta+\alpha)(\mu+\gamma)} < 1$$

$$\beta\gamma < (\mu+\delta+\alpha)(\mu+\gamma)$$

$$\beta\gamma < \gamma(\mu+\delta+\alpha) + \mu(\mu+\delta+\alpha) \quad (7)$$

$$\beta\gamma - \gamma(\mu+\delta+\alpha) < \mu(\mu+\delta+\alpha)$$

$$\gamma < \frac{\mu(\mu+\delta+\alpha)}{\beta-(\mu+\delta+\alpha)}$$

Lemma

If $\gamma < \frac{\mu(\mu+\delta+\alpha)}{\beta-(\mu+\delta+\alpha)}$ then the disease will take hold in the Population.

When $R_0 > 1$, then $\frac{\beta\gamma}{(\mu+\delta+\alpha)(\mu+\gamma)} > 1$ (8)

3.5: Endemic equilibrium

The endemic equilibrium state is the state where the disease persistent in the population.. In this situation, if $E^*(S^*I^*L^*R) \neq 0$,and the then endemic equilibrium state is given by;

$$(S^*, I^*, L^*, R^*) = \begin{bmatrix} \frac{(\mu+\delta+\alpha)}{\beta}, \frac{\theta(\mu+\delta+\alpha)}{\beta(\mu+\gamma)}, \frac{\beta\Lambda-(\mu+\tau)(\mu+\delta+\alpha)(\mu+\tau)}{(\mu+\tau)(\mu+\delta+\alpha)+\tau\alpha}, \\ \alpha\left[\frac{\beta\Lambda-(\mu+\tau)(\mu+\delta+\alpha)(\mu+\tau)}{(\mu+\tau)(\mu+\delta+\alpha)+\tau\alpha(\mu+\tau)}\right] \end{bmatrix}$$

3.1.9 Stability of the Endemic Equilibrium

Theorem: If $R_0 > 1$ then the endemic equilibrium is asymptotically stable.

Proof: using the Jacobian matrix, $J = \begin{bmatrix} -\beta I-(\mu+\theta) & -\beta S & 0 & \tau \\ \beta S & -(\mu+\delta+\alpha) & \gamma & 0 \\ \theta & 0 & -(\mu+\gamma) & 0 \\ 0 & \alpha & 0 & -(\mu+\tau) \end{bmatrix}$

At the endemic equilibrium state E^* the Jacobian matrix becomes;

$$J^* = \begin{bmatrix} -\beta S^* - (\mu+\theta) & -\beta S^* & 0 & \tau \\ \beta I^* & -(\mu+\delta+\alpha) & \gamma & 0 \\ \theta & 0 & -(\mu+\gamma) & 0 \\ 0 & \alpha & 0 & -(\mu+\tau) \end{bmatrix}$$

The characteristic equation is $[J - I\lambda]$ =0. Taking the dominant eigen value,

the basic reproduction number is.

$$R_0 = \frac{[\mu+\delta+\alpha]\{\beta[\beta(\beta\Lambda - (\mu+\tau)(\mu+\delta+\alpha)(\mu+\tau)(\mu+\gamma))]\}}{(\mu+\tau)(\mu+\delta+\alpha)+\tau\alpha}$$

The DEE is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. On the other hand, the EE is locally asymptotically stable when $R_0 > 1$ and unstable when $R_0 < 1$.

Proof:

If $R_0 < 1$the disease endemic e

$$\frac{[\mu+\delta+\alpha]\{\beta[\beta(\beta\Lambda - (\mu+\tau)(\mu+\delta+\alpha)(\mu+\tau)(\mu+\gamma))]\}}{(\mu+\tau)(\mu+\delta+\alpha)+\tau\alpha} < 1$$

$$\gamma < \frac{(\mu+\tau)(\mu+\delta+\alpha)+\tau\alpha}{[\mu+\delta+\alpha]\{\beta[\beta(\beta\Lambda - (\mu+\tau)(\mu+\delta+\alpha)(\mu+\tau)(\mu+\gamma))]\}} - \mu$$

If $R_0 > 1$ then,

The disease endemic equilibrium is asymptotically stable. Hence the disease will continue to exist in the population. Otherwise it will die out with time if $R_0 < 1$.

Conclusion

In this paper, the effect of latently infected population on the transmission of TB was analyzed. The endemic equilibrium state of the model using basic reproduction number shows that TB can be effectively controlled if the rate of both the latently and infectious class is always less than the product of total contraction and breakdown of susceptible class.From the results, as the transmission rate increases or as the recovery rate decreases, $R_0 > 1$ and the disease free

equilibrium is unstable. This indicates that the disease will spread when there is an outbreak.Consequently, as the transmission rate decreases, or the recovery rate increases. $R_0 < 1$the DFE will be stable hence the disease will not spread

The model gave a basic reproductive number $R_0 > 1$ This means that the disease will persist in the population.

5.0 Recommendations

TB transmission can be minimized in the population if effort is made to ensure that the endemic equilibrium of the model is never stable. This can be achieved if the following recommendations are considered;

1) People should be enlightened on the mode of TB transmission dynamics and home care strategies of people with TB.
2) The government should intensify the education on TB in the churches, schools, to the individuals in the communities of its existence, free access to medical care and treatment duration.
3) The government should integrate TB programs into other existing health services such as outreach, maternal and child welfare programs among others in order to increase its awareness.

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