Original Research Article

Solving Directly Second Order Initial Value Problems with Lucas Polynomial

Abstract

This paper presents a one step hybrid numerical scheme with one off grid points for solving directly the general second order initial value problems. The scheme is developed using collocation and interpolation technique invoked on Lucas polynomial. The proposed scheme is consistent, zero stable and of order four. This scheme can estimate the approximate solution at both step and off step points simultaneously by using variable step size. Numerical results are given to show the efficiency of the proposed scheme over some existing schemes of same and higher order.

Keywords: One -step hybrid method. Initial value problems, Lucas Polynomial, Collocation, Interpolation, Approximation

1 Introduction

Ordinary Differential Equations often appear in mathematical modeling of physical phenomena such as modeling and formulation of Pricing policy for the production of goods, modeling of population growths for two or more countries, modeling of chemical reactions, etc. In the recent years, many numerical methods for approximating the solutions of initial value problems have been developed by various authors.

In this paper, we are concerned with solutions of second order initial value problem of the form:

$$y'' = f(x, y.y'), \quad y(a) = \eta_0 \quad y'(a) = \eta_1 \tag{1.1}$$

where $\Re \times \Re^m \times \Re^m \to \Re^m$ and $y, y_0, y' \in \Re$ are given real constant.

Many Authors such as Henrici [8], Simos [15] Awoyemi[6] Adeniran, Akindeinde and Ogundare [12], Adeniran, Odejide and Ogundare [10], Adeniran and Ogundare[13] have devoted lots of attention to the development of various methods for solving directly (1) without reducing it to system of first order.

Adeniran, Akindeinde and Ogundare [12] develop a five-step trigonometrically fitted scheme for approximating solutions of second order ordinary differential equation with oscillatory solutions. Stability and convergence properties of the scheme were established. Numerical implementation of the scheme shows that it performs better than some of the existing methods in the literature.

Adeyeye and Omar [5]construct a five step block block methods to numerically approximate second order ordinary differential equations, which are developed with or without the presence of higher derivatives with same order, The resulting block methods are used to solve the second order ordinary differential equations. Numerical implementation of the method shows that it display better accuracy.

Jator and King [4] in their paper title " Integrating Oscillatory General Second-Order Initial Value Problems Using a Block Hybrid Method of Order 11" considered a Block Hybrid Method of order 11 for directly solving systems of general second-order initial value problems (IVPs), including Hamiltonian systems and partial differential equations (PDEs) which arise in areas of science and engineering. The properties of the Block hybrid method are well discussed and the performance of the method is demonstrated on some numerical examples.

Several numerical methods based on the use of polynomial functions (Power series, Legendre, Chebyshev, e.t.c) and non polynomial(trigonometric) have been used as basis function to develop numerical methods for direct solution of (1) using interpolation and collocation procedure.

Lucas polynomial in one variable can be defined as

$$L_n(x) = \sum_{0 \le j \le \frac{n}{2}} \frac{n}{n-j} \left({n-2j \atop j} \right) x^{n-2j}. \qquad (1.2)$$

(0)

Unfortunately, these polynomial are not so well studied in the theory of orthogonal polynomials, the reason being that, they are special case of Chebyshev polynomial(of the first kind) where x and y in the bivariate Lucas polynomial are replaced by $2x$ and -1 yielding the similar three term recurrence

$$L_n(x) = 2x.L_{n-1}(x) - L_{n-2}(x)$$

with initial values $L_0(x) = 1$ and $L_1(x) = x$
First few Lucas polynomial by (2) is given as

$$L_2(x) = x^2 + 2$$

$$L_3(x) = x^3 + 3x$$

$$L_4(x) = x^4 + 4x^2 + 2$$

$$L_5(x) = x^5 + 5x^3 + 5x$$

$$L_6(x) = x^6 + 6x^4 + 9x^2 + 2$$

In this paper, we are motivated by the work of Anake et. al.[7] to develop a continuous hybrid one step method using Lucas Polynomial as basis function.

2 Derivation of the Scheme

We consider Lucas polynomial series of the form

$$y(x) = \sum_{n=0}^{c+i-1} a_n L_n(x), \qquad (2.1)$$

as an approximate solution to equation (1.1).

where c and i are number of distinct collocation and interpolation points respectively. Substituting the second derivative of (2.1) into (1.1) gives

$$f(x, y(x), y'(x)) = \sum_{n=0}^{c+i-1} j(j-1) a_n L''_n(x), \qquad (2.2)$$

The paper consider a grid point of step length one and off step point at $x = x_{n+\frac{1}{2}}$. Collocating (2.2) at points $x = x_n$, $x_{n+\frac{1}{2}}$ and x_{n+1} and interpolating (2.1) at $x = x_n$ and $x_{n+\frac{1}{2}}$ leads to leads to a system of five equations which is solved by any linear system solvers such as Crammers rule to obtain a_n, $n = 0, 1, \cdots, 5$. The a_ns obtained are then substituted into (2.1) to obtain the continuous form of the method

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+\frac{1}{2}} + h^2 \left[\beta_0(u) f_n + \beta_1(u) f_{n++\frac{1}{2}} + \beta_2(u) f_{n+1}\right], \qquad (2.3)$$

where α_n and β_n are continuous coefficients. The continuous method (2.3) is used to generate the main method. That is, we evaluate at $x = x_{n+1}$

$$y_{n+1} - 2y_{n+\frac{1}{2}} + y_n = \frac{h^2}{48}\left[f_{n+1} + 10 f_{n+\frac{1}{2}} + f_n\right], \qquad (2.4)$$

In order to incorporate the second initial condition at (1.1) in the derived schemes, we differentiate (2.3) and evaluate at point $x = x_n$, $x = x_{n+\frac{1}{2}}$ and $x = x_{n+1}$ to have:

$$hy'_n - 2y_{n+\frac{1}{2}} + 2y_n = h^2\left[-\frac{7}{48} f_n - \frac{1}{8} f_{n+\frac{1}{2}} + \frac{1}{48} f_{n+1}\right], \qquad (2.5)$$

$$hy'_{n+\frac{1}{2}} - 2y_{n+\frac{1}{2}} + 2y_n = h^2\left[\frac{1}{16} f_n + \frac{5}{24} f_{n+\frac{1}{2}} - \frac{1}{48} f_{n+1}\right], \qquad (2.6)$$

$$hy'_{n+1} - 2y_{n+\frac{1}{2}} + 2y_n = h^2\left[\frac{1}{48} f_n + \frac{13}{24} f_{n+\frac{1}{2}} + \frac{3}{16} f_{n+1}\right]. \qquad (2.7)$$

The methods derived in equation 2.4 to 2.7 will be combined and implemented as a block in solving numerical examples.

3 Analysis of the Scheme

In this section, we analyze the derived scheme in (2.4 - 2.7) which includes the order and error constant, consistency, zero stability, convergence of the method and region of absolute stability.

3.1 Order and Error Constant

We adopt the method proposed by Fatunla [3] and Lambert [2] to obtain the order of our scheme(2.4-2.7) as $(4, 3, 3, 3)$ and error constants as $-\frac{1}{15360}$, $-\frac{1}{720}$, $\frac{7}{5760}$, $-\frac{1}{720}$

3.2 Consistency

According to Gurjinder et al.[11], A linear multistep method is said to be consistent if it has an order of convergence greater than 1 ($p \geq 1$). Thus, our derived schemes are consistent, since the order are 4 and 3 respectively.

3.3 Zero Stability

Definition 3.1 *A linear multistep method is Zero-stable for any well behaved initial value problem provided*

- *all roots of* $\rho(r) = 0$ *lies in the unit disk,* $|r| \leq 1$
- *any roots on the unit circle*($|r| \leq 1$) *are simple Lambert [?]*

Hence

$$\rho(z) = z - 2z^{\frac{1}{2}} + 1 \tag{3.1}$$

setting equation 3.1 equal to zero and solving for z gives $z = 1$, hence the method is zero stable.

3.4 Convergence

The convergence of our one step hybrid scheme is considered in the light of the fundamental theorem of Dahlquist(Henrici [9]), we state Dahlquists theorem without proof.

Theorem 3.1 *The neccessary and sufficient condition for a linear multistep to be convergent is for it to be consistent and zero stable*

Since the method is consistent and zero stable, hence it is convergent.

4 Numerical Examples

Example 1
We consider the following problem:

$$y'' = -1001y' - 1000y, \quad y(0) = 1, y'(0) = -1$$

whose exact solution is given by y(x) = exp(-x).
Source: Abhulimen and Okunuga (2008).

Example 2
We consider the non-linear initial value problem:

$$y'' - x(y')^2 = 0, \quad y(0) = 1, y'(0) = \frac{1}{2}$$

whose exact solution is given by $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$.
Source: Adeyeye and Omar (2018).

Table 1: Showing the exact solutions and the computed results from the proposed methods for Example 1, $h = 0.1$.

x	exact	Numerical	error
0.1	0.9048374180360	0.904837414703514	3.332446×10^{-9}
0.2	0.818730753077982	0.818730746689286	6.388696×10^{-9}
0.3	0.740818220681718	0.740818211523017	9.158701×10^{-9}
0.4	0.670320046035639	0.670320034395873	1.163977×10^{-8}
0.5	0.606530659712633	0.606530645877659	1.383497×10^{-8}
0.6	0.548811636094026	0.548811620342161	1.575187×10^{-8}
0.7	0.496585303791410	0.496585286390063	1.740135×10^{-8}
0.8	0.449328964117222	0.449328945320445	1.879678×10^{-8}
0.9	0.406569659740599	0.406569639787376	1.995322×10^{-8}
1.0	0.367879441171442	0.367879420284596	2.088685×10^{-8}

The numerical result for Example 1 were presented in Tables 1. The new hybrid method displayed better accuracy within the range of integration.

Table 2: Comparison of errors for Example 1.

x	Our new method $p = 3\&4,\ k = 1$	Adeniran et. al(2015) $p = 3,\ k = 1$	Error in Abhulimen & Okunuga(2008) Exponential fitted method $p = 5$
0.1	3.33×10^{-9}	2.56×10^{-09}	
0.2	6.39×10^{-9}	1.75×10^{-08}	5.90×10^{-10}
0.3	9.16×10^{-9}	9.68×10^{-08}	
0.4	1.16×10^{-8}	5.12×10^{-07}	1.20×10^{-09}
0.5	1.38×10^{-8}	2.68×10^{-06}	
0.6	1.58×10^{-8}	1.40×10^{-05}	1.80×10^{-09}
0.7	1.74×10^{-8}	7.31×10^{-05}	
0.8	1.88×10^{-8}	3.82×10^{-04}	1.80×10^{-09}
0.9	2.00×10^{-8}	2.00×10^{-03}	
1.0	2.09×10^{-8}	1.04×10^{-02}	1.80×10^{-09}

The error obtained from Example 1 shown in Table 1 were compared to to other existing method The new one step hybrid method displayed fair accuracy within the range of integration.

Table 3: Showing the exact solutions and the computed results from the proposed methods for Example 2, $h = 0.1$.

x	exact	Numerical	error
0.1	1.05004172927849	1.05004171876598	-1.051251×10^{-8}
0.2	1.10033534773107	1.10033532596417	-2.176690×10^{-8}
0.3	1.15114043593647	1.15114040131119	-3.462528×10^{-8}
0.4	1.20273255405408	1.20273250383304	-5.022104×10^{-8}
0.5	1.25541281188300	1.25541274169931	-7.018369×10^{-8}
0.6	1.30951960420311	1.30951950719359	-9.700952×10^{-8}
0.7	1.36544375427140	1.36544361955552	$-1.3471588 \times 10^{-7}$
0.8	1.42364893019360	1.42364874013572	$-1.9005788 \times 10^{-7}$
0.9	1.48470027859405	1.48470000368455	-2.749090×10^{-7}
1.0	1.54930614433406	1.54930573314847	$-4.1118559 \times 10^{-7}$

The maximum absolute error for the new method at $h = \frac{1}{100}$ is 5.20880×10^{-14} while that of Adeyeye and Omar(Five step block method) is 2.831069×10^{-13}.Thus the new proposed scheme display better accuracy.

5 Conclusion

The hybrid one step method for solving directly second order initial value problems generated in this paper is accurate, efficient, consistent and zero stable. This method is self-starting and requires only one grid functions evaluation at each integration step. The method complete favorably with other existing methods [[7],[13], [5],] in the literature.

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