NUMERICAL SOLUTION TO ONE-DIMENSIONAL CONSOLIDATION BY THE FINITE ELEMENT METHOD

Abstract - Adequate prediction of structures settlement is of utmost importance in order to prevent future failure of civil engineering structures due to excessive settlement resulting from an inadequate settlement prediction. In this paper, laboratory consolidation test was performed on five different clay samples from different locations to determine the soil consolidation in terms of pore water pressure. A formulation of Finite Element (FE) method was also developed for solving one-dimensional consolidation problem and its validity checked out. The one-dimensional consolidation differential equation was solved using finite element analysis by Rayleigh-Ritz method to obtain an approximate solution and ten elements were used to discretize the domain. MATLAB program was used to write the finite element codes. Considering the graphs generated from the MATLAB program which compares the consolidation behavior of the soil sample from analytical and numerical point of view, it is seen that there is a good agreement between Terzaghi's exact solution to consolidation behavior of soils and numerical solution using the finite element method.

Keywords: consolidation; one-dimensional; Terzaghi's solution; Finite Element Method; MATLAB

1.0 Introduction

The increase rate of failures of civil engineering structures in Nigeria, has made it necessary to study the settlement of structures, especially within the southern part of Nigeria. This area is made up of reclaimed lands, which are comprised of highly compressible weak organic and soft soils of varying thickness. The construction of structures like high rise buildings or embankments on compressible soils with a high-level water table often leads to lateral pressures and movement, intolerable settlements as well as slope and bearing failures. This may result in long construction delays and costly remedial works. According to Aseeja (2016), any site with soft soils needs to be analyzed, so that the mitigating strategies/techniques to avoid problems associated with soft soil can be best identified and implemented. Consolidation analysis predicts the long-term behavior of structures built on soils. As a result, substantial effort has been put forth to better understand how soils consolidate and what factors affect the process of consolidation. Karl von Terzaghi was among the first to develop an analytical theory to explain and predict the process in fine-grained soils (Craig, 2004). According to Amit et al (2014), consolidation characteristics of clays play an important role in finding the settlement of foundations and simulation of other structures for their stress-deformation response. In this research work, the consolidation behavior of clays would be analyzed from an analytical (laboratory experiment) and numerical modelling (computer program) point of view and the results validated. Numerical study of consolidation has the advantage of incorporating finite strain in its consolidation analysis which improves on the small strain assumption in Terzaghi's theory of consolidation. It also considers variations of compressibility and permeability during consolidation which is another limitation of Terzaghi's theory of consolidation. Numerical study gives a more realistic settlement prediction. The numerical modelling is based on the principle of Finite Element Method (Singh, 2014). In this research, MATLAB was used to write a Finite Element Analysis program for one-dimensional analysis of clay consolidation.

2.0 One-Dimensional Consolidation Differential Equation

The basic differential equation of one-dimensional consolidation is as shown below:

$$C_v \frac{\partial^2 \overline{u}}{\partial z^2} = \frac{\partial \overline{u}}{\partial t} \qquad (1)$$

Where C_v is the coefficient of consolidation and is given by:

$$C_v = \frac{k}{\gamma_w m_v} = \frac{k}{g\rho_w m_v} \qquad (2)$$

The basic differential equation of one-dimensional consolidation given in eqn. (1) gives the distribution of excess hydrostatic pressure $\overline{u}$ with depth *z* and time *t*. The analytical solution of eqn. (1) is obtained by Fourier series and the exact solution presented as follows:

$$\overline{u} = \frac{4}{\pi}\overline{u}_i \sum_{N=0}^{N=\infty} \frac{1}{(2N+1)}\left[\sin\frac{(2N+1)nz}{H}\right] e^{-\left[(2N+1)^2 \times \frac{\pi^2}{H^2}\right]C_v t} \qquad (3)$$

In this exact solution equation obtained by Fourier series, z and H represents thickness of soil and drainage path respectively. This equation can be generally applied to any soil of initial pore water pressure $\overline{u}_i$ (Arora, 2008).

3.0 Finite Element Solution to One-Dimensional Consolidation Differential Equation

Finite Element solution to one-dimensional consolidation starts from declaring the differential equation governing it. The basic differential equation of one-dimensional consolidation is given as:

$$m_v \frac{\partial u}{\partial t}\gamma_w - k\frac{\partial^2 u}{\partial z^2} = 0 \qquad (4)$$

The weak form of equation (4) is obtained by multiplying it by a weight function W and then integrated over the domain of the problem.

$$0 = \int_\Omega W\left[m_v \frac{\partial u}{\partial t}\gamma_w - k\frac{\partial^2 u}{\partial z^2}\right]dz \qquad (5)$$

$$0 = \int_\Omega \left[W m_v \frac{\partial u}{\partial t}\gamma_w - Wk\frac{\partial^2 u}{\partial z^2}\right]dz \qquad (6)$$

Integrating by parts, we obtain

$$0 = \int_{x_A}^{x_B}\left[W m_v \frac{\partial u}{\partial t}\gamma_w + k\frac{\partial W}{\partial z}\frac{\partial u}{\partial z}\right]dz - \left[W\frac{\partial u}{\partial z}\right]_{x_A}^{x_B} \qquad (7)$$

Where the domain of the problem is between x_A and x_B
We assume an approximate solution given by the form:

$$u(z,t) \approx \sum_{j=1}^{n} u_j^e(t)\psi_j^e(z) \qquad (8)$$

Where u_j denote the value of $u(z,t)$ at the spatial location (z_j) and time t. And ψ_j is the interpolation function used for the approximation (shape function). The semi discrete model used is obtained from equation (7) by substituting the finite element approximation equation (8) into equation (7) and substituting the weight function W with ψ_i.

$$0 = \int_{x_A}^{x_B}\left[m_v\gamma_w\psi_i\left(\sum_{j=1}^{n}\frac{du_j}{dt}\psi_j\right) + k\frac{d\psi_i}{dz}\left(\sum_{j=1}^{n}\frac{d\psi_j}{dz}u_j\right)\right]dz - Q$$

(9)

Where $$Q = \frac{du}{dz} \qquad (10)$$

In matrix form, we have:

$$\left[M_{ij}^{e}\right]\left\{\dot{u}^{e}\right\}+\left[k_{ij}^{e}\right]\left\{u^{e}\right\}=\left\{Q^{e}\right\} \tag{11}$$

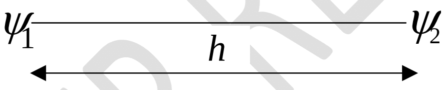
Where

$$M_{ij}^{e}=\int_{x_A}^{x_B}\left[m_v\gamma_w\psi_i\psi_j\right]dz \tag{12}$$

$$K_{ij}^{e}=\int_{x_A}^{x_B}\left[k\frac{\partial\psi_i}{\partial z}\frac{\partial\psi_j}{\partial z}\right]dz \tag{13}$$

$$Q^{e}=\int\frac{du_i}{dz}\psi_i \tag{14}$$

Linear Lagrange Interpolation function is used as given in lower coordinate below



$$\psi_1^{e}=1-\frac{z}{h} \tag{15}$$

$$\psi_2^{e}=\frac{z}{h} \tag{16}$$

The derivative of the interpolation function with respect to z are as follows:

$$\frac{d\psi_1}{dz}=\frac{-1}{h} \tag{17}$$

$$\frac{d\psi_2}{dz}=\frac{1}{h} \tag{18}$$

From equation (12), the coefficient matrix M_{ij}^{e} for an element "*e*" is evaluated

for $i=1$ and $j=1$

$$M_{11}^{e}=\int_0^h\left[m_v\gamma_w\psi_i\psi_j\right]dz \tag{19}$$

Where h is the domain of an element

$$M_{11} = \int_0^h m_v \gamma_w \left(1 - \frac{z}{h}\right)\left(1 - \frac{z}{h}\right) dz \tag{20}$$

$$M_{11} = \int_0^h \left[m_v \gamma_w \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right)\right] dz \tag{21}$$

$$M_{11} = \left[m_v \gamma_w \left(z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right)\right]_0^h \tag{22}$$

$$M_{11} = \left[m_v \gamma_w \left(h - h + \frac{h^3}{3h^2}\right)\right] \tag{23}$$

$$M_{11} = m_v \gamma_w \frac{h}{3} \tag{24}$$

Similar computation is done for M_{12}, M_{21} and M_{22}

For M_{12} we have

$$M_{12} = \int_0^h \left[m_v \gamma_w \left(1 - \frac{z}{h}\right)\left(\frac{z}{h}\right)\right] dz \tag{25}$$

$$M_{12} = \int_0^h \left[m_v \gamma_w \left(\frac{z}{h} - \frac{z^2}{h^2}\right)\right] dz \tag{26}$$

$$M_{12} = \left[m_v \gamma_w \left(\frac{z^2}{2h} - \frac{z^3}{3h^2}\right)\right] \tag{27}$$

$$M_{12} = \left[m_v \gamma_w \left(\frac{h}{2} - \frac{h}{3}\right)\right] = m_v \gamma_w \frac{h}{6} \tag{28}$$

In matrix form

$$M_{ij} = m_v \gamma_w h \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \tag{29}$$

Similarly, K_{ij} is evaluated from equation (13)

for $i = 1$ and $j=1$

$$K_{11} = \int_0^h \left[k \left(\frac{-1}{h} \right) \left(\frac{-1}{h} \right) \right] dz \tag{30}$$

$$K_{11} = \left[\frac{k^2}{h^2} \right]_0^h \tag{31}$$

$$K_{11} = \left[\frac{k}{h} \right] \tag{32}$$

for $i = 1$ and $j=2$

$$K_{12} = \int_0^h \left[k \left(\frac{-1}{h} \right) \left(\frac{1}{h} \right) \right] dz \tag{33}$$

$$K_{12} = \left[\frac{-k^2}{h^2} \right]_0^h \tag{34}$$

$$K_{12} = \left[\frac{-k}{h} \right] \tag{35}$$

Similar computation is done for K_{21} and K_{22} and the following values shown in matrix form are obtained.

$$K_{ij} = \begin{bmatrix} \frac{k}{h} & \frac{-k}{h} \\ \frac{-k}{h} & \frac{k}{h} \end{bmatrix} \tag{36}$$

For the manual finite element analysis, 8 linear elements will be used to discretize the domain.

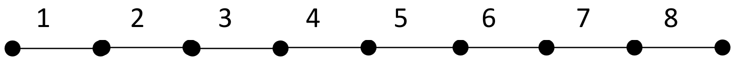


Fig. 1: Eight Element Discretization

Using 8 elements produce nine nodes.

Equation (11) can be rewritten for one element as shown

$$\begin{bmatrix} M^1{}_{11} & M^1{}_{12} \\ M^1{}_{21} & M^1{}_{22} \end{bmatrix} \begin{Bmatrix} \overset{\square}{u}_1 \\ \overset{\square}{u}_2 \end{Bmatrix} + \begin{bmatrix} K^1{}_{11} & K^1{}_{12} \\ K^1{}_{21} & K^1{}_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

(37)

The assembled equation for eight elements is given as

$$\begin{bmatrix} M_{11}^{1} & M_{12}^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{21}^{1} & M_{22}^{1}+M_{22}^{2} & M_{12}^{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{22}^{2} & M_{22}^{2}+M_{11}^{3} & M_{12}^{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{21}^{3} & M_{22}^{3}+M_{11}^{4} & M_{12}^{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{21}^{4} & M_{22}^{4}+M_{11}^{5} & M_{12}^{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{21}^{5} & M_{22}^{5}+M_{11}^{6} & M_{12}^{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{21}^{6} & M_{22}^{6}+M_{11}^{7} & M_{12}^{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{21}^{7} & M_{22}^{7}+M_{11}^{8} & M_{12}^{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{21}^{8} & M_{22}^{8} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \\ \ddot{u}_7 \\ \ddot{u}_8 \\ \ddot{u}_9 \end{Bmatrix}$$

$$+$$

$$\begin{bmatrix} K_{11}^{1} & K_{12}^{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{21}^{1} & K_{22}^{1}+K_{22}^{2} & K_{12}^{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22}^{2} & K_{22}^{2}+K_{11}^{3} & K_{12}^{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{21}^{3} & K_{22}^{3}+K_{11}^{4} & K_{12}^{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{21}^{4} & K_{22}^{4}+K_{11}^{5} & K_{12}^{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{21}^{5} & K_{22}^{5}+K_{11}^{6} & K_{12}^{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{21}^{6} & K_{22}^{6}+K_{11}^{7} & K_{12}^{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{21}^{7} & K_{22}^{7}+K_{11}^{8} & K_{12}^{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{21}^{8} & K_{22}^{8} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

$$= \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 + Q_1^5 \\ Q_2^5 + Q_1^6 \\ Q_2^6 + Q_1^7 \\ Q_2^7 + Q_1^8 \\ Q_2^8 \end{Bmatrix} \tag{38}$$

To solve equation (38), we use α - family of approximation given

$$\left(\left[M^e\right]+\Delta t\alpha\left[K^e\right]\right)\left\{u^e\right\}_{s+1}=\left(\left[M^e\right]-\Delta t\left(1-\alpha\right)\left[K^e\right]\right)\left\{u^e\right\}_s+\Delta t\left(\alpha\left\{Q^e\right\}_{s+1}+(1-\alpha)\left\{Q^e\right\}_s\right) \tag{39}$$

Where $\left[M^e\right]$ is the assembled M_{ij} matrix

$\left[K^e\right]$ is the assembled K_{ij} matrix

$\left\{Q^e\right\}$ is the assembled Q_i matrix

Δt is the time step

α could be any value between 0 and 1

$\left\{u^e\right\}_{s+1}$ is the next value of $\left\{u^e\right\}_s$ after an iteration

The boundary condition of equation (1) is

$$\begin{aligned} u\left(0,t\right)&=0 \\ u\left(7,t\right)&=0 \end{aligned} \tag{40}$$

And the initial condition is

$$u\left(z,0\right)=100 \text{ kPa} \tag{41}$$

Due to balance of flux at the connecting nodes in equation (38)

$$\begin{Bmatrix} Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 + Q_1^5 \\ Q_2^5 + Q_1^6 \\ Q_2^6 + Q_1^7 \\ Q_2^7 + Q_1^8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{42}$$

Q_1^1 and Q_2^8 are unknown, hence we eliminate row 1 and row 9. Then from equation (40)

$$u_1 = u_9 = 0 \tag{43}$$

After applying boundary condition to equation (38), we obtain:

$$\begin{bmatrix} (M^1_{22}+M^2_{11}) & M^2_{12} & 0 & 0 & 0 & 0 & 0 \\ M^2_{21} & (M^2_{22}+M^3_{11}) & M^3_{12} & 0 & 0 & 0 & 0 \\ 0 & M^3_{21} & (M^3_{22}+M^4_{11}) & M^4_{12} & 0 & 0 & 0 \\ 0 & 0 & M^4_{21} & (M^4_{22}+M^5_{11}) & M^5_{12} & 0 & 0 \\ 0 & 0 & 0 & M^5_{21} & (M^5_{22}+M^6_{11}) & M^6_{12} & 0 \\ 0 & 0 & 0 & 0 & M^6_{21} & (M^6_{22}+M^7_{11}) & M^7_{12} \\ 0 & 0 & 0 & 0 & 0 & M^7_{21} & (M^7_{22}+M^8_{11}) \end{bmatrix} \begin{Bmatrix} \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \\ \dot{u}_7 \\ \dot{u}_8 \end{Bmatrix}$$

$$+$$

$$\begin{bmatrix} (K^1_{22}+K^2_{11}) & K^2_{12} & 0 & 0 & 0 & 0 & 0 \\ K^2_{21} & (K^2_{22}+K^3_{11}) & K^3_{12} & 0 & 0 & 0 & 0 \\ 0 & K^3_{21} & (K^3_{22}+K^4_{11}) & K^4_{12} & 0 & 0 & 0 \\ 0 & 0 & K^4_{21} & (K^4_{22}+K^5_{11}) & K^5_{12} & 0 & 0 \\ 0 & 0 & 0 & K^5_{21} & (K^5_{22}+K^6_{11}) & K^6_{12} & 0 \\ 0 & 0 & 0 & 0 & K^6_{21} & (K^6_{22}+K^7_{11}) & K^7_{12} \\ 0 & 0 & 0 & 0 & 0 & K^7_{21} & (K^7_{22}+K^8_{11}) \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{44}$$

Using the following soil parameters for sample 1

soil permeability, $K = 6.28 \times 10^{-5}$ m/s
coefficient of volume compressibility, $m_v = 5.6 \times 10^{-4} m^2 / kN$
unit weight of water, $\gamma_w = 9.81 kN / m^3$
soil depth, $z = 7m$
modulus of elasticity, $E = 1.8 MN / m^2$
$\alpha = 0.5$
$\Delta t = 0.1$

Using 8 elements discretization of the domain yields $h = \frac{7}{8} = 0.875$

Equation (29) becomes

$$M_{ij} = \begin{bmatrix} 0.0016 & 0.0008 \\ 0.0008 & 0.0016 \end{bmatrix} \tag{45}$$

And equation (36) becomes

$$K_{ij} = 1\times10^{-4}\begin{bmatrix} 0.7173 & -0.7173 \\ -0.7173 & 0.7173 \end{bmatrix} \tag{46}$$

Since h is used as constant of 0.875 for the elements, then

$$K_{ij}^{1} = K_{ij}^{2} = K_{ij}^{3} = K_{ij}^{N} \tag{47}$$

$$M_{ij}^{1} = M_{ij}^{2} = M_{ij}^{3} = M_{ij}^{N} \tag{48}$$

Solving eqn. (44) using all stated parameters for the first iteration after 0.1 days, we obtain:

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}_{s=0.1} = \begin{Bmatrix} 99.7071 \\ 100.0781 \\ 99.9779 \\ 100.0109 \\ 99.9779 \\ 100.0781 \\ 99.7071 \end{Bmatrix} \tag{49}$$

Similar computations for second iteration with $s = 0.2$ days yields:

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}_{s=0.2} = \begin{Bmatrix} 99.4165 \\ 100.1541 \\ 99.9569 \\ 100.0212 \\ 99.9569 \\ 100.1541 \\ 99.4165 \end{Bmatrix} \tag{50}$$

With the help of the MATLAB program, developed iterations of over 3000 can be achieved and also a higher number of discretization (say using 50 elements) can be done. Using 8 elements manual discretization and time step of 0.1 days with the program developed after 3000 iterations, the following result is gotten at $s = 300 days$.

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}_{s=300} = \begin{Bmatrix} 20.4231 \\ 37.7296 \\ 49.2864 \\ 53.3429 \\ 49.2864 \\ 37.7296 \\ 20.4231 \end{Bmatrix}$$

4.0 Results and Discussion

Figures 1, 2, 3, 4, and 5 show the plots of soil consolidation with depth from numerical solution and from Terzaghi's solution (exact solution). These graphs show the soil consolidation in terms of pore water pressure after 1000 days (10,000 iterations) using 10 elements discretization of the domain for the five soil samples with different soil parameters.

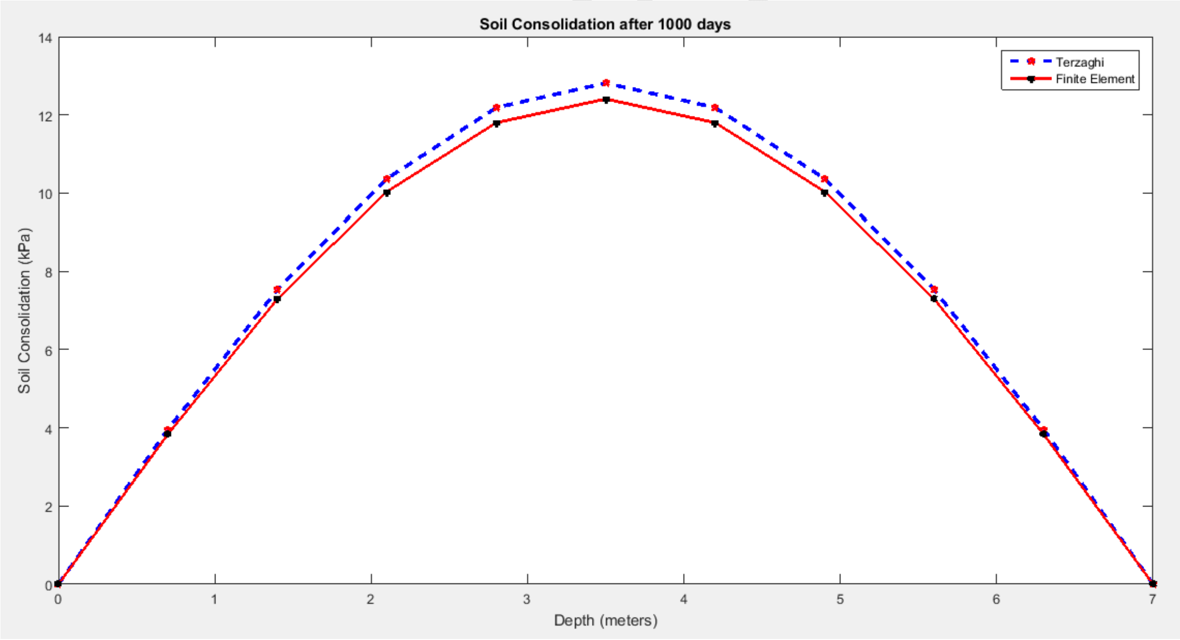


Figure 1. Comparisons of soil consolidation along depth at time, t =1000 days (Location 1)

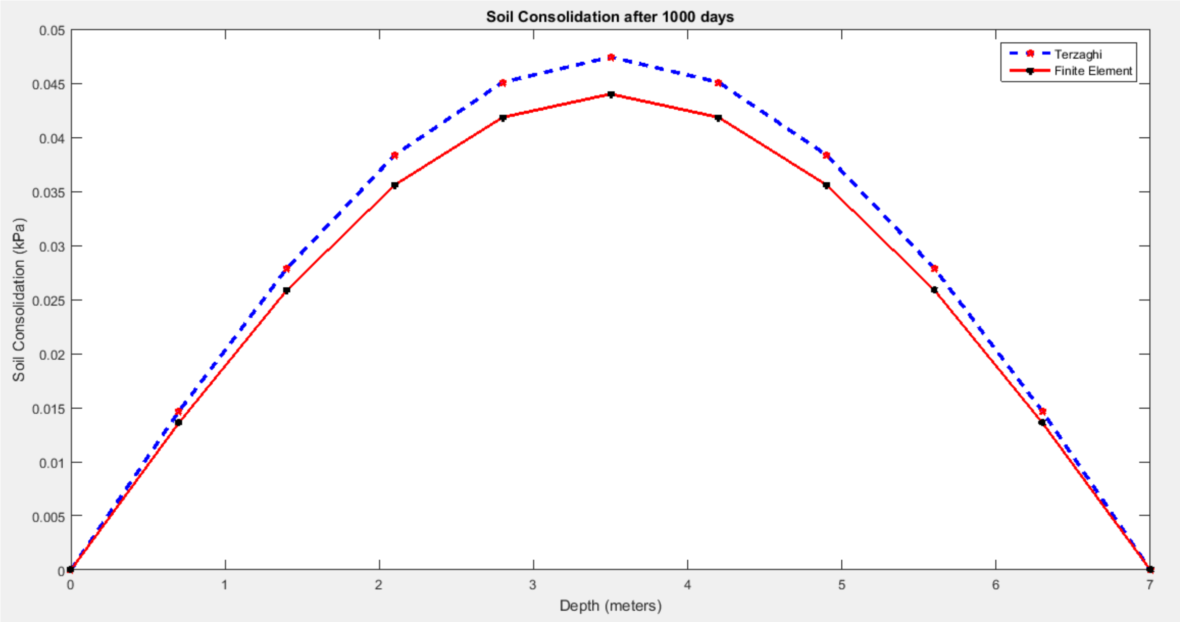


Figure 2. Comparisons of soil consolidation along depth at time, t =1000 days (Location 2)

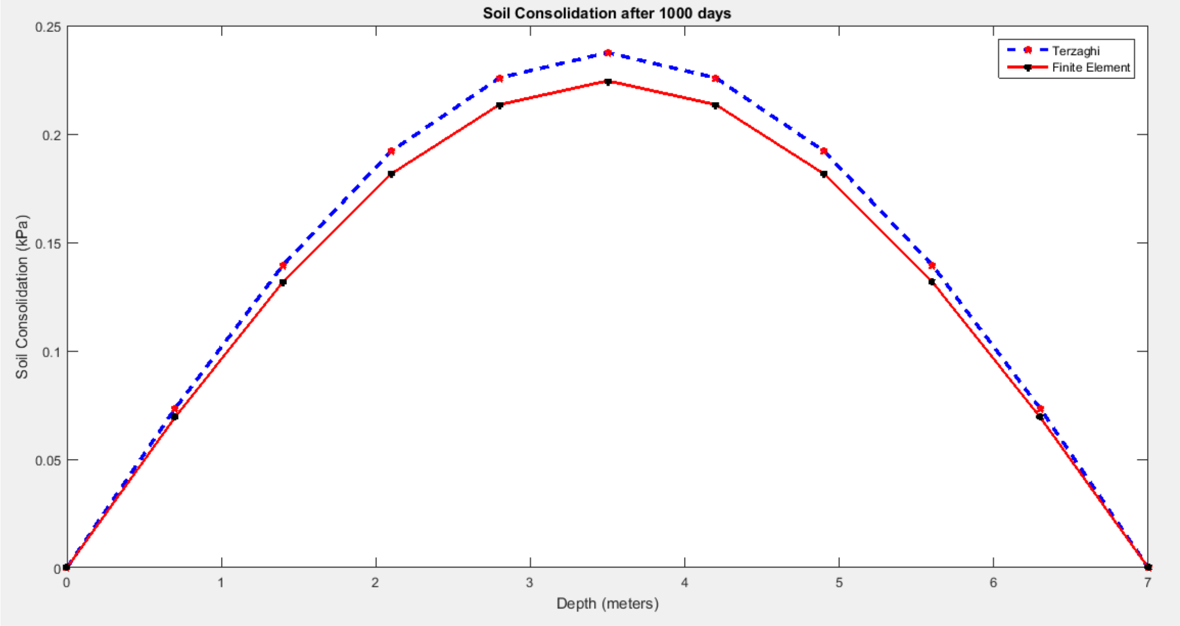


Figure 3. Comparisons of soil consolidation along depth at time, t =1000 days (Location 3)

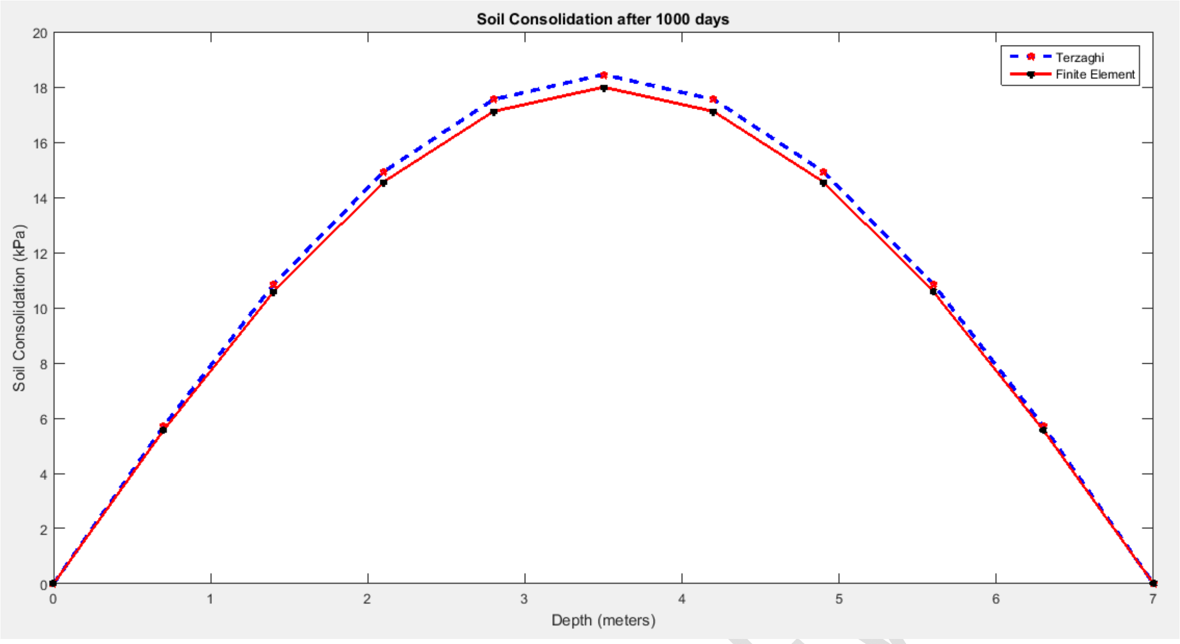


Figure 4. Comparisons of soil consolidation along depth at time, t =1000 days (Location 4)

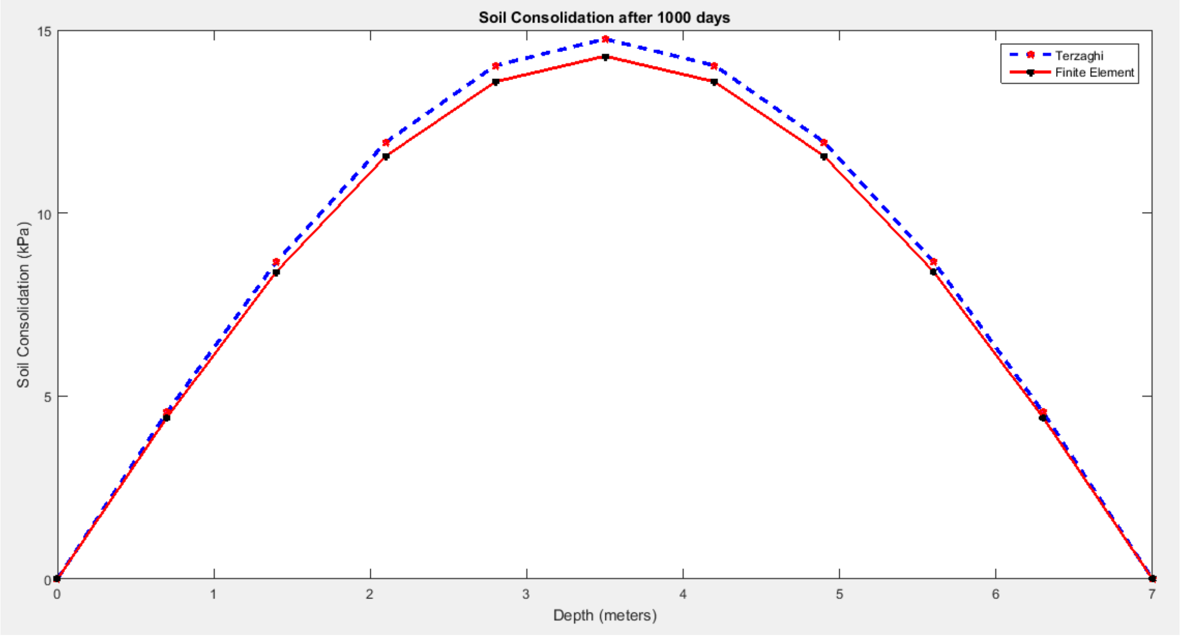


Figure 5. Comparisons of soil consolidation along depth at time, t =1000 days (Location 5)

5.0 Conclusion

One-dimensional consolidation problem was solved successfully using the finite element method in this study and the program of the formulation written with MATLAB program. Considering the graphs generated from the MATLAB program which compares the consolidation behavior of the soil sample from analytical and numerical point of view, it is seen that there is a good agreement between Terzaghi’s exact solution to consolidation behavior of soils and numerical solution using the finite element method.

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